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Hypergroups and n-ary relations

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ABSTRACT

In this paper we associate a hypergroupoid $\langle H, \otimes_\rho \rangle$ with an n-ary relation ρ defined on a nonempty set H. We investigate when it is an H_v-group, a hypergroup or a join space. Then we determine some connections between this hypergroupoid and Rosenberg's hypergroupoid associated with a binary relation.

1. Introduction

In the last few decades, the hypergroups have been studied in connection with various domains, such as: binary relations, graphs and hypergraphs, fuzzy sets, rough sets, topology, codes, cryptography, automata, probabilities, etc.; for more details see [7]. Chvalina [1] used ordered structures for the construction of semihypergroups and hypergroups. Later Rosenberg [18] extended Chvalina's definition, introducing a hypergroupoid associated with a binary relation and described all the relations ρ on a set H such that the new hypergroupoid is a join space, i.e. a hypergroup with a geometry inspired property. Then this new hyperstructure was studied by Corsini [4], Corsini and Leoreanu [8], and recently by Cristea and Ştefănescu [9,10]. Other hypergroupoids associated with binary relations were introduced and analysed by Corsini [5,3,6], De Salvo and Lo Faro [11,12], Spartalis [19,20], Spartalis and Mamaloukas [21], Konstantinidou and Serafimidis [13]. Recently Leoreanu-Fotea and Davvaz [14] studied n-hypergroups in connection with binary relations.

In this paper we continue the study on the correspondence between hypergroups and ordered sets; in particular we use n-ary relations. The n-ary relations were studied in depth by Novak and Novotny [15–17] because of their applications in the theory of dependence spaces. Ušan and Šešelja [23] gave several kinds of generalized reflexive, symmetric and transitive n-ary relations and established connections between some of these relations. Moreover, the n-ary relations are used in database theory, because they provide a convenient tool for database modeling.

In this work we use some connections between binary and n-ary relations, explained in the papers of V. Novak and M. Novotny. More precisely, an n-ary relation σ_n on a nonempty set H may be

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associated with any binary relation σ on H and conversely, a binary relation ρ^b on H may be associated with any n-ary relation ρ on H. In the following section we recall some definitions and results obtained on this argument.

Our goal is to extend the results obtained on the hypergroups connected with binary relations to the case of n-ary relations. Firstly, we define a hypergroupoid associated with a given n-ary relation and we find necessary and sufficient conditions, regarding the n-ary relation, such that this hypergroupoid is an H_v-group or a join space. Secondly, we establish some connections between hypergroupoids associated with n-ary relations and hypergroupoids associated with binary or ternary relations.

2. *n*-ary relations

In this section we present some basic notions about the n-ary relations. We suppose that $H \neq \emptyset$ is a set, $n \in \mathbb{N}$ a natural number such that $n \geq 2$, and $\rho \subseteq H^n$ an n-ary relation on H.

Definition 1 (*See [16]*). The relation ρ is said to be:

(i) *reflexive* if, for any $x \in H$, the n-tuple $(x, \ldots, x) \in \rho$;
(ii) *n-transitive* if it has the following property: if $(x_1, \ldots, x_n) \in \rho$, $(y_1, \ldots, y_n) \in \rho$ hold and if there exist natural numbers $i_0 > j_0$ such that $1 < i_0 \leq n$, $1 \leq j_0 < n$, $x_{i_0} = y_{j_0}$, then the n-tuple $(x_{i_1}, \ldots, x_{i_k}, y_{j_{k+1}}, \ldots, y_{j_n}) \in \rho$ for any natural number $1 \leq k < n$ and $i_1, \ldots, i_k, j_{k+1}, \ldots, j_n$ such that $1 \leq i_1 < \cdots < i_k < i_0, j_0 < j_{k+1} < \cdots < j_n \leq n$;
(iii) *strongly symmetric* if $(x_1, \ldots, x_n) \in \rho$ implies $(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in \rho$ for any permutation σ of the set $\{1, \ldots, n\}$;
(iv) *n-ary preordering* on H if it is reflexive and n-transitive;
(v) an *n-equivalence* on H if it is reflexive, strongly symmetric and n-transitive.

We consider the following examples.

Examples 2. (1) For $n = 2$, a binary relation is 2-transitive if and only if it is transitive in the usual sense and therefore it is a 2-equivalence if and only if it is an equivalence in the usual sense.
(2) Let $n = 3$. A ternary relation ρ is 3-transitive if and only if it satisfies the following conditions:
 (i) If $(x, y, z) \in \rho$, $(y, u, v) \in \rho$, then $(x, u, v) \in \rho$.
 (ii) If $(x, y, z) \in \rho$, $(z, u, v) \in \rho$, then $(x, y, u) \in \rho$, $(x, y, v) \in \rho$, $(x, u, v) \in \rho$, $(y, u, v) \in \rho$.
 (iii) If $(x, y, z) \in \rho$, $(u, z, v) \in \rho$, then $(x, y, v) \in \rho$.
(3) Set $H = \{a_1, a_2, \ldots, a_n\}$ and let $\rho = \{(a_i, \ldots, a_i) \mid 1 \leq i \leq n\}$, where any sequence is formed by n equal symbols, be the diagonal relation on H. Then ρ is an n-equivalence.
(4) Set $H = \{1, 2, 3\}$ and let $\rho \subseteq H \times H \times H$ be the ternary relation on H defined by $\rho = \{(1, 1, 3), (1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 3, 3)\}$.
Then ρ is a 3-ary preordering on H, but it isn't a 3-equivalence.

Definition 3 (*See [16]*). Let ρ be an n-ary relation on a set H. We may associate with ρ a binary relation ρ^b as follows. For any $(x, y) \in H^2$, we put $(x, y) \in \rho^b$ if there exist $(x_1, x_2, \ldots, x_n) \in \rho$ and natural numbers i, j such that $1 \leq i < j \leq n$, $x = x_i$, $y = x_j$.

Example 4. Let ρ be a ternary relation on H. Then the binary relation ρ^b associated with H like in Definition 3 is defined as follows: $(x, y) \in \rho^b$ if there exists $z \in H$ such that $(x, y, z) \in \rho$ or $(z, x, y) \in \rho$ or $(x, z, y) \in \rho$.

For example, the binary relation ρ^b associated with the ternary relation ρ in Examples 2, (4), is $\rho^b = \{(1, 1), (1, 3), (2, 2), (3, 3)\}$, which is a preordering but not an equivalence.

More generally, we have the following results from Novak and Novotny [16].

Theorem 5 (*Theorem 3.1 [16]*). *Let ρ be an n-ary preordering on a set H. Then ρ^b is a preordering on H.*

Theorem 6 (*Theorem 5.1 [16]*). *Let ρ be an n-equivalence on a set H. Then ρ^b is an equivalence relation on H.*

Definition 7 (*See [16]*). Let σ be a binary relation on a set H. We may associate with σ an n-ary relation σ_n on H by putting for any $(x_1, x_2, \ldots, x_n) \in H^n$, $(x_1, x_2, \ldots, x_n) \in \sigma_n$ if $(x_i, x_j) \in \sigma$ for every pair of natural numbers (i, j) such that $1 \leq i < j \leq n$.

Theorem 8 (*Theorem 4.1 [16]*). *Let σ be a preordering on a set H. Then σ_n is an n-ary preordering on H.*

Theorem 9 (*Theorem 5.2 [16]*). *Let σ be an equivalence relation on a set H. Then σ_n is an n-equivalence on H.*

It is easy to see that the following corollary holds.

Corollary 10. (i) *If ρ is an n-ary relation on a set H, then $(\rho^b)_n \supset \rho$, but the equality does not necessarily hold.*
(ii) *If σ is a binary relation on a set H, then $(\sigma_n)^b \subset \sigma$, but the equality does not necessarily hold.*

Proof. The first parts of the two statements are easily verified. We give only counterexamples for their second parts.

(i) On the set $H = \{1, 2\}$ we consider $\rho = \{(1, 2, \ldots, 2)\}$. Then $\rho^b = \{(1, 2), (2, 2)\}$ and thus $(\rho^b)_n = \{(1, 2, \ldots, 2), (2, \ldots, 2)\} \supsetneq \rho$.
(ii) On the set $H = \{1, 2, 3\}$ we take the binary relation $\sigma = \{(1, 2), (2, 1), (2, 3), (3, 3)\}$. Then $\sigma_n = \{(2, 3 \ldots, 3), (3, \ldots, 3)\}$ and therefore $(\sigma_n)^b = \{(2, 3), (3, 3)\} \subsetneq \sigma$. □

Proposition 11. (i) *If ρ_1 and ρ_2 are two n-ary relations on a set H such that $\rho_1 \subset \rho_2$, then $\rho_1^b \subset \rho_2^b$.*
(ii) *For any n-ary relation ρ on a set H, it follows that $((\rho^b)_n)^b = \rho^b$.*

Proof. (i) Set $(x, y) \in \rho_1^b$; then there exist $(x_1, x_2, \ldots, x_n) \in \rho_1$ and natural numbers i, j such that $1 \leq i < j \leq n, x = x_i, y = x_j$. Since $\rho_1 \subset \rho_2$, it follows that there exist $(x_1, x_2, \ldots, x_n) \in \rho_2$ and natural numbers i, j such that $1 \leq i < j \leq n, x = x_i, y = x_j$, that is $(x, y) \in \rho_2^b$.
(ii) For the proof of the direct inclusion we denote ρ^b by σ. By Corollary 10(ii) we have $(\sigma_n)^b \subset \sigma$, that is $((\rho^b)_n)^b \subset \rho^b$. Conversely, by Corollary 10(i), for an n-ary relation ρ, we have $\rho \subset (\rho^b)_n$ and by the item (i) of this proposition, it follows that $\rho^b \subset ((\rho^b)_n)^b$, which concludes the proof. □

Theorem 12 (*See Theorems 4.2, 4.3, 5.3 [16]*). *If ρ is an n-ary preordering on a set H, then $(\rho^b)_n = \rho$. If σ is a preordering on a set H, then $(\sigma_n)^b = \sigma$.*

3. Hypergroupoids associated with *n*-ary relations

For a nonempty set H, we denote by $\mathcal{P}^*(H)$ the set of all nonempty subsets of H. A nonempty set H, endowed with a mapping, called a *hyperoperation*, $\circ : H^2 \longrightarrow \mathcal{P}^*(H)$ is called a *hypergroupoid*. A hypergroupoid which verifies the following conditions:

(i) $(x \circ y) \circ z = x \circ (y \circ z)$, for all $x, y, z \in H$, and
(ii) $x \circ H = H = H \circ x$, for all $x \in H$ (reproduction axiom)

is called a *hypergroup*.
If A and B are nonempty subsets of H, then we define $A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b$.
A hypergroupoid $\langle H, \circ \rangle$ is called an H_v-*semigroup* if the weakly associative axiom is valid, i.e., $(x \circ y) \circ z \cap x \circ (y \circ z) \neq \emptyset$, for all $x, y, z \in H$. An H_v-semigroup is called an H_v-*group* if the reproduction axiom is valid.
For each pair $(a, b) \in H^2$, we define:
$a/b = \{x \mid a \in x \circ b\}$ and $b \setminus a = \{y \mid a \in b \circ y\}$. If A and B are nonempty subsets of H, then we denote $A/B = \bigcup_{\substack{a \in A \\ b \in B}} a/b$.
A commutative hypergroup $\langle H, \circ \rangle$ is called a *join space* if for any $(a, b, c, d) \in H^4$, the following implication holds:

$$a/b \cap c/d \neq \emptyset \Longrightarrow a \circ d \cap b \circ c \neq \emptyset \quad \text{("transposition axiom")}.$$

For more details on hypergroup theory, see [2,24].

Let ρ be an n-ary relation on a set H. In what follows we use the notation

$$L(\rho) = \{x \in H \mid \exists u_2, u_3, \dots, u_n \in H : (x, u_2, \dots, u_n) \in \rho\}$$
$$R(\rho) = \{x \in H \mid \exists u_1, u_2, \dots, u_{n-1} \in H : (u_1, u_2, \dots, u_{n-1}, x) \in \rho\}.$$

Moreover, for any $x \in H$, set

$$\begin{aligned} L(x) = \{y \in H \mid \exists u_1, \dots, u_{n-2} \in H : (y, x, u_1, \dots, u_{n-2}) \in \rho \\ \vee (u_1, \dots, u_{n-2}, y, x) \in \rho \vee (u_1, \dots, u_k, y, x, u_{k+1}, \dots, u_{n-2}) \in \rho, \\ \text{for any } k \in \{1, \dots, n-3\}\} \end{aligned}$$

and similarly

$$\begin{aligned} R(x) = \{y \in H \mid \exists u_1, \dots, u_{n-2} \in H : (x, y, u_1, \dots, u_{n-2}) \in \rho \\ \vee (u_1, \dots, u_{n-2}, x, y) \in \rho \vee (u_1, \dots, u_k, x, y, u_{k+1}, \dots, u_{n-2}) \in \rho, \\ \text{for any } k \in \{1, \dots, n-3\}\}. \end{aligned}$$

Remark 1. It is obvious that

(i) $y \in L(x)$ if and only if $x \in R(y)$, for any $(x, y) \in H^2$.
(ii) $\bigcup_{x \in H} L(x) \neq H$ if and only if there exists $y \in H$ such that $R(y) = \emptyset$.
(iii) $\bigcup_{x \in H} R(x) \neq H$ if and only if there exists $y \in H$ such that $L(y) = \emptyset$.

Indeed, $\bigcup_{x \in H} L(x) \neq H$ if and only if there exists $y \in H$ such that $y \notin \bigcup_{x \in H} L(x)$, which is equivalent to the fact that there exists $y \in H$ such that $y \notin L(x)$, for any $x \in H$, that is there exists $y \in H$ such that $x \notin R(y)$, for any $x \in H$, equivalent to the fact that there exists $y \in H$ such that $R(y) = \emptyset$. □

Let ρ be an n-ary relation on a nonempty set H. We define on H the following hyperoperation:

$$x \otimes_\rho y = L(x) \cup R(y) \tag{1}$$

and we notice that if $\langle H, \otimes_\rho \rangle$ is a hypergroupoid then, for any $x \in H$, $L(x) \neq \emptyset$ or $R(x) \neq \emptyset$. The converse is not true as the following counterexample shows: let $H = \{x, y, z, t\}$ and $\rho = \{(x, y, z), (x, z, t), (x, y, t)\}$. We have $L(x) = R(t) = \emptyset$ and so $x \otimes_\rho t = \emptyset$, that is $\langle H, \otimes_\rho \rangle$ is not a hypergroupoid.

Lemma 13. *The hypergroupoid $\langle H, \otimes_\rho \rangle$ is a quasihypergroup if and only if, for any $x \in H$, $L(x) \neq \emptyset$ and $R(x) \neq \emptyset$.*

Proof. The reproducibility law means: for any $x \in H$, $x \otimes_\rho H = H \otimes_\rho x = H$, that is, for any $x, y \in H$, there exist $z_1, z_2 \in H$ such that $y \in [L(x) \cup R(z_1)] \cap [L(z_2) \cup R(x)]$.

First we suppose that, for any $x \in H$, $L(x) \neq \emptyset$ and $R(x) \neq \emptyset$. Then $\bigcup_{x \in H} L(x) = H = \bigcup_{x \in H} R(x)$ and therefore, for any $y \in H$, there exist $z_1, z_2 \in H$ such that $y \in R(z_1) \cap L(z_2)$; thus $\langle H, \otimes_\rho \rangle$ is reproductive, so it is a quasihypergroup.

Now we consider $\langle H, \otimes_\rho \rangle$ a quasihypergroup and we suppose that there exists $y \in H$ such that $L(y) = \emptyset$ or $R(y) = \emptyset$. If $R(y) = \emptyset$ then, for any $x \in H$, $y \notin L(x)$ and thus $L(x) \cup R(y) = x \otimes_\rho y = L(x) \not\ni y$, for any $x \in H$, and then $H \otimes_\rho y \not\ni y$; so $H \otimes_\rho y \neq H$ which contradicts the reproducibility law. Similarly, if there exists $y \in H$ such that $L(y) = \emptyset$, then $y \otimes_\rho H \neq H$ and again we obtain a contradiction. □

Proposition 14. *Let ρ be an n-ary relation on a set H. Then $\langle H, \otimes_\rho \rangle$ is an H_v-group if and only if, for any $x \in H$, $L(x) \neq \emptyset$ and $R(x) \neq \emptyset$.*

Proof. If $\langle H, \otimes_\rho \rangle$ is an H_v-group, then it is a quasihypergroup and, by Lemma 13, it follows that $L(x) \neq \emptyset$ and $R(x) \neq \emptyset$, for any $x \in H$.

Conversely, we suppose that for any $x \in H$, $L(x) \neq \emptyset$ and $R(x) \neq \emptyset$. By Lemma 13, it follows that $\langle H, \otimes_\rho \rangle$ is a quasihypergroup. It remains to prove that the hyperoperation "$\otimes_\rho$" is weakly associative; for this we show that, for any $x, y, z \in H$,

$$(x \otimes_\rho y) \otimes_\rho z \cap x \otimes_\rho (y \otimes_\rho z) \ni y.$$

Since

$$(x \otimes_\rho y) \otimes_\rho z = \{L(u) \cup R(z) \mid u \in L(x) \cup R(y)\} \supseteq \{L(u) \mid u \in R(y)\} = \{L(u) \mid y \in L(u)\} \ni y$$

and

$$x \otimes_\rho (y \otimes_\rho z) = \{L(x) \cup R(v) \mid v \in L(y) \cup R(z)\} \supseteq \{R(v) \mid v \in L(y)\} = \{R(v) \mid y \in R(v)\} \ni y$$

it follows that $\langle H, \otimes_\rho \rangle$ is an H_v-group. □

Corollary 15. *Let ρ be an n-ary relation on a set H. Then $\langle H, \otimes_\rho \rangle$ is an H_v-group if and only if it is a quasihypergroup.*

Proposition 16. *Let $\rho = \{(x, x, \ldots, x) \mid x \in H\}$ be the diagonal n-ary relation on a set H. Then $\langle H, \otimes_\rho \rangle$ is a join space.*

Proof. For any $x \in H$ we obtain $L(x) = R(x) = \{x\}$ and thus, for any $x, y \in H$, it follows that

$$x \otimes_\rho y = y \otimes_\rho x = \{x, y\}$$

and then $x \otimes_\rho H = H \otimes_\rho = H$, for any $x \in H$. Moreover, for any $(x, y, z) \in H^3$, we obtain

$$(x \otimes_\rho y) \otimes_\rho z = x \otimes_\rho (y \otimes_\rho z) = \{x, y, z\},$$

so $\langle H, \otimes_\rho \rangle$ is a commutative hypergroup.

It remains to prove that, for any $a, b, c, d \in H$ such that $a/b \cap c/d \neq \emptyset$, it follows that $a \otimes_\rho d \cap b \otimes_\rho c \neq \emptyset$. We find that $a/a = \{x \in H \mid a \in x \otimes_\rho a\} = H$ and for $a \neq b \in H$, $a/b = \{x \in H \mid a \in x \otimes_\rho b\} = \{a\}$.
Let $a, b, c, d \in H$ be such that $a/b \cap c/d \neq \emptyset$. If $a = b$ or $c = d$ then $a \otimes_\rho d \cap b \otimes_\rho c \ni a$ or $a \otimes_\rho d \cap b \otimes_\rho c \ni d$. If $a \neq b$ and $c \neq d$, then $a/b \cap c/d \neq \emptyset$ if and only if $a = c$ and thus $a \otimes_\rho d \cap b \otimes_\rho c \ni a$. In both cases $a \otimes_\rho d \cap b \otimes_\rho c \neq \emptyset$ and therefore $\langle H, \otimes_\rho \rangle$ is a join space. □

Lemma 17. *If ρ is an n-ary preordering on a set H, then, for any $a, x, u \in H$ such that $a \in L(u)$ and $u \in L(x)$, it follows that $a \in L(x)$.*

Proof. Let a, x, u be in H such that $a \in L(u)$ and $u \in L(x)$. Then there exist $a_1, \ldots, a_{n-2} \in H$, $b_1, \ldots, b_{n-2} \in H$ such that $(a, u, a_1, \ldots, a_{n-2}) \in \rho \vee (a_1, \ldots, a_{n-2}, a, u) \in \rho \vee (a_1, \ldots, a_k, a, u, a_{k+1}, \ldots, a_{n-2}) \in \rho$ and $(u, x, b_1, \ldots, b_{n-2}) \in \rho \vee (b_1, \ldots, b_{n-2}, u, x) \in \rho \vee (b_1, \ldots, b_k, u, x, b_{k+1}, \ldots, b_{n-2}) \in \rho$ with $k \in \{1, \ldots, n-3\}$. We distinguish the following situations:

(1) If $(a, u, a_1, \ldots, a_{n-2}) \in \rho \vee (a_1, \ldots, a_{n-2}, a, u) \in \rho \vee (a_1, \ldots, a_k, a, u, a_{k+1}, \ldots, a_{n-2}) \in \rho$ and $(u, x, b_1, \ldots, b_{n-2}) \in \rho$ then, by the n-transitivity, it results that $(a, x, b_1, \ldots, b_{n-2}) \in \rho$, that is $a \in L(x)$.
(2) If $(b_1, \ldots, b_{n-2}, u, x) \in \rho \vee (b_1, \ldots, b_k, u, x, b_{k+1}, \ldots, b_{n-2}) \in \rho$ with $k \in \{1, \ldots, n-3\}$ and since $(x, \ldots, x) \in \rho$ by the reflexivity, then $(u, x, \ldots, x) \in \rho$ by the n-transitivity. Now, if $(a, u, a_1, \ldots, a_{n-2}) \in \rho \vee (a_1, \ldots, a_{n-2}, a, u) \in \rho \vee (a_1, \ldots, a_k, a, u, a_{k+1}, \ldots, a_{n-2}) \in \rho$, with $k \in \{1, \ldots, n-3\}$, and since $(u, x, \ldots, x) \in \rho$, it follows that $(a, x, \ldots, x) \in \rho$, again by the n-transitivity. Therefore $a \in L(x)$. □

Now we give some conditions for an n-ary preordering to induce a join space.

Proposition 18. *If ρ is an n-ary preordering on H such that $L(x) = R(x)$, for any $x \in H$, then $\langle H, \otimes_\rho \rangle$ is a join space.*

Proof. Since ρ is reflexive, it follows, by Lemma 13, that $\langle H, \otimes_\rho \rangle$ is a quasihypergroup. Moreover, since $L(x) = R(x)$, for any $x \in H$, it follows that

$$x \otimes_\rho y = y \otimes_\rho x = L(x) \cup L(y),$$

for any $x, y \in H$, and therefore $\langle H, \otimes_\rho \rangle$ is commutative.
Now we prove that the hyperoperation "$\otimes_\rho$" is associative. For any $a \in (x \otimes_\rho y) \otimes_\rho z$, there exists $u \in L(x) \cup L(y)$ such that $a \in L(u) \cup L(z)$. We distinguish the following cases:

(i) If $a \in L(z)$, we take $v = z \in L(y) \cup L(z)$ and then $a \in L(x) \cup L(z) = L(x) \cup L(v)$; thus $a \in x \otimes_\rho (y \otimes_\rho z)$.

(ii) If $a \in L(u)$ with $u \in L(x)$, then, by Lemma 17, it follows that $a \in L(x)$; therefore there exists $v \in L(y) \cup L(z)$ (for example $v = y$) such that $a \in L(x) \cup L(v)$, so $a \in x \otimes_\rho (y \otimes_\rho z)$.
(iii) If $a \in L(u)$ with $u \in L(y)$, there exists $v = u \in L(y) \cup L(z)$ such that $a \in L(x) \cup L(u) = L(x) \cup L(v)$, and again $a \in x \otimes_\rho (y \otimes_\rho z)$.

We have proved that, for any $x, y, z \in H$, we have

$$(x \otimes_\rho y) \otimes_\rho z \subset x \otimes_\rho (y \otimes_\rho z).$$

Similarly we can show the other inclusion

$$x \otimes_\rho (y \otimes_\rho z) \subset (x \otimes_\rho y) \otimes_\rho z.$$

It remains to check the condition of the join space. Set $a, b, c, d \in H$ such that $a/b \cap c/d \neq \emptyset$; then there exists $x \in a/b \cap c/d$, that is $a \in x \otimes_\rho b = L(x) \cup L(b)$ and $c \in x \otimes_\rho d = L(x) \cup L(d)$. We consider the following situations:

(i) If $a \in L(x)$ and $c \in L(x)$ then $x \in R(a) = L(a)$, $x \in R(c) = L(c)$ and therefore $x \in [L(a) \cup L(d)] \cap [L(b) \cup L(c)] = a \otimes_\rho d \cap b \otimes_\rho c$.
(ii) If $c \in L(x)$ and $a \notin L(x)$ then $a \in L(b)$, and since $a \in L(a)$ (by the reflexivity), it follows that $a \in [L(a) \cup L(d)] \cap [L(b) \cup L(c)] = a \otimes_\rho d \cap b \otimes_\rho c$.
(iii) If $c \notin L(x)$ then $c \in L(d)$, and since $c \in L(c)$ (by the reflexivity), it follows that $c \in [L(a) \cup L(d)] \cap [L(b) \cup L(c)] = a \otimes_\rho d \cap b \otimes_\rho c$.

We can conclude that $a \otimes_\rho d \cap b \otimes_\rho c \neq \emptyset$, so $\langle H, \otimes_\rho \rangle$ is a join space. □

Since an n-ary equivalence satisfies the conditions of the above proposition, we have:

Corollary 19. *If ρ is an n-ary equivalence on a set H, then $\langle H, \otimes_\rho \rangle$ is a join space.*

Remark 2. Let ρ be an n-ary preordering on a set H. The condition $L(x) = R(x)$, for any $x \in H$, is a sufficient condition, but not a necessary one such that the hyperoperation "$\otimes_\rho$" is associative, as we can observe in the following examples.

Example 20. On the set $H = \{1, 2, 3\}$ we consider the ternary relation $\rho = \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 3, 3), (1, 1, 3)\}$ which is a 3-preordering. It is obvious that $L(1) = \{1\} \neq \{1, 3\} = R(1)$, $L(2) = \{2\} = R(2)$ and $L(3) = \{1, 3\} \neq \{3\} = R(3)$. Moreover, for any $x, y, z \in H$, $(x \otimes_\rho y) \otimes_\rho z = x \otimes_\rho (y \otimes_\rho z)$.

Example 21. On the set $H = \{1, 2, 3\}$ we consider the 3-preordering $\rho = \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 3, 3), (1, 1, 3), (2, 3, 3), (2, 2, 3)\}$, where $L(1) = \{1\} \neq \{1, 3\} = R(1)$, $L(2) = \{2\} \neq \{2, 3\} = R(2)$, $L(3) = \{1, 2, 3\} \neq \{3\} = R(3)$. Since $(2 \otimes_\rho 2) \otimes_\rho 3 = \{1, 2, 3\} \neq \{2, 3\} = 2 \otimes_\rho (2 \otimes_\rho 3)$, it follows that "$\otimes_\rho$" is not associative.

Remark 3. Let ρ be an n-ary reflexive relation on a set H. By Lemma 17 it follows that, if ρ is also n-transitive, then ρ satisfies the property

(T) for any $a, x, u \in H$ such that $a \in L(u)$ and $u \in L(x)$ it results that $a \in L(x)$.

Moreover, for n-ary reflexive relations ρ on H such that $L(x) = R(x)$, for any $x \in H$, the property (T) does not imply the n-transitivity, and also if $L(x) = R(x)$, for any $x \in H$, as we can see in the following example.

Example 22. On the set $H = \{1, 2, 3\}$ let us consider the following n-relation: $\rho = \{(1, \ldots, 1), (2, \ldots, 2), (3, \ldots, 3), (2, 1, \ldots, 1, 2)\}$. We find $L(1) = \{1, 2\} = R(1)$, $L(2) = \{1, 2\} = R(2)$, $L(3) = \{3\} = R(3)$, and the property (T) is easily satisfied, but ρ is not n-transitive, because we have $(2, \ldots, 2) \in \rho$, $(2, 1, \ldots, 1, 2) \in \rho$, but $(2, \ldots, 2, 1) \notin \rho$.

Proposition 23. *Let ρ be an n-ary relation on H such that $x \in L(x) = R(x)$, for any $x \in H$. If ρ satisfies the property* (T)*, then "$\otimes_\rho$" is associative and therefore $\langle H, \otimes_\rho \rangle$ is a join space.*

Proof. The reproducibility follows from Lemma 13. We suppose that "$\otimes_\rho$" is not associative; then there exist $x, y, z \in H$ such that $(x \otimes_\rho y) \otimes_\rho z \neq x \otimes_\rho (y \otimes_\rho z)$. Thus there exists $u \in (x \otimes_\rho y) \otimes_\rho z$ such that $u \notin x \otimes_\rho (y \otimes_\rho z)$ or vice versa. We consider the first situation; it follows that there exists $v \in L(x) \cup L(y)$ such that $u \in L(v) \cup L(z)$, and for any $t \in L(y) \cup L(z)$, $u \notin L(x) \cup L(t)$ (in particular $u \notin L(x) \cup L(y) \cup L(z)$). If $u \in L(v)$ with $v \in L(x)$, by the property (T) it follows that $u \in L(x)$, which is false, and similarly, if $u \in L(v)$ with $v \in L(y)$, it follows that $u \in L(y)$, again false. If $u \in L(z)$ we obtain a contradiction with the fact $u \notin L(x) \cup L(t)$, for any $t \in L(y) \cup L(z)$. □

Proposition 24. *Let ρ be an n-ary relation on a set H, with $|H| \geq 3$, such that $x \notin L(x)$, $|L(x)| = 1 = |R(x)|$, for any $x \in H$. Then "$\otimes_\rho$" is not associative.*

Proof. Since $|L(x)| = 1 = |R(x)|$, for any $x \in H$, it follows that $L(x) \neq \emptyset$ and $R(x) \neq \emptyset$, for any $x \in H$, that is $\bigcup_{x\in H} L(x) = H = \bigcup_{x\in H} R(x)$. Moreover, there exists a unique $y_x \in H \setminus \{x\}$ such that $x \in L(y_x)$ (if there exist $y_x \neq y'_x \in H \setminus \{x\}$ such that $x \in L(y_x) \cap L(y'_x)$, then $y_x, y'_x \in R(x)$, so $|R(x)| \geq 2$, which is false) and similarly there exists a unique $z_x \in H \setminus \{x\}$ such that $x \in R(z_x)$. We distinguish the following situations:

(1) If $L(x) \cap R(x) = \emptyset$, for any $x \in H$, then it is clear that $y_x \neq z_x$ and $L(y_x) = R(z_x) = \{x\}$, $R(x) = \{y_x\}$, $L(x) = \{z_x\}$. Now it follows that

$$\begin{aligned} (x \otimes_\rho x) \otimes_\rho x &= \{y_x, z_x\} \otimes_\rho x = L(y_x) \cup L(z_x) \cup R(x) \\ &= \{x, y_x\} \cup L(z_x) \not\ni z_x; \\ x \otimes_\rho (x \otimes_\rho x) &= x \otimes_\rho \{y_x, z_x\} = L(x) \cup R(y_x) \cup R(z_x) \\ &= \{z_x, x\} \cup R(y_x) \ni z_x, \end{aligned}$$

so the hyperoperation "$\otimes_\rho$" is not associative; it is only weakly associative.

(2) There exists $\bar{x} \in H$ such that $L(\bar{x}) = R(\bar{x}) = \{y_{\bar{x}}\}$. Then

$$(\bar{x} \otimes_\rho \bar{x}) \otimes_\rho y_{\bar{x}} = y_{\bar{x}} \otimes_\rho y_{\bar{x}} = L(y_{\bar{x}}) \cup R(y_{\bar{x}}) \not\ni y_{\bar{x}},$$
$$\bar{x} \otimes_\rho (\bar{x} \otimes_\rho y_{\bar{x}}) = \bar{x} \otimes_\rho (\{y_{\bar{x}}\} \cup R(y_{\bar{x}})) = L(\bar{x}) \cup R(y_{\bar{x}}) \cup R(u) \ni y_{\bar{x}},$$

where $u \in R(y_{\bar{x}})$, which means that the hyperoperation "$\otimes_\rho$" is not associative. □

4. Hypergroupoids associated with ternary relations

Another approach to the connections between hypergroups and ordered sets is given by Ştefănescu [22]: given a hypergroupoid $\langle H, \circ \rangle$, we may consider the ternary relation ρ on H associated with the hyperoperation

$$(a, b, c) \in \rho \quad \text{if and only if} \quad c \in a \circ b.$$

This is the most natural way to define a ternary relation associated with a hyperoperation.

If $\langle H, \circ \rangle$ is a hypergroup, then Ştefănescu [22] has shown that ρ satisfies the following three conditions:

(1) For all $a, b \in H$, there exists at least one element $c \in H$, such that $(a, b, c) \in \rho$.
(2) If, for $a, b, c, z \in H$ there exists $x \in H$ such that $(a, b, x), (x, c, z) \in \rho$, then there exists $y \in H$ such that $(a, y, z), (b, c, y) \in \rho$ and conversely.
(3) For all $a, b \in H$ there exist $x, y \in H$ such that $(a, x, b) \in \rho$ and $(y, a, b) \in \rho$.

Conversely, if ρ is a ternary relation on a nonempty set H such that the conditions (1)–(3) are satisfied, then, on taking the hyperoperation

$$x \circ y = \{a \in H \mid (x, y, a) \in \rho\},$$

$\langle H, \circ \rangle$ is a hypergroup.

In the following we present two properties of the ternary relations ρ which satisfy the condition (1).

With any binary relation σ on a set H we associate a ternary relation denoted by $\sigma_t \subseteq H \times H \times H$ as follows:

$$(x, y, z) \in \sigma_t \Longleftrightarrow (x, y) \in \sigma \wedge (y, z) \in \sigma \wedge (x, z) \in \sigma. \tag{2}$$

Proposition 25. *The unique ternary relation σ_t obtained from a binary relation σ using the method (2) and such that*

$$\forall(a, b) \in H^2, \quad \exists c \in H : (a, b, c) \in \sigma_t \tag{3}$$

is the total relation $\sigma_t = H \times H \times H$.

Proof. The condition (3) is equivalent to the following one: for any $(a, b) \in H^2$, $(a, b) \in \sigma$, so the binary relation σ is the total relation $H \times H$ and thus $\sigma_t = H \times H \times H$. □

Moreover, the hypergroupoid obtained from σ_t taking

$$x \circ y = \{z \in H \mid (x, y, z) \in \sigma_t\}$$

is the total hypergroup on H.

Conversely, with any ternary relation ρ on H we associate a binary relation $\rho^b \subseteq H \times H$ as follows:

$$(x, y) \in \rho^b \iff \exists z \in H : (x, y, z) \in \rho. \tag{4}$$

Let $\langle H, \circ\rangle$ be an arbitrary hypergroupoid which determines the ternary relation ρ defined by

$$(x, y, z) \in \rho \iff z \in x \circ y.$$

Since $\langle H, \circ\rangle$ is a hypergroupoid, it follows that, for any $(x, y) \in H^2$, there exists $z \in H$ such that $z \in x \circ y$, that is $(x, y, z) \in \rho$; therefore, for any $(x, y) \in H^2$, we obtain $(x, y) \in \rho^b$, that is $\rho^b = H \times H$. So we have proved the following result.

Proposition 26. *The unique binary relation ρ^b obtained, using the method (4), from the ternary relation ρ associated with any hypergroupoid $\langle H, \circ\rangle$ as follows:*

$$(x, y, z) \in \rho \iff z \in x \circ y,$$

is the total relation $H \times H$.

5. Connections with Rosenberg's hypergroupoid

Let ρ be a binary relation on a nonempty set H. We denote by

$$D(\rho) = \{x \in H \mid (x, y) \in \rho, \text{ for some } y \in H\}$$
$$R(\rho) = \{x \in H \mid (y, x) \in \rho, \text{ for some } y \in H\}$$

the domain and the range of ρ. Rosenberg [18] defined the following hyperoperation on H. For any $x \in H$, set $U_x = \{y \in H \mid (x, y) \in \rho\}$ and, for any $x, y \in H$,

$$x \circ y = U_x \cup U_y.$$

He proved that $\mathbb{H}_\rho = \langle H, \circ\rangle$ is a hypergroupoid if and only if $D(\rho) = H$. Moreover $\mathbb{H}_\rho$ is a quasihypergroup if and only if $D(\rho) = R(\rho) = H$. In the same paper, Rosenberg characterized all binary relations ρ such that $\mathbb{H}_\rho$ is a semihypergroup, a hypergroup or a join space. Here we are interested only in weak associativity: for any $x, y, z \in H$, there exists $a \in H$ such that

$$a \in (x \circ y) \circ z \cap x \circ (y \circ z),$$

which is equivalent to: for any $x, y, z \in H$, there exist $a \in H$ and $b \in U_x \cup U_y$ such that $a \in U_b \cup U_z$ and there exists $c \in U_y \cup U_z$ such that $a \in U_x \cup U_c$. This means that there exists $a \in H$ such that

$$(x, a) \in \rho^2 \vee (y, a) \in \rho^2 \vee (z, a) \in \rho$$

and

$$(x, a) \in \rho \vee (y, a) \in \rho^2 \vee (z, a) \in \rho^2.$$

Proposition 27. *If ρ is a binary relation on a set H, with full domain and full range, then $\mathbb{H}_\rho$ is an H_v-group.*

Proof. Since $D(\rho) = H$, it follows that, for any $x \in H$, there exists $z \in H$ such that $(x, z) \in \rho$ and there exists $y \in H$ such that $(z, y) \in \rho$; so, for any $x \in H$, there exists $y \in H$ such that $(x, y) \in \rho^2$; thus $D(\rho^2) = H$. It follows that, for any $y \in H$ there exists $a \in H$ such that $(y, a) \in \rho^2$ and then $a \in (x \circ y) \circ z \cap x \circ (y \circ z)$, which means that $\mathbb{H}_\rho$ is an H_v-group. □

Let ρ be an n-ary relation on H such that, for any $x \in H$, $L(x) \neq \emptyset$ and $R(x) \neq \emptyset$. From Proposition 14, it follows that the hypergroupoid $\langle H, \otimes_\rho\rangle$ defined in the third section is an H_v-group. Then the binary relation ρ^b associated with ρ as in Definition 3 ($(x, y) \in \rho^b$ if there exist $(x_1, x_2, \ldots, x_n) \in \rho$ and natural numbers i, j such that $1 \leq i < j \leq n$, $x = x_i$, $y = x_j$) has full domain and full range, so, by Proposition 27, Rosenberg's hypergroupoid $\mathbb{H}_{\rho^b}$ is an H_v-group. In conclusion, we have proved the following result.

Proposition 28. *If ρ is an n-ary relation on H such that, for any $x \in H$, $L(x) \neq \emptyset$ and $R(x) \neq \emptyset$, then the hypergroupoids $\langle H, \otimes_\rho\rangle$ and $\mathbb{H}_{\rho^b}$ are H_v-groups.*

Conversely, let σ be a binary relation on H, with full domain and full range. Then the n-ary relation σ_n associated with σ as in Definition 7 ($(x_1, \ldots, x_n) \in \sigma_n$ if $(x_i, x_j) \in \sigma$, for any i, j, $1 \leq i < j \leq n$) has the property that, for any $x \in H$, $L(x) \neq \emptyset$ and $R(x) \neq \emptyset$. Indeed, for any $x \in H$, there exists $y \in H$ such that $(x, y) \in \sigma$; then $y \in R(x)$. Similarly, there exists $z \in H$ such that $(z, x) \in \sigma$; thus $z \in L(x)$. We can give the following result.

Proposition 29. *If σ is a binary relation on H, with full domain and full range, then the hypergroupoids $\mathbb{H}_\sigma$ and $\langle H, \otimes_{\sigma_n}\rangle$ are H_v-groups.*

6. Conclusions

Several connections between hypergroups and binary relations have been established so far. Here we have defined a hypergroupoid $\langle H, \otimes_\rho\rangle$ associated with an n-ary relation ρ on H. For any $x \in H$, defining

$$\begin{aligned} L(x) = \{&y \in H \mid \exists u_1, \ldots, u_{n-2} \in H : (y, x, u_1, \ldots, u_{n-2}) \in \rho \\ &\vee (u_1, \ldots, u_{n-2}, y, x) \in \rho \vee (u_1, \ldots, u_k, y, x, u_{k+1}, \ldots, u_{n-2}) \in \rho, \\ &\text{for any } k \in \{1, \ldots, n-3\}\} \end{aligned}$$

and similarly

$$\begin{aligned} R(x) = \{&y \in H \mid \exists u_1, \ldots, u_{n-2} \in H : (x, y, u_1, \ldots, u_{n-2}) \in \rho \\ &\vee (u_1, \ldots, u_{n-2}, x, y) \in \rho \vee (u_1, \ldots, u_k, x, y, u_{k+1}, \ldots, u_{n-2}) \in \rho, \\ &\text{for any } k \in \{1, \ldots, n-3\}\}, \end{aligned}$$

we set

$$x \otimes_\rho y = L(x) \cup R(y).$$

We have characterized all the n-ary relations ρ such that the hypergroupoid $\langle H, \otimes_\rho\rangle$ is an H_v-group, and we have stated some connections between ρ and hypergroups or join spaces. Moreover, we have presented some correspondences between this hypergroupoid and the hypergroupoid obtained by Rosenberg from a binary relation, or the hypergroupoid obtained by Ştefănescu by a certain ternary relation.

It is natural to consider also the hyperoperations $x \circ_\rho y = R(x) \cup R(y)$ or $x \odot_\rho y = R(x) \cap L(y)$; the first one is a generalization of Rosenberg's hyperoperation to the case of n-ary relations and the second one is a generalization of Corsini's hyperoperation introduced in [5]. In a future work we intend to analyze the hypergroupoids obtained in this manner and to study their properties in connection with the union, intersection, and Cartesian product of n-ary relations. Moreover we try to generalize the association between hypergroupoids and hypergraphs using the properties of n-ary relations (see [3]).

References

[1] J. Chvalina, Commutative hypergroups in the sense of Marty and ordered sets, in: Proc. Summer School on General Algebra and Ordered Sets, Olomouc, Czech Republic, 1994, pp. 19–30.
[2] P. Corsini, Prolegomena of Hypergroups Theory, Aviani Editore, 1993.
[3] P. Corsini, Hypergraphs and hypergroups, Algebra Universalis 35 (1996) 548–555.
[4] P. Corsini, On the hypergroups associated with binary relations, Multi. Val. Logic 5 (2000) 407–419.
[5] P. Corsini, Binary relations and hypergroupoids, Ital. J. Pure Appl. Math. 7 (2000) 11–18.
[6] P. Corsini, Binary relations, interval structured and join spaces, J. Appl. Math. Comput. 10 (1–2) (2002) 209–216.
[7] P. Corsini, V. Leoreanu, Applications of Hyperstructure Theory, in: Advances in Mathematics, Kluwer Academic Publishers, 2003.
[8] P. Corsini, V. Leoreanu, Hypergroups and binary relations, Algebra Universalis 43 (2000) 321–330.
[9] I. Cristea, M. Ştefănescu, Binary relations and reduced hypergroups, Discrete Math. 308 (2008) 3537–3544.
[10] I. Cristea, Several aspects on the hypergroups associated with n-ary relations, An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat. (2009) (in press).
[11] M. De Salvo, G. Lo Faro, Hypergroups and binary relations, Mult. Val. Logic 8 (5-6) (2002) 645–657.
[12] M. De Salvo, G. Lo Faro, A new class of hypergroupoids associated to binary relations, J. Mult.-Valued Logic Soft Comput. 9 (4) (2003) 361–375.
[13] M. Konstantinidou, K. Serafimidis, Sur les P-supertreillis, in: T. Vougiouklis (Ed.), New Frontiers in Hyperstructures and Rel. Algebras, Hadronic Press, Palm Harbor, USA, 1996, pp. 139–148.
[14] V. Leoreanu-Fotea, B. Davvaz, n-hypergroups and binary relations, European J. Combin. 29 (5) (2008) 1207–1218.
[15] V. Novak, On representation of ternary structures, Math. Slovaca 45 (5) (1995) 469–480.
[16] V. Novak, M. Novotny, Pseudodimension of relational structures, Czechoslovak. Math. J. 49 (124) (1999) 547–560.
[17] M. Novotny, Ternary structures and groupoids, Czech. Math. J. 41 (116) (1991) 90–98.
[18] I.G. Rosenberg, Hypergroups and join spaces determined by relations, Ital. J. Pure Appl. Math. 4 (1998) 93–101.
[19] S. Spartalis, Hypergroupoids obtained from groupoids with binary relations, Ital. J. Pure Appl. Math. 16 (2004) 201–210.
[20] S. Spartalis, The Hyperoperation Relation and the Corsini's partial or not-partial Hypergroupoids (A classification), Ital. J. Pure Appl. Math. 24 (2008) 97–112.
[21] S. Spartalis, C. Mamaloukas, On hyperstructures associated with binary relations, Comput. Math. Appl. 51 (1) (2006) 41–50.
[22] M. Ştefănescu, Some interpretations of hypergroups, Bull. Math. Soc. Sci. Math. Roumanie 49(97) (1) (2006) 99–104.
[23] J. Ušan, B. Šešelja, Transitive n-ary relations and characterizations of generalized equivalences, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 11 (1981) 231–245.
[24] T. Vougiouklis, Hyperstructures and their Representation, Hadronic Press, Palm Harbor, U.S.A, 1994.