

INDEPENDENT CONNECTIONS: AN EASY CHARACTERIZATION OF BASELINE-EQUIVALENT MULTISTAGE INTERCONNECTION NETWORKS*

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Abstract. In this paper we study topological properties of multistage interconnection networks. We define the interconnection scheme between two stages as a pair of functions (f, g) which associate with a cell or node x , labelled as a binary string, his two sons $f(x)$ and $g(x)$. We consider a class of connections called independent connections. Using our graph characterization of networks topologically equivalent to the Baseline network, we show that all the Banyan networks built with independent connections are isomorphic. Finally, we show that if the interconnection scheme is defined by using permutations on the links induced by a permutation of the index digit (PIPID permutation) then the connection is independent. As these PIPID are associated with a very simple bit directed routing, they are used to define most of the networks introduced in the literature. Therefore all these networks are easily shown to be equivalent.

1. Introduction

Several multistage interconnection networks have been proposed for communication in parallel architectures. They are typically designed using at least $n = \log_2(N)$ stages of $\frac{1}{2}N \times 2$ switching cells to connect N inputs to N outputs [1]. The interconnection scheme between two stages is often defined (see [2]) using permutations between the outlinks and inlinks. The topological properties of these networks have been studied extensively, as only a few parameters (number of stages, type and number of cells, connections between stages) may drastically change their functionalities. Topological equivalence between the “classical” networks (Omega [3], Flip [4], Indirect Binary Cube [5], Modified Data Manipulator [6], Baseline and Reverse Baseline (see Fig. 1) [7]) has been proved by Wu and Feng [7] who have exhibited one-to-one mappings of the nodes between each network and the Baseline network.

Another approach consists of modeling the networks by graphs or directed graphs. Such an approach was considered by Agrawal in [8] (see also [9]). He proposed a

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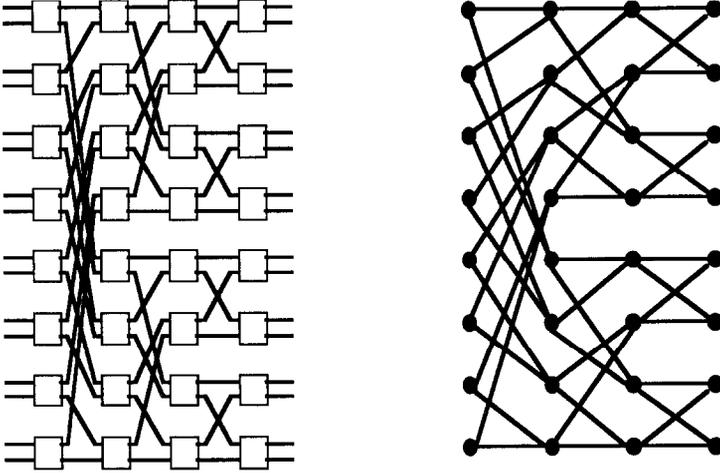


Fig. 1. Baseline network and Baseline MI-digraph.

characterization of this class of networks by “Buddy Properties”; unfortunately, the assertion of Theorem 1 of [8] is not sufficient to prove equivalence as has been shown in [10].

Kruskal and Snir [11], within the graph theory framework, used a labeling scheme to describe routing in the network. They defined a network isomorphism as a graph isomorphism which furthermore preserves the vertex labels. They obtained a sufficient condition, called the bidelta property, to insure that a network is isomorphic, in their sense, to the classical ones.

In [12] we obtained a graph theoretical characterization of the networks equivalent to the Baseline network using connected components of families of subgraphs. This characterization is difficult to apply if the networks are defined by permutations on the links. The aim of this paper is to show the relation between our graph characterization and the usual definitions of multistage interconnection networks using a set of permutations, via the key notion of “independent connections”.

In Section 2, we introduce some notation, and we state the characterization in terms of graph theory. Section 3 is devoted to the study of Independent Connections: our link between graph theory and networks definition using permutations. We define the interconnection scheme between two stages as a pair of functions (f, g) which associate with a cell or node x , labelled as a binary string, his two sons $f(x)$ and $g(x)$. We consider a class of connections called independent connections. Using our graph characterization of networks topologically equivalent to the Baseline network, we show that all the Banyan networks built with independent connections are isomorphic. In Section 4 we consider the PIPID field, a useful set of permutations which allow the construction of the six “classical” networks. PIPID permutations on N symbols are defined by a permutation of the index digit of the binary

representation of these symbols. We show that PIPID permutations used to build Banyan networks may easily be modeled by independent connections. Therefore we obtain the main result of this paper that Banyan networks built with PIPID permutations are topologically equivalent to the Baseline network. Note that the six networks studied by Wu and Feng [7] are designed using a subset of PIPID and that such a design allows an efficient bit directed routing.

2. A graph characterization

Multistage interconnection networks may easily be modeled by directed graphs (digraphs) in which nodes represent the switching cells and arcs the communication links. We do not add extra nodes for the inputs and the outputs of the network as they do not play any role in the graph isomorphism.

Let C be a set of nodes; we will denote by $I^+(C)$ the set of the children of the nodes in C , and by $I^-(C)$ the set of the parents of the nodes in C .

A multistage interconnection digraph (MI-digraph) with n stages is a digraph whose nodes are partitioned into n ordered stages. We denote by V_i the nodes of the i th stage. There are arcs only from nodes of the i th stage to nodes of the $(i+1)$ th stage (i.e. from V_i to V_{i+1}). The nodes are of indegree 2 and outdegree 2 except the nodes from the first and the last stage. And every stage has $N/2$ nodes where $N = 2^n$.

With this definition, we say that two multistage interconnection networks are topologically equivalent if and only if their MI-digraphs are isomorphic. Two digraphs are isomorphic if and only if there exists a bijection from the nodes of the first digraph into the nodes of the second digraph, which preserves the adjacency relationship.

Remark. We omit in all figures the direction of the arcs as they are all directed from the left to the right.

Baseline network definition. The n -stage Baseline network is built in a recursive manner. The subnetwork between stages 2 and n consists of two $(n-1)$ -stage Baseline networks. These components are connected via the first stage such that nodes $2i$ and $2i+1$ of stage 1 are connected to the i th nodes of the two subnetworks ($i = 0, \dots, 2^{n-1} - 1$). This property is known as the left-recursive construction of the Baseline network [13]. Figure 1 shows the 4th stage Baseline and its associate MI digraph.

Banyan property definition. One minimal requirement is to allow a connection between any pair of input and output nodes. We say that a network has the Banyan property if and only if for any input and any output there exists a unique path connecting them.

Definition. The connected components of an MI-digraph are those of the undirected underlying graph, obtained from the digraph by deleting the orientation of the arcs.

Definition. We denote by $(G)_{i,j}$ the subgraph of G that contains the vertices of the stages from i to j : $V_i \cup V_{i+1} \cup \dots \cup V_j$.

$P(i, j)$ property definition. We say that an MI-digraph with n stages satisfies the $P(i, j)$ property for $1 \leq i \leq j \leq n$ if and only if the subdigraph $(G)_{i,j}$ has exactly $2^{n-1-(j-i)}$ connected components.

$P(1, *)$ and $P(*, n)$ property definition. We say that an MI-digraph satisfies property $P(1, *)$ if and only if it satisfies $P(1, j)$ for every j such that $1 \leq j \leq n$. Similarly it satisfies property $P(*, n)$ if and only if it satisfies $P(i, n)$ for every i .

Using this notation, the next theorem states a condition of topological equivalence for multistage interconnection networks.

Theorem. *All the MI-digraphs with n stages satisfying the Banyan property $P(*, n)$ and $P(1, *)$ are isomorphic.*

For a proof see [12].

The assumptions of the theorem are very easy to check using a breadth first search algorithm to compute the number of connected components and the number of nodes at distance k . Unfortunately, these conditions are hardly related to numerical definitions of multistage interconnection networks (i.e. the permutations realized at each stage). For instance, the Omega network is defined by n perfect shuffles, and it is not obvious to understand why this type of definition implies the $P(1, *)$ and $P(*, n)$ topological properties.

In the next sections, we define independent connections as a pair of mappings satisfying numerical constraints. We prove, using the former theorem, that Banyan graphs built with these connections are isomorphic to the Baseline MI-digraph. Furthermore, we show that the set of permutations on N symbols, defined by a permutation of the binary digits of the symbol representation, may easily be associated with independent connections. Like the perfect shuffle, permutations used to design multistage interconnection networks often exhibit this property, and the equivalence relationship between "classical" networks becomes clear.

3. Independent connections

As we consider networks defined in terms of permutations, we add a labeling of the nodes in the graph. At each stage, nodes are labeled from 0 to $2^{n-1} - 1$, following the natural order of the drawing (Fig. 2). The label of a node is an $n - 1$ tuple

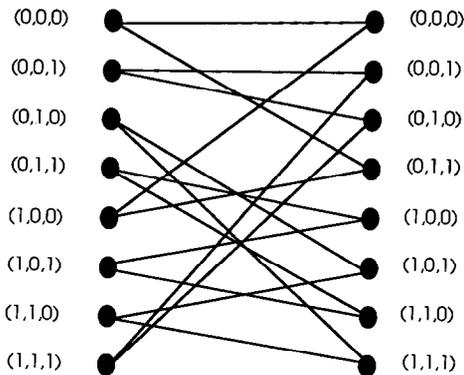


Fig. 2. Labeling of an MI-digraph.

(x_{n-1}, \dots, x_1) , in base 2, so $(x_{n-1}, \dots, x_1)^{-1}$. We consider the usual addition in the group (Z_2^{n-1}, \oplus) , called bitwise addition (or exclusive or).

Now, we define connections and independent connections. The major result of this section is that Banyan networks built with independent connections are topologically equivalent.

Definition of a connection. For all $i \neq n$, a connection (f, g) between the i th stage and the $(i+1)$ st stage of the MI-digraph G is a pair of functions f and g defined on Z_2^{n-1} such that, if x is a node of the i th stage of G (i.e. V_i) then the two children of x in the $(i+1)$ st stage (i.e. V_{i+1}) are $f(x)$ and $g(x)$ (i.e. $F^+(x) = \{f(x), g(x)\}$).

Such a decomposition of the adjacency relationship exists as the outdegree of a node is always two, except in the last stage.

Definition of an independent connection. A connection (f, g) is independent if and only if

$$\begin{cases} \forall \alpha, \alpha \neq (0, \dots, 0), \exists \beta \text{ such that } \forall x \\ f(x \oplus \alpha) = \beta \oplus f(x) \text{ and } g(x \oplus \alpha) = \beta \oplus g(x). \end{cases}$$

We exhibit in Section 4 some examples of independent connections and show that the classical connections are independent.

The following definitions will be useful.

Definition of a translated set. Let A be a subset of V_i , and ν a vector in Z_2^{n-1} . We call the ν -translated set of A , the set of the nodes $\{a_i \oplus \nu\}$ for all a_i in A .

Definition of the reverse graph G^{-1} . The digraph G^{-1} is obtained from G by changing the orientation of all the arcs. The digraph G^{-1} is associated with what is called the “reverse network” in the literature [7].

Proposition 1. *Let (f, g) be an independent connection between V_i and V_{i+1} in an MI-digraph G then we can define an independent connection (ϕ, ψ) between V_{i+1} and V_i in the reverse graph G^{-1} .*

Proof. We will say that a vertex y of V_{i+1} is of type (h_1, h_2) if the two arcs entering y are of type h_1 and h_2 that is $y = h_1(x) = h_2(x')$. There cannot exist a vertex y of type (f, g) and a vertex z of type (f, f) (or (g, g)). Indeed suppose $y = f(x) = g(x')$ and $z = f(u) = f(u')$. Let $\alpha = x \oplus u$; then $f(x) = f(u \oplus \alpha) = \beta \oplus z = y$. But $f(u' \oplus \alpha) = \beta \oplus z = y$. And so y receives 3 arcs, a contradiction. Therefore either all the vertices of V_{i+1} are of the type (f, g) or half the vertices of V_{i+1} are of type (f, f) and half are of type (g, g) .

- In the first case, f and g are one-to-one mappings and we design the reverse connection using the converse functions $\phi = f^{-1}$ and $\psi = g^{-1}$. Obviously the connection (ϕ, ψ) is an independent connection.
- Consider now the second case. Let F (resp. G) be the set of vertices of type (f, f) (resp. (g, g)). Let α_1 be such that $f(x \oplus \alpha_1) = f(x)$. Note that $g(x \oplus \alpha_1) = g(x)$. Let $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ be a basis of Z_2^{n-1} . Let A be the set of vertices generated by $(\alpha_2, \alpha_3, \dots, \alpha_{n-1})$. Then A has 2^{n-2} nodes and $B = V_i - A$ is the α_1 translated of A . Now, f (resp. g) restricted to A induces a bijection between A and F (resp. G). Similarly f (resp. g) restricted to B induces a bijection between B and F (resp. G). We can now define ϕ and ψ as follows:

$$\text{if } y \quad \phi(y) = \Gamma^{-1}(y) \cap A \text{ and } \psi(y) = \Gamma^{-}(y) \cap B.$$

Note that (ϕ, ψ) is a connection as $|\Gamma^{-}(y) \cap A| = |\Gamma^{-}(y) \cap B| = 1$. ϕ is the converse of f (resp. g) for the arcs between A and F (resp. G). A similar property holds for ψ with the set B instead of the set A . It is easy to check that (ϕ, ψ) is an independent connection: to each $\alpha_i, i = 2, \dots, n - 1$, is associated a unique β_i and these β_i generate a subspace of dimension $n - 2$. If we let $\beta_1 = g(0, \dots, 0) \oplus f(0, \dots, 0)$ then β_1 is not in this subspace and $\phi(y \oplus \beta_1) = \phi(y), \psi(t \oplus \beta_1) = \psi(y)$ and G is the β_1 translated set of F . \square

Lemma 2. *A Banyan MI-digraph G built with independent connections satisfies the $P(*, n)$ property.*

Proof. We proceed by induction. We prove in fact that the connected components of the subdigraph $(G)_{j,n}$ intersect each stage V_i for $j \leq i \leq n$ in exactly 2^{n-j} nodes (see Fig. 3). Let us now consider a component of $(G_{j-1,n})$, C . It consists of a set of nodes X in V_{j-1} , a connected component of $(G)_{j,n}$ called A and a remaining part.

- Let $A_i = A \cap V_i, j \leq i \leq n$. By induction hypothesis A_j has 2^{n-j} nodes.
- Two vertices of A_j cannot have a parent in common otherwise there will be two different paths between this parent and any vertex of V_n contradicting the Banyan property. So X has 2^{n+1-j} nodes.

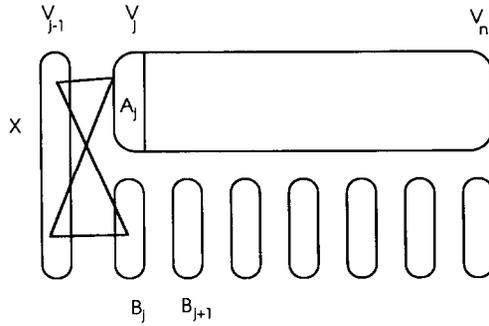


Fig. 3. Construction of G .

- Following [8] let us say that two nodes y and y' are buddy if they have the same father:

$$y = h_1(x) \text{ and } y' = h_2(x) \quad \text{where } \{h_1, h_2\} = \{f, g\}.$$

Let B_j be the set of buddy nodes of A_j . By the property above $B_j \cap A_j = \emptyset$. So, as

$$C \cap V_j = A_j \cup B_j, \quad |C \cap V_j| = 2^{n+1-j}.$$

We prove that B_j is a translated set of A_j . Let x and y be any pair of nodes of X ; then as the connection (f, g) is independent $f(x) \oplus g(x) = f(y) \oplus g(y)$. But as we are in Z_2^{n-1} we have

$$g(x) \oplus f(x) = f(x) \oplus g(x) = f(y) \oplus g(y) = g(y) \oplus f(y).$$

Hence the difference between the labels of the nodes in A_j and B_j is constant. B_j is a translated set of A_j . Let B_{j+1} be the set of children of the nodes of B_j . As B_j is a translated set of A_j the definition of an independent connection implies that B_{j+1} is a translated set of A_{j+1} . Therefore they have the same number of nodes, and so

$$|C \cap V_{j+1}| = 2^{n+1-j}.$$

And so on by induction B_k is a translated set of A_k , for $j \leq k \leq n$ and so

$$|C \cap V_k| = 2^{n+1-j}.$$

Therefore C intersects each stage in 2^{n+1-j} nodes and so the induction hypothesis is satisfied. \square

According to our graph characterization, we can state the announced result whose corollaries are developed in the next section. We consider in the following section some connections defined by a permutation of the digital representation (i.e. the tuple (x_{n-1}, \dots, x_1)) and we show the relations between these connections and the

PIPID set of permutations. Fortunately enough, these connections are independent connections, allowing us to use our main theorem:

Theorem 3. *A Banyan MI-digraph built with independent connections is isomorphic to the Baseline MI-digraph.*

Proof. By Lemma 2, it satisfies $P(*, n)$. Applying Lemma 2 to the reverse MI-digraph G^{-1} , which satisfies the hypothesis by Proposition 1, G also satisfies $P(1, *)$. According to our characterization, G is isomorphic to the Baseline MI-digraph. \square

4. Pipid permutations

Let us now label the links going out of the cells and the links going in following the natural order of the drawing (see Fig. 4). To each outlink (resp. inlink) is associated a number between 0 and $N - 1$ whose binary representation is denoted by $(x_{n-1}, \dots, x_1, x_0)$. The interconnection scheme between V_i and V_{i+1} is defined by a permutation of these N labels.

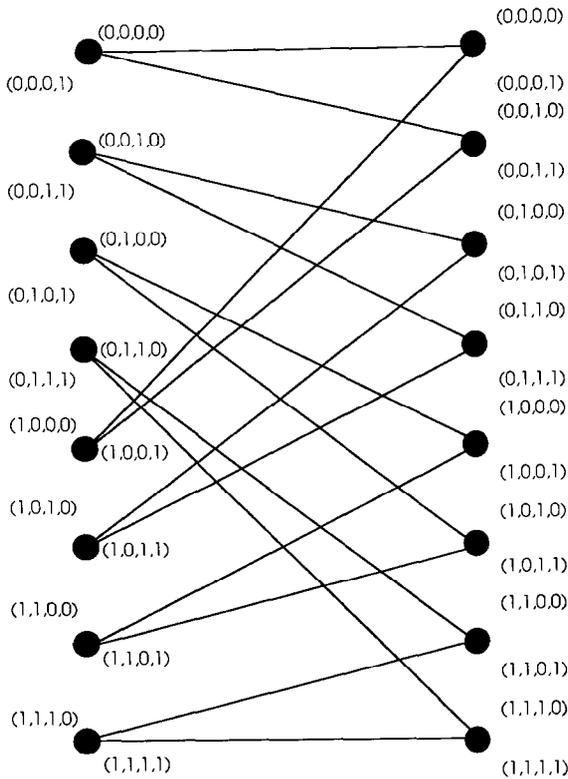


Fig. 4. Labels and permutation.

Multistage interconnection networks have often been defined using these permutations and functional properties have been derived from this model [14]. For instance, the Omega network is defined using n perfect shuffles. A perfect shuffle σ may be defined as a circular left shift of the binary representation of the operand:

$$\sigma(x_{n-1}, x_{n-2}, \dots, x_0) = (x_{n-2}, \dots, x_0, x_{n-1}).$$

Similarly, the k -subshuffle σ_k , the k -butterfly β_k , and the bit reversal ρ , are easily defined by permutations on the bits of the number representation (see [2] for more definitions). These permutations have been used to design the six networks studied by Wu and Feng and one may ask if this scheme of construction is the reason of the topological equivalence of the networks.

Consider numbers from 0 to $N-1$ and their binary representation $(x_{n-1}, \dots, x_1, x_0)$. Following [15], we define a Permutation Induced by a Permutation on the Index Digits (PIPID) as a permutation on the index of the representation.

$$\begin{cases} \lambda \in \text{PIPID}(2^n) \leftrightarrow \exists \theta \in S_n / \text{permutatio} \\ \lambda(x_{n-1}, \dots, x_1, x_0) = (x_{\theta(n-1)}, \dots, x_{\theta(1)}, x_{\theta(0)}). \end{cases}$$

As has previously been stated, perfect shuffle, bit reversal and butterfly are examples of PIPID. We prove in the following that these permutations are also associated to a family of very simple independent connections.

Compare now the label of the cell (or node) used in Section 3 and the labels of the links leaving these nodes as stated in the beginning of this section. The $n-1$ first bits of a link label are exactly the binary representation of the label of the incident node.

Let λ be an arbitrary permutation of PIPID used to design a stage of a network, and let θ be the associated permutation of the index. Let x be a node or cell label, $x = (x_{n-1}, \dots, x_1)$. The links connected to this cell are labeled:

$$\begin{cases} y^0 = (x_{n-1}, \dots, x_1, 0), \\ y^1 = (x_{n-1}, \dots, x_1, 1). \end{cases}$$

Applying permutation λ on these two labels gives the two labels of the links (z^0, z^1) entering the next stage:

$$\begin{cases} z^0 = \lambda(y_0), \\ z^1 = \lambda(y^1). \end{cases}$$

Let $k = \theta^{-1}(0)$. We have

$$\begin{cases} z^0 = (x_{\theta(n-1)}, \dots, x_{\theta(k+1)}, 0, x_{\theta(k-1)}, \dots, x_{\theta(1)}, x_{\theta(0)}), \\ z^1 = (x_{\theta(n-1)}, \dots, x_{\theta(k+1)}, 1, x_{\theta(k-1)}, \dots, x_{\theta(1)}, x_{\theta(0)}). \end{cases}$$

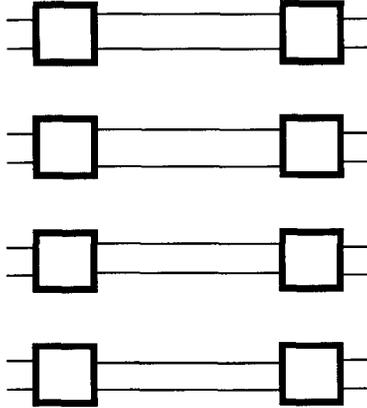


Fig. 5. Stage such as $\theta^{-1}(0) = 0$.

So if we consider only the $(n - 1)$ first digits, we obtain the labels of the cells connected to cell x :

$$\begin{cases} (x_{\theta(n-1)}, \dots, x_{\theta(k+1)}, 0, x_{\theta(k-1)}, \dots, x_{\theta(1)}), \\ (x_{\theta(n-1)}, \dots, x_{\theta(k+1)}, 1, x_{\theta(k-1)}, \dots, x_{\theta(1)}). \end{cases}$$

Note that we had supposed in the former equations that k is not zero. Indeed, if k is zero, then there are two links between the cells (Fig. 5), and the graph does not obviously satisfy the Banyan property.

Now, we have to identify the two mappings f and g and to check that the connection (f, g) satisfies the independence property. For example, we can choose:

$$\begin{cases} f(x_{n-1}, \dots, x_1) = (x_{\theta(n-1)}, \dots, x_{\theta(k-1)}, 0, x_{\theta(k-1)}, x_{\theta(1)}), \\ g(x_{n-1}, \dots, x_1) = (x_{\theta(n-1)}, \dots, x_{\theta(k+1)}, 1, x_{\theta(k-1)}, x_{\theta(1)}). \end{cases}$$

(f, g) is an independent connection; indeed,

$$\begin{cases} f(x \oplus \alpha) = f(x) \oplus \beta & \text{and} & g(x \oplus \alpha) = g(x) \oplus \beta \\ \text{where } \beta = (\alpha_{\theta(n-1)}, \dots, \alpha_{\theta(k+1)}, 0, \alpha_{\theta(k-1)}, \alpha_{\theta(1)}). \end{cases}$$

So, we can associate independent connections to the PIPID permutations used to build Banyan networks. We have now an easy way to check sufficient conditions of equivalence with the Baseline network: all Banyan multistage networks built with PIPID permutations are topologically equivalent to the Baseline network. As Omega, Baseline, Reverse Baseline, Flip, Indirect Binary Cube and Modified Data Manipulator networks are designed using PIPID permutations, they are all equivalent.

5. Conclusion

We state in [12] a characterization of Baseline equivalent networks using a graph model of multistage interconnection networks. As this characterization is difficult

to apply to networks defined by permutations, we design a new tool, independent connections, to prove that Banyan networks built with these connections are equivalent. Independent Connections use the binary labels of the cells. We show that the independence property is satisfied by the networks defined by permutations in PIPID. As these permutations are associated to a very simple bit directed routing, they have been used to design most of the multistage interconnection networks presented in the literature.

Note that the results obtained here apply only to networks built with 2×2 switching cells, whereas our graph characterization has been generalized to arbitrary size of cells.

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