Tableau Systems for Some Paraconsistent Modal Logics

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Abstract

In this paper we present tableau-style proof theories for some modal extensions of two paraconsistent propositional logics: RM3, which allows for truth value gluts, and the weaker BN4, which also allows for truth value gaps. These proof theories are shown to be sound and complete with respect to their corresponding semantics. For comparison, we then present some Hilbert-style axiomatizations of these systems proposed by Lou Goble, and bring out some of the comparative advantages and disadvantages of these vis-à-vis our tableau systems.

Keywords: paraconsistent logic, modal logic, tableaux

1 Introduction

Modal logics are logics that formalize various notions of necessity and possibility (alethic, deontic, epistemic, temporal, etc.). Paraconsistent logics are logics which, unlike e.g. classical and intuitionist logics, reject explosion \(A, \neg A/B\) and thus can be used to formalize inconconsistent but non-trivial bodies of information. There are numerous reasons for combining these two types of logics. For example, classically-based deontic logics typically validate deontic hyper-explosion \(OA, O\neg A/B\).

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2 We say “typically” because there are classically-based deontic logics (constructed in non-standard ways) that reject deontic explosion. (See, e.g. [6], [7], [8], [12], [11], [15].) However, any normal modal logic interpreted as a deontic logic validates deontic explosion. Worse, any normal modal logic containing the “D” axiom, \(OA \rightarrow \neg O\neg A\), validates deontic hyper-explosion \((OA, O\neg A/B)\).
date the problematic principle of deontic explosion \((OA, O\neg A/OB)\), according to which conflicting obligations render *everything* obligatory; paraconsistent deontic logics do not.\(^3\) As another example: some paraconsistent epistemic logics arguably provide a solution to Fitch’s “knowability paradox” \(^4\), which purports to show that if all truths are knowable then all truths are known.\(^4\)

In “Paraconsistent Modal Logic” \(^5\), Lou Goble presents Hilbert-style axiomatizations of some modal extensions of the paraconsistent logics \(\text{RM}3\) (which allows for truth value “gluts,” i.e. sentences that are both true and false) and the weaker \(\text{BN}4\) (which also allows for truth value “gaps,” i.e. sentences that are neither true nor false),\(^6\) and shows these axiomatizations to be sound and complete with respect to their corresponding semantics. In this paper, we present tableau-style proof theories for these systems, and demonstrate their soundness and completeness with respect to the corresponding semantics.

For comparison, we then present Goble’s axiomatizations of the systems, bringing out some of the comparative advantages and disadvantages of these vis-à-vis our tableau systems.

\section{Semantics}

Our object language, \(\mathcal{L}\), is \(\text{Atom} = \{p_i : i \in \mathbb{N}\}\) closed under \(\neg, \land, \rightarrow\), and \(\Box, \lor, \leftrightarrow, \text{and } \Diamond\) are defined in the usual ways. We use \(\Sigma, \Delta\) to range over subsets of \(\mathcal{L}\).

**Definition 2.1 [KN4 model]** A KN4 model is a triple \((W, R, v)\), where \(W\) is a non-empty set, \(R \subseteq W^2\), and \(v : \text{Atom} \times W \mapsto \wp\{1, 0\}\). \(v\) is extended to \(\bar{v} : \mathcal{L} \times W \mapsto \wp\{1, 0\}\) as follows. For all \(p \in \text{Atom}, A, B \in \mathcal{L}, w \in W\):

\(^3\) Paraconsistent deontic logics are presented and discussed in, e.g. [3], [5], [10], [11], [13].

\(^4\) Attempts to resolve the knowability paradox by appeal to paraconsistent epistemic logics are found in, e.g., [1], [17].

\(^5\) Goble has graciously granted me permission to cite, and to quote from, this as-yet unpublished paper.

\(^6\) \(\text{RM}3\) and \(\text{BN}4\) are presented in [2].
\[ \bar{v}(p, w) = v(p, w) \]

\[ 1 \in \bar{v}(\neg A, w) \iff 0 \in \bar{v}(A, w) \]

\[ 0 \in \bar{v}(\neg A, w) \iff 1 \in \bar{v}(A, w) \]

\[ 1 \in \bar{v}(A \land B, w) \iff 1 \in \bar{v}(A, w) \text{ and } 1 \in \bar{v}(B, w) \]

\[ 0 \in \bar{v}(A \land B, w) \iff 0 \in \bar{v}(A, w) \text{ or } 0 \in \bar{v}(B, w) \]

\[ 1 \in \bar{v}(A \rightarrow B, w) \iff 1 \in \bar{v}(A, w) \Rightarrow 1 \in \bar{v}(B, w); \text{ and} \]

\[ 0 \in \bar{v}(B, w) \Rightarrow 0 \in \bar{v}(A, w) \]

\[ 0 \in \bar{v}(A \rightarrow B, w) \iff 1 \in \bar{v}(A, w) \text{ and } 0 \in \bar{v}(B, w) \]

\[ 1 \in \bar{v}(\Box A, w) \iff \forall w'(wRw' \Rightarrow 1 \in \bar{v}(A, w')) \]

\[ 0 \in \bar{v}(\Box A, w) \iff \exists w'(wRw' \text{ and } 0 \in \bar{v}(A, w')) \]

Here, of course, 1 represents truth and 0 represents falsity. Note that, breaking from the “classical” tradition, we do not equate truth with non-falsity, nor falsity with non-truth.

**Definition 2.2 [KM3 model]** A KM3 model is just like a KN4 model except that it precludes gaps, i.e. it requires that for all \( p \in \text{Atom} \) and \( w \in W \),

\[ 1 \in v(p, w) \text{ or } 0 \in v(p, w) \]

It is easy to show that the exclusion of gaps in KM3 extends to all elements of \( L \), i.e. for all KM3 models \( \langle W, R, v \rangle, w \in W, A \in L : \bar{v}(A, w) \neq \emptyset \).

**Definition 2.3** [semantic consequence] \( A \) is a semantic consequence of \( \Sigma \) in KN4 (in symbols, \( \Sigma \vdash_{\text{KN4}} A \)) iff for all KN4 models \( \langle W, R, v \rangle \) and all \( w \in W \), if \( 1 \in \bar{v}(B, w) \) for all \( B \in \Sigma \), then \( 1 \in \bar{v}(A, w) \). We write \( \emptyset \vdash_{\text{KN4}} A \) to abbreviate \( \emptyset \vdash_{\text{KN4}} A \). Semantic consequence for KM3 is defined in the same way, but with respect to KM3 models. (Since all KM3 models are KN4 models, \( \vdash_{\text{KN4}} \subseteq \vdash_{\text{KM3}} \).

KN4 is to BN4, and KM3 is to RM3, as the familiar modal logic K is to classical propositional logic (CPL): just as CPL is precisely the non-modal fragment of K, BN4 (RM3) is precisely the non-modal fragment of KN4 (KM3). The following graph illustrates the inclusion relations between these six systems:
The following are some restrictions that may be placed on the $R$ relation. For all $w, w', w'' \in W$:

- **Reflexivity (r)**: $wRw$
- **Symmetry (s)**: $wRw' \Rightarrow w'Rw$
- **Transitivity (t)**: $wRw' \Rightarrow (w'Rw'' \Rightarrow wRw'')$
- **Seriality (i)**: $\exists w'.wRw'$
- **Near-reflexivity (n)**: $\exists w'.w'Rw \Rightarrow wRw$
- **Euclidicity (e)**: $wRw' \Rightarrow (wRw'' \Rightarrow w'Rw'')$

We denote extensions of $\text{KN4}$ and $\text{KM3}$ by adding lower case letters to the end of the system name. For example, $\text{KN4in}$ is the extension of $\text{KN4}$ that is obtained by requiring that $R$ be serial and near-reflexive; and $\text{KM3r}$ is the extension of $\text{KM3}$ that is obtained by requiring that $R$ be reflexive.\(^7\) We will say that a system is an “$R$-extension” of $\text{KN4}$ iff it is the result of adding one or more of our six conditions on $R$ to $\text{KN4}$ (and similarly for $\text{KM3}$).

Semantic consequence for these extensions of $\text{KN4}$ and $\text{KM3}$ is defined, as one would expect, as truth preservation at all worlds in all relevant models. For example, $\Sigma \models_{\text{KN4rs}} A$ iff for all $\text{KN4rs}$ models $\langle W, R, v \rangle$ and all $w \in W$, if $1 \in \bar{v}(B, w)$ for all $B \in \Sigma$, then $1 \in \bar{v}(A, w)$.

Note that we have specified $128 (= 2 \times 2^6)$ distinct semantical systems (though some of these, e.g. $\text{KN4r}$ and $\text{KN4er}$, define equivalent consequence relations).

### 3 Tableaux

In defining our tableau systems, we adopt some of the techniques of [14].

**Definition 3.1** [initial list] An *initial list* for an inference $\Sigma/A$ is any list of

\(^7\) Goble uses a more syntactocentric nomenclature, e.g. using ‘$\text{KN4.DU}$’ to denote $\text{KN4en}$. 

the form

\[
B_0 \ 0^+ \\
\vdots \\
B_n \ 0^+ \\
A \ 0^-
\]

where \( \{B_0, \ldots, B_n\} \subseteq \Sigma \). We use \( I(\Sigma/A) \) to denote the set of all initial lists for \( \Sigma/A \).

**Definition 3.2** [maximal initial list] Where \( \Sigma = \{B_0, \ldots, B_n\} \), the *maximal* initial list for \( \Sigma/A \), \( I_m(\Sigma/A) \), is

\[
B_0 \ 0^+ \\
\vdots \\
B_n \ 0^+ \\
A \ 0^-
\]

**Definition 3.3** [basic rules] Our basic tableau rules, the elements of \( Basic \), are as follows:

\[
\begin{array}{c|c|c|c|c|c}
& [\neg\neg] & [\land+] & [\land-] & [\neg\land+] \\
\neg\neg A \ x\pm & A \land B \ x^+ & A \ x^+ & A \land B \ x^- & \neg (A \land B) \ x^+ \\
A \ x\pm & B \ x^+ & & & -A \ x^+ & -B \ x^+
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c}
& [\neg\land-] & [\rightarrow +] \\
\neg(A \land B) \ x^- & A \rightarrow B \ x^+ & A \ x^- & B \ x^+ & \neg B \ x^- & \neg A \ x^+ \\
\neg A \ x^- & \neg B \ x^- & \neg A \ x^+ & \neg B \ x^- & \neg A \ x^+
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c}
& [\rightarrow -] & [\neg\rightarrow] & [\Box+] & [\Box-] \\
A \rightarrow B \ x^- & A \rightarrow B \ x^+ & \neg(A \rightarrow B) \ x^\pm & \Box A \ x^+ & \Box A \ x^- \\
A \ x^+ & \neg B \ x^+ & A \land \neg B \ x^\pm & xy & y^+ & i^-
\end{array}
\]
The basic import of the rules is this: If a branch contains nodes of the forms appearing above the horizontal line, it can be extended to include nodes of the forms appearing below the horizontal line. Vertical lines indicate that the branch is to be split. In these rules, \( x \) and \( y \) are any natural numbers; \( i \) is a natural number that is new to the branch. We use ‘±’ to condense rules in an obvious way; e.g. \([-→+]\) is a condensation of the following two rules:

\[
\begin{array}{c}
[¬→+] \\
\hline
¬(A → B) x+ \\
A \land ¬B x+
\end{array}
\quad
\begin{array}{c}
[¬→-] \\
\hline
¬(A → B) x− \\
A \land ¬B x−
\end{array}
\]

A node of the form \( A x+ \) indicates that (on the branch we are assuming) \( A \) is true at the world corresponding to \( x \). A node of the form \( A x− \) indicates that \( A \) is not true at the world corresponding to \( x \). A node of the form \( ¬A x+ \) indicates that \( ¬A \) is true, i.e. \( A \) is false, at the world corresponding to \( x \). A node of the form \( ¬A x− \) indicates that \( ¬A \) is not true, i.e. \( ¬A \) is not false, at the world corresponding to \( x \). (Keep in mind that in our semantics truth \( ≠ \) non-falsity and falsity \( ≠ \) non-truth!) A node of the form \( xy \) indicates that the world corresponding to \( x \) bears \( R \) to the world corresponding to \( y \).

**Definition 3.4** [closed branch] A branch of a KN4 tableau is closed just in case nodes of the forms \( A x+ \) and \( A x− \) occur on it. In KM3 we have an additional closure condition: \( A x− \) and \( ¬A x− \).

**Definition 3.5** [tableau] The set \( T(Σ/A) \) of tableaux for an inference \( Σ/A \) is the smallest superset of \( I(Σ/A) \) such that if \( t \in T(Σ/A) \) and \( t′ \) is the result of applying a tableau rule to an open (i.e. non-closed) branch of \( t \), then \( t′ \in T(Σ/A) \).

**Definition 3.6** [maximal tableau] The set \( T_m(Σ/A) \) of maximal tableaux for an inference \( Σ/A \) is the smallest superset of \( \{I_m(Σ/A)\} \) such that if \( t \in T_m(Σ/A) \) and \( t′ \) is the result of applying a tableau rule to an open branch of \( t \), then \( t′ \in T_m(Σ/A) \).

**Definition 3.7** [closed tableau] A tableau is closed iff all of its branches are.
Notation.

We write ‘×’ at the bottom of a branch to indicate that it is closed, and ‘↑’ to indicate that it is open (i.e. not closed).

Definition 3.8 [proof-theoretic consequence] A is a proof-theoretic consequence of Σ in $\text{KN}_4$, in symbols $\Sigma \vdash_{\text{KN}_4} A$, iff there is a closed $\text{KN}_4$ tableau for $\Sigma/A$. Similarly, $\Sigma \vdash_{\text{KM}_3} A$ iff there is a closed $\text{KM}_3$ tableau for $\Sigma/A$. (Since all closed $\text{KN}_4$ tableaux for $\Sigma/A$ are closed $\text{KM}_3$ tableaux for $\Sigma/A$, $\vdash_{\text{KN}_4} \subseteq \vdash_{\text{KM}_3}$.)

Definition 3.9 [additional rules] We have the following additional rules for extensions of $\text{KN}_4$ and $\text{KM}_3$: \footnote{We include the ‘$A$’ in the rules $[r]$, $[t]$, and $[e]$ in order to avoid infinite tableaus.}

\[
\begin{array}{cccc}
[r] & [s] & [t] & [e] \\
A x\pm & xy & A x\pm & xy \\
xx & yx & yz & xx \\
xz & yz & \\
\end{array}
\]

Tableau systems for $R$-extensions of $\text{KN}_4$ and $\text{KM}_3$ are defined as follows. The rules for a system are Basic plus the additional rules corresponding to the additional restrictions on $R$. For example, the set of rules for $\text{KN}_4\text{in}$ is $\text{Basic} \cup \{[i], [n]\}$ and the set of rules for $\text{KM}_3r$ is $\text{Basic} \cup \{[r]\}$. Four-valued extensions (i.e. the ones whose names begin with ‘$\text{KN}$’) have the same closure conditions as $\text{KN}_4$. Three valued extensions (i.e. the ones whose names begin with ‘$\text{KM}$’) have the same closure conditions as $\text{KM}_3$.

Proof-theoretic consequence for these $R$-extensions of $\text{KN}_4$ and $\text{KM}_3$ is defined as one would expect. For example, $\Sigma \vdash_{\text{KM}_3e} A$ iff there is a closed $\text{KM}_3e$ tableau for $\Sigma/A$.

Example 3.10 Here is a proof that $\Box(A \rightarrow B), \Diamond A \vdash_{\text{KN}_4} \Diamond B$:
1. $\square(A \to B) \ 0+$ initial list
2. $\neg\square\neg A \ 0+$ ”
3. $\neg\square\neg B \ 0−$ ”
4. 01 2 $[
eg\square+]$
5. $\neg\neg A \ 1+$ ”
6. $A \ 1+$ 5 $[\neg−]$
7. $\neg\neg B \ 1−$ 3,4 $[\neg\square−]$
8. $B \ 1−$ 7 $[\neg−]$
9. $A \to B \ 1+$ 1,4 $[\square+]$
10. $A \ 1− \ | \ B \ 1+$ 9 $[\to+]$
11. $\neg B \ 1− \ | \ \neg A \ 1+ \ | \ \neg B \ 1− \ | \ \neg A \ 1+$ ”

Note that the number column, justification column, and ‘×’s are not officially parts of the tableau.

**Example 3.11** Here is a proof that $\square A, \square \neg A \not\vdash_{KM3r} \square B$:

1. $\square A \ 0+$ initial list
2. $\square \neg A \ 0+$ ”
3. $\square B \ 0−$ ”
4. 00 3 $[r]$
5. $A \ 0+$ 1,4 $[\square+]$
6. $\neg A \ 0+$ 2,4 $[\square+]$
7. 01 3 $[\square−]$
8. $B \ 1−$ ”
9. $A \ 1+$ 1,7 $[\square+]$
10. $\neg A \ 1+$ 2,7 $[\square+]$
11. 11 10 $[r]$

↑
We can read a counterexample off of this open branch using the method of the induced model (see definition 21 below). The induced model for this branch is: $W = \{w_0, w_1\}$; $R = \{\langle w_0, w_0 \rangle, \langle w_0, w_1 \rangle, \langle w_1, w_1 \rangle\}$; $v(p_0, w_0) = v(p_0, w_1) = \{1, 0\}$; $1 \notin v(p_1, w_1)$. It is easy to verify that in this model, $1 \in \bar{v}(\Box p_0, w_0)$ and $1 \notin \bar{v}(\Box \neg p_0, w_0)$ but $1 \notin \bar{v}(\Box p_1, w_0)$. Also, note that $R$ is reflexive, as required.

We now prove soundness and completeness for each of the systems we have defined.

**Notation.**

We let $S$ stand for any of our systems, $\text{KN4}^+$ for any $R$-extension of $\text{KN4}$, and $\text{KM3}^+$ for any $R$-extension of $\text{KM3}$.

**Definition 3.12 [faithful]** Let $b$ be a branch of an $S$ tableau. An $S$ model $\mathcal{M} = \langle W, R, v \rangle$ is faithful to $b$ iff there is a mapping $f : \mathbb{N} \rightarrow W$ satisfying the following conditions.

1. if $xy$ is on $b$ then $f(x)Rf(y)$
2. if $Ax^+$ is on $b$ then $1 \in \bar{v}(A, f(x))$
3. if $Ax^-$ is on $b$ then $1 \notin \bar{v}(A, f(x))$

We say that $f$ shows $\mathcal{M}$ to be faithful to $b$.

**Lemma 3.13 (soundness lemma)** Let $b$ be an open branch of an $S$ tableau, and let $\mathcal{M} = \langle W, R, v \rangle$ be an $S$ model that is faithful to $b$. If an $S$ tableau rule is applied to $b$, then $\mathcal{M}$ is faithful to at least one of the branches thereby generated.

**Proof.** The proof is by cases. There are 18 cases to consider—one for each tableau rule. We omit the details due to lack of space. □

**Theorem 3.14 (soundness for $\text{KN4}$ and its $R$-extensions)** $\Sigma \vdash_{\text{KN4}^+} A$ only if $\Sigma \models_{\text{KN4}^+} A$.

**Proof.** Suppose $\Sigma \not\models_{\text{KN4}^+} A$. Then there is a $\text{KN4}^+$ model $\mathcal{M} = \langle W, R, v \rangle$ and $w \in W$ such that $1 \in \bar{v}(B, w)$ for all $B \in \Sigma$ and $1 \notin \bar{v}(A, w)$. Clearly $\mathcal{M}$ is faithful to each element of $I(\Sigma/A)$. Moreover, by the soundness lemma, each subsequent application of a $\text{KN4}^+$ tableau rule to an element of $I(\Sigma/A)$ will yield at least one branch to which $\mathcal{M}$ is faithful. Thus every $t \in T(\Sigma/A)$ has at least one branch to which $\mathcal{M}$ is faithful. Let $t \in T(\Sigma/A)$, and let $b$ be a branch of $t$ to which $\mathcal{M}$ is faithful. Suppose $b$ is closed: then nodes of the forms $C x^+$ and $C x^-$ occur on $b$. Thus, since $\mathcal{M}$ is faithful to $b$, $1 \in \bar{v}(C, w(x))$

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Note that we convert all schematic letters $A$, $B$, etc. to atomic formulas $p_0$, $p_1$, etc. Otherwise we wouldn’t really be defining a model.
and $1 \notin v(C, w(x))$. (Contradiction.) Thus $b$ is open. Thus every element of $T(\Sigma/A)$ has at least one open branch. Thus there are no closed tableaux for $T(\Sigma/A)$. Thus $\Sigma \not\vdash_{\text{KN4}} A$. □

**Theorem 3.15** (soundness for KM3 and extensions) $\Sigma \vdash_{\text{KM3}} A$ only if $\Sigma \models_{\text{KM3}} A$.

**Proof.** Just as for KN4$, except we now must take an additional closure condition into account when showing that $b$ is open. Suppose nodes of the forms $Cx$ and $\neg Cx$ occur on $b$. Then, since $\mathcal{M}$ is faithful to $b$, $1 \notin \bar{v}(C, w(x))$ and $0 \notin \bar{v}(C, w(x))$, i.e. $\bar{v}(C, w(x)) = \emptyset$. But this contradicts the fact that there can be no truth value gaps in a KM3$^+$ model. □

**Definition 3.16** [induced model] Let $b$ be an open, complete branch of an $S$ tableau. The $S$ model induced by $b$ is the $S$ model $\mathcal{M} = \langle W, R, v \rangle$ such that for all $p \in \text{Atom}$:

1. $W = \{w_x : x \text{ is a natural number occurring on } b\}$
2. $R = \{(w_x, w_y) : xy \text{ occurs on } b\}$
3. if $px+$ is on $b$, then $1 \in v(p, w_x)$
4. if $px-$ is on $b$, then $1 \notin v(p, w_x)$
5. if $\neg px+$ is on $b$, then $0 \in v(p, w_x)$
6. if $\neg px-$ is on $b$, then $0 \notin v(p, w_x)$

We assume some arbitrary but fixed method of specifying any parameters not determined by the above; this justifies our speaking of “the” $S$ model induced by $b$. Strictly speaking we need to verify that the $S$ model induced by $b$ really is an $S$ model, but this is not difficult.

**Lemma 3.17** (completeness lemma) Let $b$ be an open, complete branch of a tableau, and let $\mathcal{M} = \langle W, R, v \rangle$ be the model induced by $b$. Then:

1. if $Ax+$ is on $b$, then $1 \in \bar{v}(A, w_x)$
2. if $Ax-$ is on $b$, then $1 \notin \bar{v}(A, w_x)$
3. if $\neg Ax+$ is on $b$, then $0 \in \bar{v}(A, w_x)$
4. if $\neg Ax-$ is on $b$, then $0 \notin \bar{v}(A, w_x)$

**Proof.** The proof is by induction on the length of $A$. We omit the details due to lack of space. □

**Theorem 3.18** (completeness for finite $\Sigma$) For finite $\Sigma$, if $\Sigma \vdash_{\text{S}} A$ then $\Sigma \models_{\text{S}} A$.

**Proof.** Suppose $\Sigma \not\vdash_{\text{S}} A$. Then there is no closed $S$ tableau for $\Sigma/A$. Let $t$ be any open, complete, maximal $S$ tableau for $\Sigma/A$, and choose some open branch, $b$, of $t$. Let $\mathcal{M} = \langle W, R, v \rangle$ be the $S$ model induced by $b$. Since $t$
is a maximal S tableau for $\Sigma/A$, $B$ $0^+$ is on $b$ for all $B \in \Sigma$. Also, $A$ $0^-$ is on $b$. Thus, by the completeness lemma, $1 \in \bar{v}(B, w_0)$ for all $B \in \Sigma$ and $1 \notin \bar{v}(A, w_0)$. Thus $\Sigma \not\models_S A$. ⊢

In order to extend our completeness theorem to allow for infinite $\Sigma$ we will appeal to compactness:

**Theorem 3.19 (compactness)** If $\Sigma \models_S A$ then there is a finite $\Delta \subseteq \Sigma$ such that $\Delta \models_S A$.

The compactness of our systems follows more or less trivially from the fact that (as Goble has shown) they are axiomatizable. However, we need not take a detour through Goble’s axiomatizations to prove compactness; we can prove it directly in terms of our tableaux, using a technique similar to that used in [16, chap. III].

The unqualified completeness theorem now follows straightforwardly:

**Theorem 3.20 (completeness)** Where $S$ is any of the logics we have defined herein, if $\Sigma \models_S A$ then $\Sigma \vdash_S A$.

**Proof.** Suppose $\Sigma \models_S A$. Then, by compactness, $\Delta \models_S A$ for some finite $\Delta \subseteq \Sigma$. Thus, by the finite completeness theorem, $\Delta \vdash_S A$. Thus, by the monotonicity of $\vdash_S$, $\Sigma \vdash_S A$. ⊢

The following fact comes in handy, as it allows us to conclude that an inference is invalid from the fact that there is a complete, open tableau for it.

**Corollary 3.21 (all or nothing)** If there is a complete, open $S$ tableau for $\Sigma/A$, then there is no closed $S$ tableau for $\Sigma/A$.

**Proof.** Suppose there is an open, complete $S$ tableau for $\Sigma/A$. Pick an open branch, $b$, of this tableau. Let $\mathcal{M} = \langle W, R, v \rangle$ be the $S$ model induced by $b$. By the completeness lemma and the definition of $S$ tableaux for an inference, $1 \in \bar{v}(B, w_0)$ for all $B \in \Sigma$, but $1 \notin \bar{v}(A, w_0)$. Thus $\Sigma \not\models_S A$. Thus, by the soundness theorem, $\Sigma \not\vdash_S A$. Thus, by the definition of proof-theoretic consequence, there is no closed $S$ tableau for $\Sigma/A$. ⊢

### 4 Axiomatics

For purposes of comparison, we now present Goble’s Hilbert-style axiomatizations of the systems presented in Section 2.

We’ll start with Goble’s axiomatization of $\text{KN4}$. For our basic (non-modal) axioms, we have all instances of:
(1) \(A \rightarrow A\)
(2) \((A \land B) \rightarrow A\)
(3) \((A \land B) \rightarrow B\)
(4) \(((A \rightarrow B) \land (A \rightarrow C)) \rightarrow (A \rightarrow (B \land C))\)
(5) \(A \rightarrow (A \lor B)\)
(6) \(B \rightarrow (A \lor B)\)
(7) \(((A \rightarrow C) \land (B \rightarrow C)) \rightarrow ((A \lor B) \rightarrow C)\)
(8) \((A \land (B \lor C)) \rightarrow ((A \land B) \lor (A \land C))\)
(9) \(A \rightarrow \neg B \rightarrow (B \rightarrow \neg A)\)
(10) \(\neg \neg A \rightarrow A\)
(11) \((\neg A \land B) \rightarrow (A \rightarrow B)\)
(12) \(\neg A \rightarrow (A \lor (A \rightarrow B))\)
(13) \(A \lor \neg B \lor (A \rightarrow B)\)
(14) \(A \rightarrow ((A \rightarrow \neg A) \rightarrow \neg A)\)
(15) \(A \lor (\neg A \rightarrow (A \rightarrow B))\)

Our basic modal axioms are all instances of:

(K) \(\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)\)
(C) \((\Box A \land \Box B) \rightarrow \Box(A \land B)\)
(Bel) \(\Box(A \lor B) \rightarrow (\Diamond A \lor \Box B)\)
(N) If \(A\) is an axiom then so is \(\Box A\)

For rules we take first:

(Adj) From \(A\) and \(B\), infer \(A \land B\)
(MP) From \(A\) and \(A \rightarrow B\), infer \(B\)
(Prefix) From \(A \rightarrow B\), infer \((C \rightarrow A) \rightarrow (C \rightarrow B)\)
(Suffix) From \(A \rightarrow B\), infer \((B \rightarrow C) \rightarrow (A \rightarrow C)\)
These are not sufficient for completeness, however. Goble finds it necessary to include an infinite set of “extended modus ponens” rules, $XMP$, which is defined recursively as the smallest set of rules such that:

- All instances of the rule $MP^*$, From $(A \land (A \to B))$, infer $(A \land (A \to B)) \land B$, are in $XMP$, and
- If a rule $R$ is in $XMP$, then so are all the instances of $CR$, $DR$, $NR$, and $MR$.

$CR$, $DR$, $NR$, and $MR$ are defined as follows. Given a rule $R$, From $A$, infer $B$, let its disjunctive forms, $DR$, be: From $C \lor A$, infer $C \lor B$, for every formula $C$. Likewise, let its conjunctive forms, $CR$, be: From $C \land A$, infer $C \land B$, and its necessitative forms, $NR$, be: From $\Box A$, infer $\Box B$, and its possibilative forms, $MR$, be: From $\Diamond A$, infer $\Diamond B$.

Referring to $XMP$ Goble writes: “These rules are not pretty, and I would rather not posit such a set of them, but they seem to be required, all because of the absence of the theorem form of modus ponens, $(A \land (A \to B)) \to B$, from $BN4$. I invite anyone to find a more elegant formulation of the system.” [9, p. 6]

Proof theoretic consequence is defined in the usual way for axiomatic systems: $\Sigma \vdash_S A$ iff there is a finite sequence of formulas $\langle B_1, \cdots, B_n \rangle$ such that each $B_i$ is either an element of $\Sigma$, an axiom of $S$, or follows from preceding members of the sequence by a rule of $S$; and $B_n = A$.

Goble’s axiomatization of $KM3$ is quite a bit simpler, due to the fact that $KM3$, unlike $KN4$, validates the theorem form of modus ponens.

For our basic (non-modal) axioms, we have all instances of:
The modal axioms for KM3 are the same as for KN4. The rules are the same as well, except that it is not necessary to include XMP in KM3; as Goble demonstrates, all of the elements of XMP are derivable in KM3, thanks to the fact that KM3 includes \((A \land (A \rightarrow B)) \rightarrow B\) as a theorem.

Extensions of KN4 and KM3 are obtained, as one would expect, by adding the standard axioms corresponding to the conditions on \(R\):

\[
\begin{align*}
\text{T} & : \Box A \rightarrow A \quad \text{reflexivity} \\
\text{B} & : A \rightarrow \Box \Diamond A \quad \text{symmetry} \\
\text{D} & : \Box A \rightarrow \Diamond A \quad \text{seriality} \\
\text{U} & : \Box(\Box A \rightarrow A) \quad \text{near-reflexivity} \\
\text{5} & : \Diamond A \rightarrow \Box \Diamond A \quad \text{euclidianity}
\end{align*}
\]

For illustration’s sake, let’s look at an axiomatic proof that \(\Box(A \rightarrow B), \Diamond A \vdash_{KN4} \Diamond B\). (We will help ourselves to an alternative form of contraposition, \((A \rightarrow \)
B) → (¬B → ¬A), which is easily derived. We also condense some steps of the proof. Note that the actual proof, sans shortcuts, is much longer.)

1. □(A → B) premise
2. ¬□¬A premise
3. □((A → B) → (¬B → ¬A)) contraposition (N)
4. □(A → B) → □(¬B → ¬A) 3 (K) (MP)
5. □(¬B → ¬A) 1,4 (MP)
6. □¬B → □¬A 5 (K) (MP)
7. ¬□¬A → ¬□¬B 6 contraposition (MP)
8. ¬□¬B 2,7 (MP)

5 Conclusion

We conclude by bringing out some of the comparative advantages and disadvantages of our tableau systems vis-à-vis Goble’s axiomatizations.

There are five main factors to take into account when comparing proof theories for a given logical system (or set of related systems):

• Simplicity. How complicated is the proof theory?
• Ease of use. How much work/ingenuity is required to construct proofs?
• Counterexamples. Does the proof theory provide a method of generating counterexamples to invalid inferences?
• Insight. How much insight does the proof theory provide into the inferential power of the system(s)?
• Metatheory. How difficult is it to prove metatheorems (esp. soundness and completeness)?

We consider each of these factors in turn.

Simplicity. Our tableau systems are simpler than Goble’s axiomatizations, especially with respect to KN4 and its R-extensions. It is particularly interesting to note that while the axiomitizations of the four-valued systems are significantly more complicated than the axiomatizations of the three valued systems, the tableau systems for the four-valued systems are actually slightly simpler than those for the three-valued systems. Advantage: tableaux.

Ease of use. Tableau proofs require no ingenuity to construct (a computer could be programmed to construct them), though they can sometimes
be tedious to construct. Axiomatic proofs on the other hand, often require significant ingenuity to construct and are almost always tedious to construct. Advantage: tableaux.

Counterexamples. Our tableau systems provide a simple method of generate counterexamples to invalid inferences. Goble’s axiomatic systems do not provide any such method. Advantage: tableaux.

Insight. Goble’s axiomatizations have a clear advantage over our tableau systems here. In general, the biggest shortcoming of tableau proof theories is the fact that they do not illustrate much more about the logic than one could glean from the semantics alone. Advantage: axioms.

Metatheory. The soundness and completeness proofs for our tableau systems are roughly equal in length and difficulty. Soundness for Goble’s axiomatizations is more or less trivial, while completeness is rather involved and difficult. Advantage: tableaux (by a hair).

Tableaux have the advantage in four of the five factors we have considered. Of course, we are not taking into account the relative weighting of these factors. Still, we think that overall, our tableau systems compare quite favorably with Goble’s axiomatizations. It is nice to have both types of systems at our disposal, though.

References


