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Parametric Representations of Nonsteady One-Dimensional Flows: A Correction

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1. INTRODUCTION

Our recently published paper [1] contains a fundamental logical error which invalidates our assertions about the ease with which certain parametric representations of nonsteady one-dimensional flows could be constructed. Of course, this grievously restricts the prospects for application of such representations.

In this note we shall (i) expurgate [1]; (ii) describe our error; (iii) correct it; and (iv) develop a family of correct examples of our parametric representations.

The following changes are required in [1]:

Section 1: Delete the last three paragraphs.

Section 4: Delete all material starting with the paragraph that contains Eq. (4.9) and continuing to the end of Case 1.

Section 5: Delete the last paragraph.

Sections 6-8: Proposals to apply the method suggested in Section 4 are absurd and should be deleted.

The nature of our error can be summarized as follows. One-dimensional flows can be characterized by means of solutions of a family of Monge-Ampère equations that involve a single nonconstant coefficient, determined by the equation of state and by the form of the distribution of entropy among the various particle paths. By means of this coefficient we can subdivide the set of one-dimensional flows into mutually exclusive subsets. If we consider any two flows of the same subset we can identify the values of times, geometrical coordinates, and flow functions that correspond to identical values of the pressure, p, and of a Lagrangian variable, ψ. The mapping of one ut-plane onto another, defined in this way, preserves area. A well-known representation of the general area-preserving map in terms of parameters, α

and β, involves an arbitrary function $H(\alpha, \beta)$. In attempting to apply this result to the comparison of two flows in the same subset, we determined a *necessary condition* that relates $H(\alpha, \beta)$ to a function $z(p, \psi)$ such that $\alpha = z_p$ and $\beta = z_\psi$. We assumed, erroneously, that $H(\alpha, \beta)$ remains arbitrary in our application. A *necessary and sufficient condition*, which will be developed in this note, restricts the permissible function $H(\alpha, \beta)$ to be any solution of a certain quasi-linear, second-order, hyperbolic partial differential equation.

It is not easy to guess solutions for the equation that defines H. Nevertheless, our representation retains a little value as a source of novelties, since for an important class of equations of state, which includes that of the perfect gases, we have been able to determine a family of separated-variable solutions of a suitably transformed version of the equation for H.

2. The Fallacy in [1]

We shall require the following extract from the valid and relevant parts of [1].

M. H. Martin [2] has developed the following formulation for the equations of all one-dimensional flows, except for an easily discussed special class. Let us define a Lagrangian variable, ψ, by

$$d\psi = \rho\, dx - \rho u\, dt. \tag{2.1}$$

Then the specific entropy must be of the form

$$s = s(\psi), \tag{2.2}$$

and by the equation of state we can express the density in the form

$$\rho = \rho(p, s(\psi)). \tag{2.3}$$

Assume that p and ψ are functionally independent, and let $\xi(p, \psi)$ be any solution of

$$\xi_{pp}\xi_{\psi\psi} - \xi_{p\psi}^2 = - A^2(p, \psi), \tag{2.4}$$

where

$$A^2(p, \psi) = - \left(\frac{1}{\rho}\right)_p \neq 0. \tag{2.5}$$

Then the description of a one-dimensional flow is completed by

$$t = \xi_p, \qquad u = \xi_\psi, \tag{2.6}$$

$$dx = \xi_\psi\, d\xi_p + \left(\frac{1}{\rho}\right) d\psi, \tag{2.7}$$

where t denotes time, u particle velocity and x an Eulerian coordinate.

Now let us suppose $\xi(p, \psi)$ and $\xi^*(p, \psi)$ are two different solutions of (2.4) that correspond to the same $A(p, \psi)$. The mapping of the u^*t^*-plane onto the ut-plane, defined by identifying points with identical values of p and ψ preserves area. Hence we must have

$$\xi_p = \alpha + H_\beta , \qquad \xi_\psi = \beta - H_\alpha , \tag{2.8}$$

$$\xi_p^* = \alpha - H_\beta , \qquad \xi_\psi^* = \beta + H_\alpha , \tag{2.9}$$

for some function $H(\alpha, \beta)$ of some parameters α and β. Since we are actually interested in t^* and u^*, rather than ξ^* for its own sake, it would suffice to determine an acceptable H, or even just H_α and H_β. If we set

$$2z(p, \psi) = \xi + \xi^*, \qquad 2w(p, \psi) = \xi - \xi^*, \tag{2.10}$$

then by (2.8) to (2.10)

$$\alpha = z_p , \qquad \beta = z_\psi , \tag{2.11}$$

$$H_\alpha = - w_\psi , \qquad H_\beta = w_p . \tag{2.12}$$

If we eliminate w from (2.12) we obtain

$$\left(\frac{\partial}{\partial p}\frac{\partial}{\partial z_p} + \frac{\partial}{\partial \psi}\frac{\partial}{\partial z_\psi}\right) H(z_p , z_\psi) = 0. \tag{2.13}$$

Up to this point in [1] all of our reasoning has been legitimate.

In [1] we assumed that H was arbitrary. This is incorrect since, as we shall show in the following section, $H(\alpha, \beta)$ must satisfy the quasi-linear partial differential equation (3.16).

3. On the Determination of $H(\alpha, \beta)$

Let us continue to assume that $\xi(p, \psi)$ is a known solution of (2.4) for a given $A(p, \psi) \neq 0$. Recall that by (2.6), (2.8), and (2.11) we have

$$\xi_p(p, \psi) = t = \alpha + H_\beta , \qquad \xi_\psi(p, \psi) = u = \beta - H_\alpha , \tag{3.1}$$

and

$$\alpha = z_p(p, \psi), \qquad \beta = z_\psi(p, \psi), \tag{3.2}$$

for some $H(\alpha, \beta)$ and $z(p, \psi)$. Since ξ_p and ξ_ψ are functionally independent by (2.4), (3.1) implicitly defines

$$p = p(t, u), \qquad \psi = \psi(t, u). \tag{3.3}$$

Since the functions (3.3) are the inverses of the functions (3.1), we must have

$$\begin{pmatrix} \xi_{pp} & \xi_{p\psi} \\ \xi_{p\psi} & \xi_{\psi\psi} \end{pmatrix}^{-1} = \begin{pmatrix} p_t & p_u \\ \psi_t & \psi_u \end{pmatrix}.$$

Thus

$$\frac{p_t}{\xi_{\psi\psi}} = -\frac{\psi_t}{\xi_{p\psi}} = -\frac{p_u}{\xi_{p\psi}} = \frac{\psi_u}{\xi_{pp}}. \tag{3.4}$$

By (3.1) and (3.3) we can express p and ψ as functions of α and β. Since we have assumed that p and ψ are independent, then α and β must also be independent. Now let us make the Legendre transformation defined by (3.2) and

$$Z(\alpha, \beta) = pz_p + \psi z_\psi - z = \alpha p + \beta\psi - z. \tag{3.5}$$

Then we must have

$$p = Z_\alpha, \qquad \psi = Z_\beta, \tag{3.6}$$

and now by (3.5) and (3.6)

$$z(p, \psi) = \alpha Z_\alpha + \beta Z_\beta - Z = \alpha p + \beta\psi - Z. \tag{3.7}$$

Furthermore, by a well-known property of Legendre transformations

$$\frac{z_{pp}}{Z_{\beta\beta}} = -\frac{z_{p\psi}}{Z_{\alpha\beta}} = \frac{z_{\psi\psi}}{Z_{\alpha\alpha}}. \tag{3.8}$$

By (3.1) and (3.6) we have

$$\xi_p(Z_\alpha, Z_\beta) = \alpha + H_\beta, \qquad \xi_\psi(Z_\alpha, Z_\beta) = \beta - H_\alpha. \tag{3.9}$$

For a known $\xi(p, \psi)$ let the pair $H(\alpha, \beta)$, $Z(\alpha, \beta)$ be any solution of the system (3.9). Define p and ψ by (3.6) and $z(p, \psi)$ by (3.7). Then (3.2) follows from the Legendre transformation (3.6) and (3.7). Finally, (3.9) and (3.6) imply (3.1). Thus (3.1) and (3.2) are equivalent to (3.6) and (3.9).

If we eliminate H from (3.9) we obtain

$$(\xi_p - \alpha)_\alpha + (\xi_\psi - \beta)_\beta = 0, \tag{3.10}$$

which is equivalent to

$$\xi_{pp} Z_{\alpha\alpha} + 2\xi_{p\psi} Z_{\alpha\beta} + \xi_{\psi\psi} Z_{\beta\beta} = 2, \tag{3.11}$$

where the arguments of ξ_{pp}, $\xi_{p\psi}$, and $\xi_{\psi\psi}$ have been replaced by the expressions (3.6). In general, (3.11) is a nonlinear partial differential equation fo $Z(\alpha, \beta)$. By (2.4) it is of *hyperbolic type*.

If we let $Z(\alpha, \beta)$ be any solution of (3.11) such that Z_α and Z_β are independent, and if we define p and ψ by (3.6), then (3.11) is equivalent to (3.10)

This, in turn, implies (3.9) for some $H(\alpha, \beta)$. A possible $H(\alpha, \beta)$ could be defined by

$$H(\alpha, \beta) = \int [(\xi_p - \alpha)\, d\beta - (\xi_\psi - \beta)\, d\alpha]. \tag{3.12}$$

If we know a solution $Z(\alpha, \beta)$, we need not actually determine $H(\alpha, \beta)$. For, by (2.8), (2.9), and (2.11)

$$\xi^*(p, \psi) = 2z(p, \psi) - \xi(p, \psi). \tag{3.13}$$

To determine $\xi^*(p, \psi)$, it would suffice to find $z(p, \psi)$. But the latter can be defined by (3.7).

Instead of eliminating H from (3.9), let us solve for Z_α and Z_β to obtain

$$Z_\alpha = p(\alpha + H_\beta\,, \beta - H_\alpha), \qquad Z_\beta = \psi(\alpha + H_\beta\,, \beta - H_\alpha), \tag{3.14}$$

in terms of the inverse functions p and ψ defined by (3.3). If we eliminate Z from (3.14) we obtain

$$\frac{\partial p}{\partial \beta} - \frac{\partial \psi}{\partial \alpha} = 0, \tag{3.15}$$

or in expanded form

$$p_t H_{\beta\beta} + p_u(1 - H_{\alpha\beta}) - \psi_t(1 + H_{\alpha\beta}) + \psi_u H_{\alpha\alpha} = 0.$$

By (3.4) this becomes

$$\xi_{pp} H_{\alpha\alpha} + 2\xi_{p\psi} H_{\alpha\beta} + \xi_{\psi\psi} H_{\beta\beta} = 0. \tag{3.16}$$

If we replace the arguments of ξ_{pp}, $\xi_{p\psi}$, and $\xi_{\psi\psi}$ by the right members of (3.14), (3.16) becomes a quasi-linear partial differential equation for $H(\alpha, \beta)$. All steps from (3.14) to (3.16) are reversible. Hence, for any solution H of (3.16) there exists a $Z(\alpha, \beta)$ which satisfies (3.14).

The problem of constructing a new solution $\xi^*(p, \psi)$ of (2.4) from a previously determined solution $\xi(p, \psi)$ has been transformed into that of solving the quasi-linear equation (3.16). For most equations of state (3.16) will still be nonlinear. Thus nothing has been gained unless we can at least guess some solutions $H(\alpha, \beta)$. This will be done in Section 4 and 5 for an important special class of flows.

In our discussion up to this point we have assumed $\xi(p, \psi)$ is known. As a by-product we have discovered the parametric representation (3.1), (3.6) for t, u, p, ψ in terms of suitable functions $H(\alpha, \beta)$ and $Z(\alpha, \beta)$. Prior knowledge of $\xi(p, \psi)$ is not really essential for this parametric representation, since we can determine a system of partial differential equations for H and Z that does not depend on ξ. First, note that by (2.4) we must have

$$\frac{\partial(\xi_p, \xi_\psi)}{\partial(\alpha, \beta)} = -A^2(p, \psi) \frac{\partial(p, \psi)}{\partial(\alpha, \beta)}.$$

By (3.1) and (3.6) this is equivalent to

$$H_{\alpha\alpha}H_{\beta\beta} - H_{\alpha\beta}^2 + 1 = -A^2(Z_\alpha, Z_\beta)(Z_{\alpha\alpha}Z_{\beta\beta} - Z_{\alpha\beta}^2). \tag{3.17}$$

On the other hand, if we eliminate ξ from (3.1), we obtain

$$(\alpha + H_\beta)_\psi - (\beta - H_\alpha)_p = 0.$$

In expanded form this becomes

$$(1 + H_{\alpha\beta})\alpha_\psi + H_{\beta\beta}\beta_\psi + H_{\alpha\alpha}\alpha_p - (1 - H_{\alpha\beta})\beta_p = 0,$$

whence by (3.2) and (3.8)

$$Z_{\beta\beta}H_{\alpha\alpha} - 2Z_{\alpha\beta}H_{\alpha\beta} + Z_{\alpha\alpha}H_{\beta\beta} = 0. \tag{3.18}$$

Thus, in the present case the pair H, Z must be a solution of the system (3.17), (3.18).

To complete our parametric representation note that by (2.7), (3.1), and (3.6).

$$x_\alpha = (\beta - H_\alpha)(1 + H_{\alpha\beta}) + \rho^{-1}Z_{\alpha\beta},$$
$$x_\beta = (\beta - H_\alpha)H_{\beta\beta} + \rho^{-1}Z_{\beta\beta}. \tag{3.19}$$

It might be worth mentioning that for $H \equiv$ constant (3.18) is certainly satisfied. By (3.1) we have

$$\alpha = t = \xi_p, \qquad \beta = u = \xi_\psi. \tag{3.20}$$

Now (3.2) yields

$$\xi = z \tag{3.21}$$

and (3.6) becomes

$$Z(t, u) = pt + \beta u - \xi. \tag{3.22}$$

Then (3.17) reduces, as one would expect, to the equation that would be obtained from (2.4) under the Legendre transformation (3.20), (3.22).

4. Flows Associated with $\xi = K(p) + L(\psi)$

Equation (2.4) will have the solution

$$\xi = K(p) + L(\psi) \tag{4.1}$$

if

$$- A^2(p, \psi) = K''(p) L''(\psi), \tag{4.2}$$

where primes denote differentiation with respect to the appropriate argument, and by (2.5)

$$K''(p) L''(\psi) \neq 0. \tag{4.3}$$

By (2.5) this choice of A^2 corresponds to

$$\rho = \frac{1}{[K'(p) L''(\psi) + M(\psi)]}, \tag{4.4}$$

where $M(\psi)$ is an arbitrary function of ψ. If

$$\psi = \psi(s) \tag{4.5}$$

is the inverse of the function $s(\psi)$ mentioned in (2.2), then (4.4) and (4.5) define an equation of state. The equation of state of a perfect gas,

$$\frac{\rho}{\rho_0} = e^{-s/c_p} \left(\frac{p}{p_0}\right)^{1/\gamma},$$

is in the class defined by (4.4) for $M = 0$.

By (4.1) Eq. (3.16) assumes the form

$$K''(p) H_{\alpha\alpha} + L''(\psi) H_{\beta\beta} = 0, \tag{4.6}$$

where by (2.8)

$$K'(p) = \alpha + H_\beta, \qquad L'(\psi) = \beta - H_\alpha. \tag{4.7}$$

By (4.3) Eqs. (4.7) uniquely define

$$p = p(\alpha + H_\beta), \qquad \psi = \psi(\beta - H_\alpha). \tag{4.8}$$

Since (4.6) is nonlinear we cannot hope to find the general solution for arbitrary choices of K'' and L''. However, we can develop some particular solutions, as follows:

First, it will be convenient to make one of the transformations

$$X^\pm = \alpha \pm H_\beta, \qquad Y^\pm = \beta \pm H_\alpha, \tag{4.9}$$

$$Z^\pm = 2(\alpha\beta \mp H) - P^\pm Q^\pm, \tag{4.10}$$

$$P^\pm = \beta \mp H_\alpha, \qquad Q^\pm = \alpha \mp H_\beta. \tag{4.11}$$

Then

$$\xi_p = K'(p) = X, \qquad \xi_\psi = L'(\psi) = P, \qquad \xi_p^* = Q, \qquad \xi_\psi^* = Y \tag{4.12$^+$}$$

for $+$ superscripts, and

$$\xi_p = K'(p) = Q, \qquad \xi_\psi = L'(\psi) = Y, \qquad \xi_p^* = X, \qquad \xi_\psi^* = P \tag{4.12$^-$}$$

for $-$ superscripts.

In the sequel we shall assume that one of the pairs X^+, Y^+ or X^-, Y^- is functionally independent. The exceptional case in which both pairs are functionally dependent will be discussed in Section 6. For convenience we shall omit the superscripts hereafter.

It can easily be verified that $dZ = P\,dX + Q\,dY$, so that

$$P = Z_X, \qquad Q = Z_Y,$$

and then

$$dP = Z_{XX}\,dX + Z_{XY}\,dY,$$

$$dQ = Z_{XY}\,dX + Z_{YY}\,dY. \tag{4.13}$$

From (4.9) to (4.12) we obtain

$$1 \mp H_{\alpha\beta} - (1 \pm H_{\alpha\beta})\,Z_{XY} = \pm\, Z_{XX}H_{\beta\beta} = \pm\, Z_{YY}H_{\alpha\alpha}\,.$$

Eliminate $H_{\alpha\alpha}$ and $H_{\beta\beta}$ from the latter of these equations and (4.6) to find either

$$K''(p(X))\,Z_{XX} + L''(\psi(P))\,Z_{YY} = 0 \tag{4.14$^+$}$$

for $+$ superscripts, or

$$K''(p(Q))\,Z_{XX} + L''(\psi(Y))\,Z_{YY} = 0 \tag{4.14$^-$}$$

for $-$ superscripts.

5. Separable Solutions

Now let us try to find solutions of (4.14$^-$) of the form

$$Z(X, Y) = k(X)\ell(Y). \tag{5.1}$$

As we shall eventually discover, this will impose a strong, but acceptable, restriction on the permissible functional forms for $K(p)$.

By (5.1)

$$P = k'(X)\ell(Y), \qquad Q = k(X)\ell'(Y). \tag{5.2}$$

By (4.7) and (4.9) for $-$ superscripts

$$K'(p) = Q, \qquad L'(\psi) = Y, \tag{5.3}$$

whence

$$p = p(Q), \qquad \psi = \psi(Y). \tag{5.4}$$

Now (4.14⁻) yields

$$K''(p(Q))\,k''(X)\ell(Y)+L''(\psi(Y))\,k(X)\ell''(Y)=0. \tag{5.5}$$

Next, we may assume Q and X are independent. For, if they were not, then by (4.12⁻) and (2.6) ξ_p and ξ_p^* would be dependent. Since $t^*=\xi_p^*$ must not be constant, we would have $\xi_p^*=G(\xi_p)$ for some nonconstant function G. Hence

$$\xi_{p\psi}^*=G'(\xi_p)\,\xi_{p\psi}=0$$

by (4.1). Hence $\xi^*=K^*(p)+L^*(\psi)$. Since solutions of this form have been considered in [1], this requires no further discussion.

Incidentally, if Q and X are independent, then by (5.2) $k\ell'\neq 0$, and hence $k\ell\neq 0$, in general. Then we can rewrite (5.5) as

$$K''(p(Q))\,\frac{k''(X)}{k(X)}+\frac{L''(\psi(Y))\ell''(Y)}{\ell(Y)}=0. \tag{5.6}$$

Differentiate the left-hand member of (5.6) with respect to X, and use (5.2) to find

$$\frac{K'''(p)\,p'(Q)\,Q}{K''(p)}=-\frac{k(X)}{k'(X)}\left[\log\frac{k''(X)}{k(X)}\right]'=c_1\,. \tag{5.7}$$

By (5.3)

$$K''(p)\,p'(Q)=1,\qquad L''(\psi)\,\psi'(Y)=1. \tag{5.8}$$

Thus (5.3) and the outer members of (5.7) yield

$$\frac{K'''(p)}{K''(p)}=\frac{c_1K''(p)}{K'(p)}\,,$$

whence

$$K''(p)=c_2K'^{c_1}(p). \tag{5.9}$$

Case 1. If $c_1=1$, then (5.9) implies

$$K(p)=c_3e^{c_2p}+c_4\,, \tag{5.10}$$

whence

$$K''(p)=c_2{}^2c_3e^{c_2p}=c_2K'(p)=c_2Q, \tag{5.11}$$

and then

$$p=\frac{1}{c_2}\log\frac{Q}{c_2c_3}\,. \tag{5.12}$$

Now (5.5) yields

$$\frac{L''(\psi)\ell''(Y)}{\ell(Y)\ell'(Y)} = -c_2 k''(X) = c_5 . \tag{5.13}$$

Then by (5.8) and (5.13) k and ℓ must satisfy

$$k(X) = -\frac{c_5}{2c_2} X^2 + c_6 X + c_7 , \tag{5.14}$$

$$\ell''(Y) = c_5 \psi'(Y)\ell(Y)\ell'(Y), \tag{5.15}$$

where $\psi(Y)$ is defined by (5.3).

Note that although the choice of $K(p)$ is restricted by (5.10), the choice of $L(\psi)$ is arbitrary. Any solution of (5.15) with $\ell'' \neq 0$ can be multiplied by any polynomial (5.14) with $c_5 \neq 0$ to form a product solution Z of (4.14$^-$). Then (4.12$^-$) will enable us to construct a ξ^* that differs from ξ in the following important respect. By (4.1)

$$\xi_{p\psi} = 0. \tag{5.16}$$

On the other hand, by (4.12$^-$) and (5.2)

$$\xi^*_\psi = P = k'(X)\ell(L'(\psi)).$$

By (5.2) and (5.3)

$$K'(p) = Q = k(X)\ell'(L'(\psi)).$$

Since by (5.8) $K'' \neq 0$, then by (5.14) with $c_5 \neq 0$ this actually suffices to define a function $X(p, \psi)$ such that $X_p \neq 0$. But then

$$\xi^*_{\psi p} = k''\ell(L'(\psi))\, X_p \neq 0$$

in contrast with (5.16). Thus ξ^* is not a completely trivial modification of ξ.

CASE 2. Now suppose $c_1 \neq 1$. By (5.9)

$$K(p) = \frac{1}{c_2(2-c_1)} [(1-c_1)\, c_2(p+c_3)]^{(2-c_1)/(1-c_1)} + c_4 , \tag{5.17}$$

$$Q = K'(p) = [(1-c_1)\, c_2(p+c_3)]^{1/(1-c_1)} \tag{5.18}$$

$$K''(p) = c_2 Q^{c_1}. \tag{5.19}$$

Now (5.2), (5.5), and (5.19) yield

$$\frac{L''(\psi)\ell''(Y)}{\ell(Y)\ell'^{c_1}(Y)} = -c_2 k^{c_1-1}(X)\, k''(X) = c_5 . \tag{5.20}$$

Then by (5.8) and (5.20) we obtain

$$k''(X) = - c_2^{-1} c_5 k^{1-c_1}(X), \tag{5.21}$$

$$\ell''(Y) = c_5 \psi'(Y) \ell(Y) \ell'^{c_1}(Y). \tag{5.22}$$

Equation (5.21) can be solved by quadratures, of course.

Again, the choice of $K(p)$ is restricted, this time by (5.17), but $L(\psi)$ is still arbitrary. The restriction on the form of $K(p)$ is not too serious, if we note that for $c_3 = 0$, $c_1 = \gamma + 1$, $M(\psi) = 0$, and arbitrary $L(\psi)$, (5.18) and (4.4) lead to the equation of state for a perfect gas.

By the argument presented at the end of Case 1, $\xi^*_{\psi p} \neq 0$ again.

All of the discussion in this section has dealt with (4.14$^-$). A similar analysis of separable solutions could be developed for (4.14$^+$). All that we really require are the analogs of Eqs. (5.10), (5.14), and (5.15), or of (5.17), (5.21), and (5.22). These can easily be written by interchanging X and Y; k and ℓ; K and L; and p and ψ. Now, of course, it becomes possible to choose $K(p)$ arbitrarily, but then $L(\psi)$ is restricted. This situation seems to have less physical interest than the one we have just discussed at length.

6. Both X^+, Y^+ and X^-, Y^- are Functionally Dependent

If both X^+, Y^+ and X^-, Y^- are functionally dependent, then in accordance with (4.9)

$$\frac{\partial(\alpha \pm H_\beta, \beta \pm H_\alpha)}{\partial(\alpha, \beta)} = 0.$$

Thus

$$(1 \pm H_{\alpha\beta})^2 - H_{\alpha\alpha} H_{\beta\beta} = 0.$$

These equations are equivalent to

$$H_{\alpha\beta} = 0, \tag{6.1}$$

$$H_{\alpha\alpha} H_{\beta\beta} = 1. \tag{6.2}$$

By (6.1)

$$H(\alpha, \beta) = f(\alpha) + g(\beta), \tag{6.3}$$

for some $f(\alpha)$ and $g(\beta)$. By (6.2) $f''(\alpha)\, g''(\beta) = 1$, whence

$$f'' = c_1, \qquad g'' = \frac{1}{c_1}.$$

Thus

$$f(\alpha) = \tfrac{1}{2} c_1\alpha^2 + c_2\alpha + c_3 , \tag{6.4}$$

$$g(\beta) = \frac{1}{2c_1} \beta^2 + c_4\beta + c_5 . \tag{6.5}$$

By (2.11) we can rewrite (2.8) in the form

$$(\xi - z)_p = H_\beta = g'(\beta) = \frac{1}{c_1} (z + c_1c_4\psi)_\psi ,$$

$$(\xi - z)_\psi = - H_\alpha = -f'(\alpha) = - c_1 \left(z + \frac{c_2}{c_1} p\right)_p .$$

Hence

$$\xi - z + i\left(z + \frac{c_2}{c_1} p + c_1c_4\psi\right) = f(\zeta), \tag{6.6}$$

where $f(\zeta)$ is an analytic function of the complex variable

$$\zeta = p + ic_1\psi. \tag{6.7}$$

Thus

$$\xi_{pp} + c_1^{-2}\xi_{\psi\psi} = 0.$$

If we demand that ξ be of the form (4.1), then

$$K''(p) = - c_1^{-2}L''(\psi) = c_6 .$$

Then by (4.1) and (2.4) to (2.7)

$$\begin{aligned}
t &= \xi_p = c_6p + c_7 ,\\
u &= \xi_\psi = - c_1^2c_6\psi + c_8 ,\\
A^2 &= c_1^2c_6^2,\\
\frac{1}{\rho} &= - c_1^2c_6^2p + M(\psi),\\
x &= - c_1^2c_6^2p\psi + c_6c_{10}p + \int M(\psi)\, d\psi.
\end{aligned}$$

This corresponds to a class of flows with straight particle paths on which the velocity is constant (though it varies from path to path).

References

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