EXTENDED POWERS OF SPECTRA AND A GENERALIZED KAHN–PRIDDY THEOREM

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§1. MAIN RESULTS

In [7], Kahn and Priddy showed that, localized at a prime $p$, there is a map of infinite loop spaces $QBS_{p} \rightarrow Q_{0}S^{0}$ that is a projection onto a direct factor in the category of topological spaces. Here $QX = \Omega^{\infty} \Sigma^{\infty} X$ where $\Omega^{\infty}$ is the $0$th space functor from the category of spectra to the category of spaces, right adjoint to the suspension spectrum functor $\Sigma^{\infty}$, $BS_{p}$ is the classifying space of the symmetric group $\Sigma_{p}$, and $Q_{0}S^{0}$ is the basepoint component of $QS^{0}$.

This splitting in fact "deloops" once [8], so that $Q\Sigma B_{p} \rightarrow \tilde{Q}S^{1}$ has a right inverse, where $\tilde{Q}S^{1}$ is the simply connected cover of $QS^{1}$. Note that $QS^{1}$ is the fiber of the structure map $QS^{1} \rightarrow S^{1}$ associated to the infinite loop space $S^{1}$.

In this paper we prove a generalization of the Kahn–Priddy theorem in this strong form, in which $S^{1}$ is replaced by an arbitrary connected infinite loop space. When $p = 2$, our theorem corresponds to work by Finkelstein and Kahn announced in [5].

Our proof is quite short and noncomputational. We make use of manipulations of the adjoint pair $(\Sigma^{\infty}, \Omega^{\infty})$. This allows us to work stably so that we can make use of the splitting [6], valid for any connected space $X$:

$$s_{X}: \Sigma^{\infty} QX \rightarrow \bigvee_{n \geq 1} \Sigma^{n} D_{n}X.$$  

Here $D_{n}X$ is the $n$th extended power construction on $X$. Crucial to our arguments is work by Lewis et al. [11] showing that this construction can be made in the category of spectra.

To state our main theorem we consider the following situation. Let $E$ be a connected spectrum and $X = \Omega^{\infty} E$. Let $\epsilon: \Sigma^{\infty} X \rightarrow E$ be the evaluation map and $\theta: QX \rightarrow X$ be $Q^{\infty} \epsilon$. Thus $X$ is a connected infinite loop space with structure map $\theta$. Let $E'$ be the fiber of $\epsilon$ and $X' = \Omega^{\infty} E'$. Thus $X'$ is the fiber of $\theta$. Note that $\theta$ has a right inverse $\eta: X \rightarrow QX$ so that, as spaces, $QX \cong X \times X'$. Finally, let $f_{*}: \Sigma^{n} D_{n}X \rightarrow \Sigma^{n} X$ be the composite $\Sigma^{n} D_{n}X \leftarrow \Sigma^{\infty} QX \rightarrow \Sigma^{n} X$.

**Theorem 1.1.**

(1) Localized at 2, for any space $Y$, there is an exact sequence of homotopy groups:

$$[\Sigma^{n} Y, \Sigma^{n} D_{2}X] \xrightarrow{f_{*}} [\Sigma^{n} Y, \Sigma^{n} X] \rightarrow [\Sigma^{n} Y, E] \rightarrow 0.$$  

Equivalently, $f_{*}: \Sigma^{n} D_{2}X \rightarrow \Sigma^{n} X$ lifts uniquely to a map $f_{*}: \Sigma^{n} D_{2}X \rightarrow E'$, and $\Omega^{\infty} f_{*}: QD_{n}X \rightarrow X'$ has a right inverse.

(2) Localized at an odd prime $p$, for any space $Y$, there is an exact sequence of homotopy

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groups:

\[ [\Sigma^\infty Y, \Sigma^\infty D_2X \vee D_\nu X] \xrightarrow{f_2 \vee f_\nu} [\Sigma^\infty Y, \Sigma^\infty X] \xrightarrow{\epsilon} [\Sigma^\infty Y, E] \rightarrow 0. \]

Equivalently, \( f_2 \vee f_\nu : \Sigma^\infty D_2X \vee D_\nu X \rightarrow \Sigma^\infty X \) lifts uniquely to a map \( f_2' \vee f_\nu' : \Sigma^\infty D_2X \vee D_\nu X \rightarrow E' \), and \( \Omega^\infty (f_2' \vee f_\nu') : Q(D_2X \vee D_\nu X) \rightarrow X' \) has a right inverse.

Suppose now that \( E = \Sigma HZ \), where \( HZ \) is the integral Eilenberg–MacLane spectrum. Then \( X = S^1 \), \( X' = \hat{Q}S^1 \), \( D_2S^1 = \Sigma \mathbb{R}P^\infty \), and, localized at an odd prime \( p \), \( D_\nu S^1 \simeq \Sigma B\mathbb{Z}_p \) and \( D_2S^1 \simeq \ast \). Thus Theorem 1.1 has the following corollary.

**Corollary 1.2.** ("delooped" Kahn–Priddy Theorem). Localized at a prime \( p \), \( \Omega^\infty f_\nu : Q\Sigma B\Sigma_r \rightarrow \hat{Q}S^1 \) is the projection onto a direct factor.

**Remarks 1.3**

1. When \( p = 2 \), Theorem 1.1 is essentially a theorem announced by Kahn in [5] and proved by Finkelstein in his thesis [3]. Their techniques are very different from those that we use here. They make use of quite involved homological calculations of an explicit inverse to \( \Omega^\infty f_2 \times \eta : QD_2X \times X \rightarrow QX \).

2. The equivalence of the statements in the theorem follows formally by the use of adjunctions. In our context, a framework for such arguments is given in [9].

3. In the odd primary case of the theorem, the presence of \( DJ \) is necessary except when \( E = \Sigma HZ \) as discussed above. See §3 for more details.

We begin the proof. Consider the sequence

\[ \Sigma^\infty QX \xrightarrow{\Sigma^\infty \epsilon} \Sigma^\infty X \rightarrow E, \]

where \( \epsilon : \Sigma^\infty QX \rightarrow \Sigma^\infty X \) is again the evaluation. This sequence is the beginning of a "bar resolution" of \( E \), and it follows formally that, for any space \( Y \), there is an exact sequence:

\[ [\Sigma^\infty Y, \Sigma^\infty QX] \xrightarrow{\Sigma^\infty \epsilon} [\Sigma^\infty Y, \Sigma^\infty X] \xrightarrow{\epsilon} [\Sigma^\infty Y, E] \rightarrow 0. \]

Let \( DX = V D_\nu X \) and let \( \pi : DX \rightarrow X \) be the projection onto the first factor.

**Lemma 1.4.** There is a commutative diagram:

\[ \Sigma^\infty QX \xrightarrow{\pi} \Sigma^\infty DX \]

\[ \Sigma^\infty X. \]

**Proof.** Using the explicit description of the adjoint of \( s_X \) given in [1], it is clear that the adjoint of \( \pi \circ s_X \) is homotopic to \( 1 : QX \rightarrow QX \). The lemma follows.
As a consequence of this lemma, for any space Y, there is an exact sequence:

\[
\sigma Y, \Sigma^\infty DX \xrightarrow{f_n} \sigma Y, \Sigma^\infty X \rightarrow \sigma Y, E \rightarrow 0.
\]  

(*)

where \( f = V f_n: \Sigma^\infty DX \rightarrow \Sigma^\infty X \).

Theorem 1.1 will follow if we can show that, localized at a prime \( p \), the sequence of the theorem is a direct factor of the complex (*). Since \( f_1 = \pi|_{\Sigma^\infty X} = 1: \Sigma^\infty X \rightarrow \Sigma^\infty X \), we need just show that, localized at 2, there is a lifting

\[
\begin{array}{c}
V \Sigma^\infty D_nX \\
\uparrow \phi_n \\
\Sigma^\infty D_2X \xrightarrow{f_2} \Sigma^\infty X
\end{array}
\]

and that, localized at an odd prime, there is a lifting

\[
\begin{array}{c}
V \Sigma^\infty D_nX \\
\uparrow \phi_n \\
\Sigma^\infty D_2 \vee D_pX \xrightarrow{f_2 \vee f_p} \Sigma^\infty X.
\end{array}
\]

This follows from our next theorem.

THEOREM 1.5. Localize at a prime \( p \).

1. If \( i \geq 1 \), there is a lifting

\[
\begin{array}{c}
\Sigma^\infty D_nX \\
\uparrow f_n \\
\Sigma^\infty D_2X \xrightarrow{f_2} \Sigma^\infty X.
\end{array}
\]

2. If \( n \neq p \), there is a lifting

\[
\begin{array}{c}
\Sigma^\infty D_2X \\
\uparrow f_n \\
\Sigma^\infty D_2X \xrightarrow{f_1} \Sigma^\infty X.
\end{array}
\]

Again specializing to the case \( X = S^1 \), theorem 1.5 has the following consequence.

COROLLARY 1.6. Localized at an odd prime \( p \), \( f_n: \Sigma^\infty D_nS^1 \rightarrow \Sigma^\infty S^1 \) is nullhomotopic unless \( n = p \).

We do not know whether or not this statement is true when \( p = 2 \).
Theorem 1.5 will be proved in the next section. The proof involves some elementary transfer arguments, together with some general properties of extended powers of spectra. In §3 we note some other consequences of our results and techniques.

§2. EXTENDED POWERS OF SPECTRA

Recall the definition of the $n$th extended power of $X$:

$$D_nX = E\Sigma_n \times \Sigma_{n}X^{[n]}/\Sigma_n \times \Sigma_n.$$ 

Here $E\Sigma_n$ is a contractible space acted on freely by the permutation group $\Sigma_n$ and $X^{[n]}$ denotes $X$ smashed with itself $n$ times.

There is a natural map $\mu: DDX \to DX$ constructed as follows.

First recall Nishida's observation that there is a natural homeomorphism

$$\alpha: V \to D_pY \to D_qZ \to D(Y \vee Z)$$

(where, by definition, $D_pY = D_qZ = S^n$[12, 15]. Thus $DDX$ has a natural wedge decomposition into the summands $D_{m_1}D_{m_2}X \vee D_{m_2}D_{m_3}X \vee \cdots \vee D_{m_k}D_{m_k}X$ with $m_1 < m_2 < \cdots < m_k$.

Next note that

$$D_nD_mX = E\Sigma_{nm} \times \Sigma_{nm}X^{[nm]}/E\Sigma_{nm} \times \Sigma_{nm}$$

and

$$D_nX \vee D_mX = E\Sigma_{n+m} \times \Sigma_{n+m}X^{[n+m]}/E\Sigma_{n+m} \times \Sigma_{n+m}.$$ 

Thus the wreath and ordinary product inclusions

$$\Sigma_n \to \Sigma_{nm} \quad \text{and} \quad \Sigma_n \times \Sigma_m \to \Sigma_{n+m}$$

induce maps

$$\mu: D_nD_mX \to D_{nm}X \quad \text{and} \quad \mu: D_nX \vee D_mX \to D_{n+m}X.$$ 

These, in turn, define the map $\mu: DDX \to DX$, using Nishida's decomposition. Finally, recall that there are transfers associated to finite covers, natural with respect to pullbacks (see, e.g.[7]). Thus these same group inclusions define transfer maps

$$\Sigma \cong D_{nm}X \to \Sigma \cong D_{nm}X \quad \text{and} \quad \Sigma \cong D_{n+m}X \to \Sigma \cong D_{n+m}X.$$ 

Lewis et al.[11] show that functors $D_n$ can be defined on the category of spectra, so that there are natural equivalences

$$D_n\Sigma \cong X \cong \Sigma \cong D_nX.$$ 

**Theorem 2.1.** Let $X$ be a connected infinite loop space with structure map $\theta_X: QX \to X$ and let $f: \Sigma \cong DX \to \Sigma \cong X$ be the composite $\Sigma \cong DX \to \Sigma \cong QX \to \Sigma \cong X$. There is a commutative
**Diagram:**

\[
\begin{array}{ccc}
\Sigma^\infty DX & \xrightarrow{s} & \Sigma^\infty DX \\
\downarrow & & \downarrow \\
\Sigma^\infty DX & \xrightarrow{f} & \Sigma^\infty X.
\end{array}
\]

**Proof.** Consider the following diagram:

\[
\begin{array}{ccc}
\Sigma^\infty DX & \xrightarrow{s} & \Sigma^\infty DX \\
\downarrow & & \downarrow \\
\Sigma^\infty DX & \xrightarrow{\theta_1} & \Sigma^\infty QX \\
\downarrow & & \downarrow \\
\Sigma^\infty DX & \xrightarrow{\phi_1} & \Sigma^\infty X.
\end{array}
\]

(1) commutes because \( \theta_1 \) is an infinite loop map, and (2) commutes by the naturality of the stable splitting of \( QX \). The commutativity of (3) was shown by us in [10]. The theorem follows.

**Remark 2.2.** The careful reader may be wondering about our choice of stable splitting \( s, x: \Sigma^\infty X \to \Sigma^\infty DX \). R. Cohen's elegant "stable" construction was used by us in [10], while in our proof of Lemma 1.4 above we appealed to the space level construction of [11]. The equivalence of these two constructions follows from a simple argument given in [11, remarks following Theorem 5.5].

**Proof of Theorem 1.5.** We make use of various pieces of the commutative diagram of Theorem 2.1.

(1) If \( i \geq 2 \), there is a commutative diagram

\[
\begin{array}{ccc}
\Sigma^\infty D_p D_p^{-1}X & \xrightarrow{s} & \Sigma^\infty D_p X \\
\downarrow & & \downarrow \\
\Sigma^\infty D_p X & \xrightarrow{f_p} & \Sigma^\infty X.
\end{array}
\]

Since \( \Sigma_p, \Sigma_p \to \Sigma_p \) has index prime to \( p \), standard transfer arguments imply that, localized at \( p \), \( \mu \) is the projection onto a wedge summand. It follows that \( f_p \) factors through \( f_p \).

(2) Suppose that \( n \neq p' \). We have two cases.

First suppose that \( n \neq 2p' \). Then there exist numbers \( n_1 \) and \( n_2 \) such that \( n_1 \neq n_2 \), \( n_1 + n_2 = n \) and \( \Sigma_n \times \Sigma_n \hookrightarrow \Sigma_n \) has index prime to \( p \). Consider the diagram

\[
\begin{array}{ccc}
\Sigma^n D_n X \wedge D_{2^n} X & \xrightarrow{\mu} & \Sigma^n D_{2^n} X \\
\downarrow & & \downarrow \\
\Sigma^n D_2 (D_n X \vee D_{2^n} X) & \xrightarrow{f_t} & \Sigma^n X \\
\downarrow & & \downarrow \\
\Sigma^n D_n X & \xrightarrow{f_2} & \Sigma^n X \\
\end{array}
\]

As before, \( \mu \) has a right inverse and thus \( f_2 \) factors through \( f_t \).

Finally suppose that \( n = 2p' \). then \( \Sigma_d \Sigma_n \hookrightarrow \Sigma_n \) has index prime to \( p \). The diagram

\[
\begin{array}{ccc}
\Sigma^n D_n \rho X & \xrightarrow{\mu} & \Sigma^n D_n X \\
\downarrow & & \downarrow \\
\Sigma^n D_\rho X & \xrightarrow{f_2} & \Sigma^n X \\
\end{array}
\]

then shows that \( f_n \) factors through \( f_2 \).

§3. FURTHER REMARKS

Here we prove a variety of statements which follow from the ideas in the previous two sections. The reader is warned that these statements are not particularly interrelated.

We begin by elaborating on Remark 1.3 (3). We use the notation of Theorem 1.1.

**Proposition 3.1.** Localized at an odd prime \( p \), if \( E \) is any connected spectrum except \( \Sigma H \mathbb{Z} \), the map \( \Omega^n f_\rho; QD_{\rho} X \rightarrow X' \) does not have a right inverse.

**Proof.** If \( \Omega^n f_\rho \) did have a right inverse, it would follow, as in the proof of Theorem 1.1, that there would be a lifting

\[
\begin{array}{ccc}
\Sigma^n Q X & \xleftarrow{\sigma} & \Sigma^n D_{\rho} X \\
\downarrow \Sigma^n & & \downarrow \Sigma^n \\
\Sigma^n D_{\rho} X & \xrightarrow{f_\rho} & \Sigma^n X \\
\end{array}
\]

We claim that if \( X \neq S^1 \) no such lift can exist, even in mod \( p \) homology. To see this, note that if \( X \neq S^1 \) there will be decomposables in the Hopf algebra \( H_*(X; \mathbb{Z}/p) \). But \( \text{Im}(\theta - \epsilon)_* \) contains all decomposables, while any decomposables in \( \text{Im} f_\rho \) will be decomposable into \( p \)-fold products.
Next we note that the sequence (*) of §1 extends to a long exact sequence. Let $D^*$ denote the functor $D$ iterated $n$ times, and, for $i = 1, \ldots, n$, let $\mu_i: D^{*+1}X \to D^*X$ be $\mu$ applied to the $i$th and $(i+1)$st $D$'s.

**Proposition 3.2.** For any space $Y$, there is a long exact sequence

$$\cdots \to [\Sigma \infty Y, \Sigma \infty D^3X] \xrightarrow{d_2} [\Sigma \infty Y, \Sigma \infty D^2X] \xrightarrow{d_1} [\Sigma \infty Y, \Sigma \infty DX] \xrightarrow{d_0}$$

$$[\Sigma \infty Y, \Sigma \infty X] \xrightarrow{i} [\Sigma \infty Y, E] \to 0,$$

where

$$d_n = n + \sum_{i=1}^{n} (-1)^i \mu_i + (-1)^n D^n f: \Sigma \infty D^{*+1}X \to \Sigma \infty D^*X.$$

**Proof.** In the language of [9], we are claiming that

$$\cdots \to \Sigma \infty DDX \xrightarrow{\delta-f} \Sigma \infty DX \xrightarrow{\delta} \Sigma \infty X \xrightarrow{\delta} E$$

is a spacelike resolution of $E$. As we did in §1, we note that there is a canonical spacelike resolution

$$\cdots \to \Sigma \infty QQX \xrightarrow{\delta+f+\delta} \Sigma \infty QX \xrightarrow{\delta} \Sigma \infty X \xrightarrow{\delta} E.$$

The proposition follows by "changing $Q$'s into $D$'s", using Lemma 1.4 and the commutative diagram (3) of the proof of Theorem 2.1.

**Proposition 3.3.** Localize at $p$. Suppose that $f_p: \Sigma \infty D_pX \to \Sigma \infty X$ induces an isomorphism in a generalized homology theory $h_*$. Then the maps

$$f_p: \Sigma \infty D_pX \to \Sigma \infty X$$

and

$$\mu: D_{i_1}D_{i_2} \cdots D_{i_v}X \to D_pX \quad \text{with} \quad i_1 + \cdots + i_v = i$$

all induce $h_*$-isomorphisms.

**Proof.** We first note by a standard spectral sequence argument, if a map $g: E_1 \to E_2$ induces an $h_*$-isomorphism then so does $D_n g: D_n E_1 \to D_n E_2$ [10]. The proposition then follows by inductive use of the diagrams used in proving Theorem 1.5 (1), since, again by a transfer argument, all the maps $\mu$ will induce $h_*$-epimorphisms.

**Example 3.4.** $f_p: \Sigma \infty D_p S^1 \to \Sigma \infty S^1$ is an isomorphism in mod $p$ $K$-theory (see, e.g. [10]). Thus so is $f_p: \Sigma \infty D_p S^1 \to \Sigma \infty S^1$. This allows us to conclude that, if $x \in K_1(S^1; \mathbb{Z}/p^t)$ is the generator, then $Q(x, +\infty) = x$, where $Q: K_1(S^1; \mathbb{Z}/p^t) \to K_1(S^1; \mathbb{Z}/p^t)$ is McClure's $K$-theory Dyer–Lashof operation [13]. Note that this contrasts sharply with the situation in ordinary mod $p$ homology.
Example 3.5. Let $SP^n(S)$ denote the $n$th symmetric product of the sphere spectrum $S$. Localized at $p$, it can be shown that $SP^n(S)$ is the fiber of $f_p: \Sigma^\infty D_pS^1 \to \Sigma^\infty S^1$ and $\Sigma^{-1}SP^n(S)/SP^0(S)$ is the fiber of $\mu: \Sigma^\infty D_p\omega S^1 \to \Sigma^\infty D_p S^1$. (The first result is "classical" [4, 16]. The second is a simple consequence of work in progress of the author’s, inspired by [14], where it is shown that $\Sigma^{-1}SP^n(S)/SP^0(S)$ is a wedge summand of $\Sigma^\infty D_pD_pS^\omega \simeq \Sigma^\omega B\mathbb{Z} \cup S^\omega$.) Proposition 3.3 thus implies that 

$$h_*(SP^n(S)) = 0 = h_*(SP^0(S)) = 0.$$ 

This fits well with results in [10].

In contrast to Example 3.4 we have a final remark.

Remark 3.6. $f_p: \Sigma^\infty D_pS^1 \to \Sigma^\infty S^1$ has Adams filtration $i$. This follows because $f_p$ factors through the composite 

$$f_p = (f_p \circ D_0f_p \circ \cdots \circ D_{-1}f_p): \Sigma^\infty D_pS^1 \to \Sigma^\infty S^1.$$ 

$f_p$ is, in turn, determined by $f_p$ since $f_p = f_p \circ \mu$. Thus, for example, 

$$\text{Im}[f_p: \pi_*^\mathbb{Z}(D_pS^1) \to \pi_*^\mathbb{Z}(S^1)] = \text{Im}[f_p: \pi_*^\mathbb{Z}(D_pS^1) \to \pi_*^\mathbb{Z}(S^1)].$$

We conclude that the maps $f_p$ determine a decreasing filtration of $\pi_*^\mathbb{Z}(S^1)$ mapping to the Adams filtration.

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