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Attraction Principle for Nonlinear Integral Operators of the Volterra Type

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1. INTRODUCTION

We are studying the integral equation of the form

$$u(x) = \int_0^x a(x, y)\, \phi(u(y))\, dy. \tag{1}$$

All functions appearing here are nonnegative and defined for $0 \leqslant y \leqslant x$. Equation (1) has the trivial solution $u(x) \equiv 0$. It can have also other solutions. We prove, using the method due to Okrasiński, that under certain conditions upon $a(x, y)$ and $\phi(x)$ there can be at most one solution which does not vanish identically in a neighborhood of 0. Our main result is the attraction property of this nonnegative solution, provided that it exists. Namely we show that the iterations $T^n u$ of the operator

$$Tu(x) = \int_0^x a(x, y)\, \phi(u(y))\, dy$$

tend to the unique nonnegative solution for every function u, strictly positive in a neighborhood of 0.

A similar equation was studied in [4, 1, 2], under the conditions that $a(x, y)$ is invariant and $\phi(x)$ is concave.

2. THE RESULTS

We will deal with the integral operators T of the form

$$Tu(x) = \int_0^x a(x, y)\, \phi(u(y))\, dy.$$

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The functions u and ϕ are assumed to be nonnegative and strictly increasing on the half-axis $[0, +\infty)$ and $u(0)=0$, $\phi(0)=0$. Let the kernel $a(x, y)$, $x > y$, be positive and satisfy the conditions

$$\begin{aligned} &\frac{\partial a}{\partial x} \geqslant 0 \\ &\frac{\partial a}{\partial x} + \frac{\partial a}{\partial y} \geqslant 0. \end{aligned} \tag{2}$$

We also assume that $a(x, x)=0$. If not specified otherwise all the functions we introduce are smooth on the open half-axis $(0, +\infty)$ and continuous on $[0, +\infty)$. The kernel $a(x, y)$ is to be smooth for $x > y$ and continuous for $x \geqslant y$. The task we are going to address is the study of the equation

$$Tu(x) = u(x),$$

where u is nonnegative, strictly increasing, and $u(0)=0$. Obviously the conditions (1) imply that if $u(x)$ is strictly positive for $x>0$ and satisfies (2), then u is strictly increasing. Observe that the conditions (1) are equivalent to

$$a(x, y) \geqslant a(s, t)$$
$$\text{for} \quad 0 \leqslant s \leqslant x, 0 \leqslant t \leqslant y, y \leqslant x, \text{and } x-y > s-t. \tag{3}$$

LEMMA 1. *Let u and h be increasing functions on $[0, +\infty)$ such that $u(0)=h(0)=0$. Assume also that $h(x)$ is a continuous and piecewise smooth function on $[0, +\infty)$. Put $\tilde{u}(x)=u(h(x))$.*

(i) *If $Tu(x) \geqslant u(x)$ and $h'(x) \leqslant 1$, then $T\tilde{u}(x) \geqslant \tilde{u}(x)$.*

(ii) *If $Tu(x) \leqslant u(x)$ and $h'(x) \geqslant 1$, then $T\tilde{u}(x) \leqslant \tilde{u}(x)$.*

Proof. We will only prove the first part of the lemma. The proof of the second part is similar. Observe that if $0 < y < x$ then

$$a(h(x), h(y)) \leqslant a(x, y). \tag{4}$$

Indeed, since $h' \leqslant 1$ and $h(0)=0$ we have $h(x) \leqslant x$, $h(y) \leqslant y$, and $h(x) - h(y) \leqslant x-y$ for $0 < y < x$. Applying (3) we get the inequality (4). Therefore

$$\begin{aligned} T\tilde{u}(x) &= \int_0^x a(x, y)\phi(u(h(y)))\,dy \\ &\geq \int_0^x a(x, y)\,\phi(u(h(y)))\,h'(y)\,dy \\ &= \int_0^{h(x)} a(x, h^{-1}(s))\,\phi(u(s))\,ds \\ &\geq \int_0^{h(x)} a(h(x), s)\,\phi(u(s))\,ds \\ &= Tu(h(x)) \geq u(h(x)) = \tilde{u}(x). \end{aligned}$$

By applying Lemma 1 with

$$h(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq c \\ x - c & \text{if } c < x \end{cases}$$

we get the following.

COROLLARY 1. *Assume that u satisfies $Tu(x) \geq u(x)$. For a given $c > 0$ let*

$$u_c(x) = \begin{cases} 0 & \textit{if } 0 \leq x \leq c \\ u(x - c) & \textit{if } c < x. \end{cases}$$

Then $Tu_c(x) \geq u_c(x)$.

EXAMPLE. Let $f(x)$ be an increasing function such that $f(0) = 0$. Then the invariant kernel

$$a(x, y) = f(x - y)$$

satisfies the conditions (1). Observe that if $Tu = u$ then $Tu_c = u_c$ in this case.

Before stating the main result about the attraction principle for the equation

$$Tu(x) = u(x) \tag{5}$$

we need some auxiliary lemmas.

LEMMA 2. *Assume that the function $u(x)$ satisfies $Tu(x) \geq u(x)$ and let*

$$v(x) = \begin{cases} u(x) & \textit{if } 0 \leq x \leq c \\ u(c) & \textit{if } c < x. \end{cases}$$

Then there exists $\varepsilon > 0$ *such that*

$$\liminf_{n \to \infty} T^n v(x) \geqslant u(x),$$

for $c < x < c + \varepsilon$.

Proof. Assume that $\varepsilon < 1$. Let

$$c_a = \sup_{y \leqslant x \leqslant c+1} a(x, y),$$

$$c_\phi = \sup_{u(c) \leqslant x \leqslant u(c+1)} \phi'(x),$$

$$c_u = \sup_{c \leqslant x \leqslant c+1} u'(x).$$

Then for $c < x < c+1$ we have

$$\begin{aligned} u(x) - Tv(x) &\leqslant Tu(x) - Tv(x) \\ &= \int_0^x a(x, y)[\phi(u(y)) - \phi(v(y))]\, dy \\ &= \int_c^x a(x, y)[\phi(u(y)) - \phi(u(c))]\, dy \\ &\leqslant c_a c_\phi [u(x) - u(c)](x - c) \\ &\leqslant c_a c_\phi c_u (x - c)^2. \end{aligned}$$

Similarly we get

$$\begin{aligned} u(x) - T^n v(x) &\leqslant T^n u(x) - T^n v(x) \\ &= \int_c^x a(x, y)[\phi(u(y)) - \phi(T^{n-1} v(y))]\, dy \\ &\leqslant c_a c_\phi (x - c) \sup_{c < y < c+1} [\phi(u(y)) - \phi(T^{n-1} v(y))]. \end{aligned}$$

Thus by induction we can prove that

$$u(x) - T^n v(x) \leqslant c_u (c_a c_\phi)^n (x - c)^{n+1}.$$

This implies

$$\liminf_{n \to \infty} T^n v(x) \geqslant u(x),$$

if $x - c < c_a^{-1} c_\phi^{-1}$ and $x - c < 1$.

LEMMA 3. *Assume that* $Tu(x) = u(x)$ *and let*

$$v(x) = \begin{cases} u(x) & \text{if } 0 \leqslant x \leqslant c \\ u(c) & \text{if } c < x. \end{cases}$$

Then there is $\varepsilon > 0$ *such that*

$$\lim_{n \to \infty} T^n v(x) = u(x),$$

for $c < x < c + \varepsilon$.

Proof. From the preceding lemma we have that $\liminf_{n \to \infty} T^n v(x) \geqslant u(x)$, for $c < x < c + \varepsilon$, for some $\varepsilon > 0$. On the other hand

$$\limsup_{n \to \infty} T^n v(x) \leqslant u(x).$$

This is because $u(x) \geqslant v(x)$ and T is monotonic.

The idea of the proof of the next proposition is due to Okrasiński.

PROPOSITION 1. *Equation* (2) *can have at most one positive solution.*

Proof. Suppose $u(x)$ and $v(x)$ are two different positive solutions of (2). Without loss of generality we may assume that $u \nleqslant v$. Then there is $d > 0$ such that $u(x - d) > v(x)$ for some $x > 0$. If not, then we would have $u(x - d) \leqslant v(x)$ for every x and d, which would imply $u \leqslant v$. Thus let $u(x - d) > v(x)$. This can be written as $u_d(x) > v(x)$. Let c be the lower bound of the numbers x for which $u_d(x) > v(x)$. Thus $u_d(x) \leqslant v(x)$ for $0 \leqslant x \leqslant c$. Define the function $\tilde{u}(x)$ as

$$\tilde{u}(x) = \begin{cases} u_d(x) & \text{if } 0 \leqslant x \leqslant c \\ u_d(c) & \text{if } c < x. \end{cases}$$

By Corollary 1 we have $Tu_d(x) \geqslant u_d(x)$. Moreover $\tilde{u}(x) \leqslant v(x)$. Therefore

$$\limsup_{n \to \infty} T^n \tilde{u}(x) \leqslant v(x).$$

On the other hand by Lemma 2

$$\liminf_{n \to \infty} T^n \tilde{u}(x) \geqslant u_d(x),$$

for $c < x < c + \varepsilon$. This implies that $u_d(x) \leqslant v(x)$ for $c < x < c + \varepsilon$. The latter contradicts the choice of c.

We are now ready to prove the attraction principle for Equation (2).

THEOREM 1. *Let $u(x)$ be a positive solution of* (2). *Assume $v(x)$, $x>0$, is a positive function satisfying $v(0)=0$. Then*

$$\lim_{n\to\infty} T^n v(x) = u(x),$$

for $x \geqslant 0$. The convergence is uniform on every bounded interval.

Proof. Suppose first that

$$Tv(x) \geqslant v(x)$$

and $0 \leqslant v(x) \leqslant u(x)$. Then the sequence of functions $\{T^n v(x)\}$ is increasing and bounded by $u(x)$. Thus the limit

$$\tilde{u}(x) = \lim_{n\to\infty} T^n v(x)$$

defines the solution $\tilde{u}(x)$ of (2). By Proposition 1 we have $\tilde{u}(x) = u(x)$. This proves the theorem in the case when $Tv \geqslant v$.

A similar reasoning shows that if

$$Tv(x) \leqslant v(x)$$

and $0 \leqslant u(x) \leqslant v(x)$, then

$$\lim_{n\to\infty} T^n v(x) = u(x),$$

for $x \geqslant 0$.

We will complete the proof by showing that there exist increasing positive functions w_1 and w_2 such that

$$w_1(x) \leqslant v(x) \leqslant w_2(x), \qquad w_1(x) \leqslant u(x) \leqslant w_2(x),$$

and

$$Tw_1(x) \geqslant w_1(x), \qquad Tw_2(x) \leqslant w_2(x).$$

We can assume that $v(x)$ is a strictly increasing function. If not, then $Tv(x)$ is such. Obviously the solution $u(x)$ is strictly increasing. Introduce the increasing function $w_1(x)$ by

$$w_1^{-1}(x) = v^{-1}(x) + u^{-1}(x).$$

Then

$$0 \leqslant w_1(x) \leqslant v(x) \qquad \text{and} \qquad w_1(x) \leqslant u(x).$$

Since the functions u^{-1}, v^{-1}, w_1^{-1} are increasing

$$(w_1^{-1})' \geqslant (u^{-1})'. \tag{6}$$

Write w_1 in the form $w_1(x) = u(h_1(x))$. Then $h_1(x) = u^{-1}(w_1(x))$ and by (6),

$$h_1'(x) = (u^{-1})' (w_1(x)) \, w_1'(x) \leqslant 1.$$

By Lemma 1 we then have

$$Tw_1(x) \geqslant w_1(x).$$

Define the function $w_2(x)$ as

$$w_2^{-1}(x) = \int_0^x \min\{(u^{-1})' (y), (v^{-1})' (y)\} \, dy.$$

Then

$$w_2^{-1}(x) \leqslant \int_0^x (v^{-1})' (y) \, dy = v^{-1}(x),$$

$$w_2^{-1}(x) \leqslant \int_0^x (u^{-1})' (y) \, dy = u^{-1}(x).$$

Thus $w_2(x) \geqslant \max\{u(x), v(x)\}$. Moreover,

$$(w_2^{-1})' \leqslant (u^{-1})'. \tag{7}$$

Thus w_2 can be written as $w_2(x) = u(h_2(x))$, where $h_2(x) = u^{-1}(w_2(x))$. By (7) we have

$$(h_2)' (x) = (u^{-1})' (w_2(x)) \, w_2'(x) \geqslant 1.$$

Again by Lemma 1,

$$Tw_2(x) \geqslant w_2(x).$$

Summarizing we proved that there are w_1 and w_2 such that

$$w_1(x) \leqslant v(x) \leqslant w_2(x),$$

$$\lim_{n \to \infty} T^n w_i(x) = u(x), \qquad i = 1, 2.$$

Thus

$$\lim_{n \to \infty} T^n v(x) = u(x).$$

Furthermore, the sequences $T^n w_1$ and $T^n w_2$ are increasing and decreasing, respectively. Hence by Dini's theorem both converge to $u(x)$ uniformly on bounded intervals. So does $T_n v$ as

$$T^n w_1(x) \leqslant T^n v(x) \leqslant T^n w_2(x).$$

This completes the proof.

Remark. By Theorem 1 we can get an estimate for the nonzero solution $u(x)$, if it exists. Assume that the function $v(x)$ satisfies

$$Tv(x) \leqslant v(x), \qquad \text{for} \quad 0 \leqslant x \leqslant c.$$

Then

$$u(x) \leqslant v(x) \qquad \text{for} \quad 0 \leqslant x \leqslant c.$$

In particular we have the following.

COROLLARY 2. *Let* $\{v_n(x)\}_{n=1}^{\infty}$ *be a sequence of positive increasing functions such that*

$$\lim_{n \to \infty} v_n(x) = 0, \qquad \text{for} \quad x \geqslant 0,$$

and

$$Tv_n(x) \leqslant v_n(x), \qquad \text{for} \quad x \geqslant 0.$$

Then the equation $Tu(x) = u(x)$ *has no positive solutions.*

In a forthcoming paper we will use Corollary 2 to prove that if $\phi(x) = \sqrt{x}$ and $a(x, y) = f(x - y)$ is an invariant kernel given by the function

$$f(x) = e^{-e^{1/x}},$$

then Eq. (2) admits no nonzero solutions.

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