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Metrization and stratification of squares of topological spaces

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Abstract

Various separation properties, from normality to monotone normality to proto-metrizability, are presented on the common framework of neighbourhood assignments. Two hybrid separation properties, incorporating features from all of them, but in weak concentrations, are defined and shown to be equivalent to metrizability and stratifiability for squares of Hausdorff spaces with embeddings of $\omega + 1$. © 1998 Elsevier Science B.V.

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Respectfully dedicated to the memory of Professor K. Morita

The classical theorem of Katětov [13], asserting that, for compact (Hausdorff) cubes, metrizability is no more than *hereditary normality*, is a result that has not ceased to fascinate in the half century since its publication in 1948 (see Introduction of [12] and [15, p. 24], for example). Hereditary normality lacking even a suggestion of first countability in its constitution, one wonders how it can be equivalent to metrizability. The answer lies of course in the *structure*, and in the compactness, of the cube. Specifically, hereditary normality on a compact cube ensures a G_δ*-diagonal* on the square and thus metrizability on the space itself, according to Šneĭder [16]. Zenor [17] pointed out that *hereditary countable paracompactness* can also play the role that hereditary normality plays. More recently, Gruenhage showed that, for compact (Hausdorff) squares, metrizability is no more than *hereditary paracompactness* [5, Theorem 2.6], it being that, on compact squares, hereditary paracompactness ensures a G_δ*-diagonal.* This result was later extended to squares of completely regular p-spaces by Gruenhage and Pelant [6].

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In all these theorems, metrizability is equated with some *hereditary separation axioms* [1] in the presence of *compactness*. [2] Compactness being a very strong property, any contribution on the part of the (peculiarity of the) *structure* of cubes and squares in the bringing about of metrizability is obscured. It is therefore of considerable interest to do away with the compactness assumption as much as possible and see, on squares, what *part* of metrizability determines the whole.

We are thus going to study the metrizability of squares of topological spaces, each with at least one copy of $\omega+1$ embedded in it (spaces where no accumulation is possible and spaces where capacity to accumulate is limited to subsets of uncountable cardinality being either trivially metrizable or obviously otherwise). We are to use the common framework of neighbourhood assignments for the description of various established separation properties, from normality to proto-metrizability (Section 1.4), providing the backdrop against which we can see the relative strength, in various directions, of the separation property which we define (Section 1.5) and which on squares is enough to account for metrizability. There is a parallel treatment of stratifiability on squares, and we have a sharpening of the theorem of Zenor [7], [4, Theorem 5.22], and its corollaries. These results, both the one on metrizability and the one on stratifiability, can, of course, be formulated on the countable power, X^{ω}, in the manner of Zenor.

No separation axioms are assumed. We write $\overline{\mathcal{U}}$ for the family $\{\mathrm{Cl}\, U\colon\ U \in \mathcal{U}\}$.

1. Preliminaries

(1) Given an open subset B on a topological space Z. If, for every $x \in B$, we have a neighbourhood $(x)_B \subset B$ *assigned* to x, we say we have a *neighbourhood assignment* $\{(x)_B\colon\ x \in B\}$ on B.

(i) If all these neighbourhoods are open, we say the assignment is *open*.

(ii) If, for every $x \in B$, $(x)^*_B \equiv \{y \in B\colon\ x \in (y)_B\}$ is also a neighbourhood of x, we say the assignment is *topologically symmetric* and speak of a *dual assignment*.

(iii) If, for all $x, y \in B$, $x \in (y)_B \Leftrightarrow y \in (x)_B$, we say the assignment is *symmetric*.

(iv) If there is a family $\mathcal{U}$ of open subsets such that $\bigcup \mathcal{U} = B$ and, for every $x \in B$, we have $(x)_B = \mathrm{St}\,(x, \mathcal{U})$, we say the assignment is *stellar*. Clearly, a *stellar* assignment is both *open* and *symmetric*.

(v) If, for each $n \in \omega$, we have a neighbourhood assignment $\{(x)_{B,n}\colon\ x \in B\}$, we say we have a *countable family of assignments* for the open subset B.

(2) Let a *family* $\mathcal{B}$ of open subsets be given. If, for every $B \in \mathcal{B}$, we have a neighbourhood assignment $\{(x)_B\colon\ x \in B\}$ on B, we say we have a *neighbourhood assignment* $\{\{(x)_B\colon\ x \in B\}\colon\ B \in \mathcal{B}\}$ on $\mathcal{B}$ and speak of countable families of open, topologically symmetric, symmetric or stellar assignments *on the family* $\mathcal{B}$.

[1] Countable paracompactness and the like can be viewed as separation axioms (see [1, Section 2.3]). G_δ-diagonal can also be so viewed (see Section 1.6).

[2] Completely regular p-spaces, in the presence of paracompactness, are perfect preimages of metrizable spaces, where the preimages of points are all compact ([4, Corollary 3.7], [14, Corollary 2 to Theorem VI.29]).

(3) A neighbourhood assignment on $\mathcal{B}$ is said to be *symmetrically T_2-separating* if, given $A, B \in \mathcal{B}$, $x \in A\backslash B$, $y \in B\backslash A$, we have $(x)_A \cap (y)_B = \emptyset$; *$T_2$-separating* (respectively, *weakly* T_2-separating) if, given $\xi \in Z$, any neighbourhood Ξ of ξ and any $\mathcal{A} \subset \mathcal{B}$ such that $\xi \notin \bigcup \mathcal{A}$ (respectively, $\xi \notin \bigcup \overline{\mathcal{A}}$), there is a neighbourhood Υ of ξ such that $(b)_B \cap \Upsilon = \emptyset$ when $b \in B\backslash\Xi$ for some $B \in \mathcal{A}$.

(4) In the terminology of the above, we see that, if symmetrically T_2-separating neighbourhood assignments are imposed on

(i) all[3] the binary open covers $\mathcal{C}$,

(ii) all the pairs $\mathcal{P}$ of open subsets, and

(iii) the family $\mathcal{B}$ of all open sets,

we have, in turn, *normality*, *hereditary normality* and *monotone normality*.

If, in the case of monotone normality, the assignments are *stellar*, we have *proto-metrizability* [9]. If, on the other hand, the assignments are *only* topologically symmetric and T_2-separating, we have *metrical normality* [10]. It is well known that monotone normality does not imply paracompactness (see, e.g., [2]) while proto-metrizability does. It is interesting to note that, if, in the case of metrical normality, the assignments are further required to be *open*, we also have paracompactness.

(5) The concept of metrical normality can be weakened by breaking up the comprehensive family of all open sets and by a restriction on the extent of the pieces and by a relaxation of the T_2-separation (required of the assignments) to beyond the *weak* T_2-separation and of the requirements on the assignments themselves. We have thus the property of Weak Assigned Separation (WAS):

On every *countable* family $\mathcal{B}$ of *pairwise disjoint* open subsets, there is a *countable* family of topologically symmetric neighbourhood assignments

$$\langle\{\{(b)_{B,n}\colon\ b \in B\}\colon\ B \in \mathcal{B}\}\rangle,$$

to be called a Weak Separation Assignment (WSA), so that, for any $\xi \notin \bigcup\overline{\mathcal{B}}$ and any neighbourhood Ξ of ξ, there are

(i) a neighbourhood Υ of ξ,

(ii) an infinite subfamily $\mathcal{C} \subset \mathcal{B}$, and

(iii) an $n(B) \in \omega$ for every $B \in \mathcal{C}$,

so that $(b)_{B,n(B)} \cap \Upsilon = \emptyset$ whenever $b \in B\backslash\Xi$ for some $B \in \mathcal{C}$.

Clearly, the P-spaces of Gillman and Henriksen [3] (where the G_δ's are open) have the property of WAS. If, in the above, the requirement of *topological symmetry* on the neighbourhood assignments is absent, we speak, instead, of a WSA$^-$ and of the property WAS$^-$.

(6) In the above, separation is demanded of points only if they are *to some degree* separated already (see, e.g., (3) above where $x \in A\backslash B$, $y \in B\backslash A \Rightarrow (x)_A \cap (y)_B = \emptyset$). Thus we are really looking at some *intensification* of separation. If, on the other hand, we forgo the initial requirement of separation, we can account for the G_δ-diagonal and

[3] Even if $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$ for two distinct binary open covers $\mathcal{C}_1$ and $\mathcal{C}_2$ are equal, it is *not* required that the assignments on them be identical.

metrizability (among others) with the same framework of neighbourhood assignments. Thus, the existence of a G_δ-*diagonal* on (the square of) X is equivalent to the existence on (the open set) X of a countable family of stellar assignments $\langle\{(x)_n\colon x \in X\}\rangle$ such that, given $x \neq y$, there exists such an $n \in \omega$ that $y \notin (x)_n$.

Metrizability of a T_0-*space* X is equivalent to the existence on (the open set) X of a countable family of topologically symmetric neighbourhood assignments $\langle\{(x)_n\colon x \in X\}\rangle$ such that, given $x \in X$ and a neighbourhood W, there are an $n \in \omega$ and a neighbourhood V of x such that $(y)_n \cap V = \emptyset$ if $y \notin W$ (see [8, part of Corollary 2.4], [11, Theorem 4.19] and [15, Theorem 2]). If, in the above, the requirement of topological symmetry on the neighbourhood assignments is omitted, we have, instead, the stratifiability of T_1-spaces. (Conventions are that stratifiable spaces are T_1, see, e.g., [14, Definition VI.9].)

2. Main results

Theorem 1. *Given a* T_0-*space* X. *On* $X \times (\omega + 1)$, *metrizability is equivalent to Weak Assigned Separation.*

Proof. It suffices to prove the metrizability of X, given WAS on $X \times (\omega + 1)$. Let $B_i \equiv X \times \{i\}$, for every $i \in \omega$. Let $\mathcal{B} = \{B_i\colon i \in \omega\}$. Let there be a WSA

$$\langle\{\{(b)_{B,n}\colon b \in B\}\colon B \in \mathcal{B}\}\rangle$$

on $\mathcal{B}$. We are to construct the neighbourhoods $[x]_{i,n}$ of x on X for all $i, n \in \omega$ and $x \in X$. For any $i, n \in \omega$ and $x \in X$, let $[x]_{i,n}$ be such that $((x,i))_{B_i,n} = [x]_{i,n} \times \{i\}$. Clearly, given $x \in X$ and an open neighbourhood W of x, $\Xi \equiv W \times (\omega + 1)$ is a neighbourhood of $\xi \equiv (x, \omega)$ such that $\xi \notin \bigcup \overline{\mathcal{B}}$ and there are, by hypothesis,

(i) a neighbourhood Υ of ξ that can be taken to be $V \times ((\omega+1) \backslash N)$ for some $N \in \omega$,

(ii) a $j > N$, and

(iii) an $n \in \omega$,

so that $((y,j))_{B_j,n} \cap \Upsilon = \emptyset$, whenever $y \notin W$. We have therefore $[y]_{j,n} \cap V = \emptyset$ whenever $y \notin W$. Metrizability of X follows (Section 1.6). □

Theorem 2. *Given a* T_1-*space* X. *On* $X \times (\omega + 1)$, *stratifiability is equivalent to* WAS^- (*Section* 1.5).

Remarks. (1) Theorem 2 strengthens the result of Zenor [7]. Corollaries 4.4–4.6 in [7] are similarly strengthened via Theorem 4 below. We have therefore the result that, on squares of completely regular p-spaces, metrizability is equivalent to WAS^- (cf. Corollary 5 of [6] cited in the Introduction).

(2) Even if, in the definition of a WSA^-, the assignment on $\mathcal{B}$ is made *dependent* on the choice of Ξ, but not on $\xi \in \Xi$, we already have a property on $X \times (\omega + 1)$ that

ensures countable paracompactness on X. For, given any decreasing sequence $\langle F_i \rangle$ of closed subsets of X with empty intersection, *dependent* on

$$\Xi \equiv (X \times \{\omega\}) \cup \bigcup \{(X \backslash F_i) \times \{i\}\colon\ i \in \omega\},$$

an open neighbourhood of $X \times \{\omega\}$, there is an assignment on the $\mathcal{B}$ as defined in the proof of Theorem 1. With the $[x]_{i,n}$'s similarly defined and with

$$U_j \equiv \bigcap \Big\{ \bigcup \{[x]_{i,n}\colon\ x \in F_i\}\colon\ i, n \leqslant j \Big\},$$

we have in the sequence $\langle U_j \rangle$ the requirement for countable paracompactness according to Ishikawa [14, Theorem V.6].

Lemma 3. *Given a Hausdorff space X. If can be embedded $\omega + 1$ in X, there is such a family of pairwise disjoint open subsets $\{U_n\colon\ n \in \omega\}$ in X that $n \in U_n$ and that $\omega \notin \operatorname{Cl} U_n$ for every $n \in \omega$.*

Theorem 4. *Given a Hausdorff space X, embedded in which is a copy of $\omega + 1$. The property of Weak Assigned Separation on X^2 is a (necessary and) sufficient condition for metrizability. The property of WAS$^-$ on X^2 is a (necessary and) sufficient condition for stratifiability.*

Proof. To prove metrizability of X, given WAS on X^2, let $B_i \equiv X \times U_i$ (U_i being the open sets the existence of which is asserted in Lemma 3) for every $i \in \omega$. Let $\mathcal{B} = \{B_i\colon\ i \in \omega\}$. Let there be a WSA $\langle \{\{(b)_{B,n}\colon\ b \in B\}\colon\ B \in \mathcal{B}\} \rangle$ on $\mathcal{B}$. For every $i, n \in \omega$ and $x \in X$, let $[x]_{i,n}$ be such that $((x,i))_{B_i,n} \cap (X \times \{i\}) = [x]_{i,n} \times \{i\}$. Clearly, in the same manner that the same conclusion is arrived at in the proof of Theorem 1, we have, given $x \in X$ and an open neighbourhood W, a neighbourhood V of x such that $[y]_{j,n} \cap V = \emptyset$, for some $j, n \in \omega$, whenever $y \notin W$. Metrizability of X follows (Section 1.6). Similarly for stratifiability of X. □

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