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Cluster tilting for higher Auslander algebras

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Abstract

The concept of cluster tilting gives a higher analogue of classical Auslander correspondence between representation-finite algebras and Auslander algebras. The n-Auslander–Reiten translation functor τ_n plays an important role in the study of n-cluster tilting subcategories. We study the category $\mathcal{M}_n$ of preinjective-like modules obtained by applying τ_n to injective modules repeatedly. We call a finite-dimensional algebra Λ *n-complete* if $\mathcal{M}_n = \operatorname{add} M$ for an n-cluster tilting object M. Our main result asserts that the endomorphism algebra $\operatorname{End}_\Lambda(M)$ is $(n+1)$-complete. This gives an inductive construction of n-complete algebras. For example, any representation-finite hereditary algebra $\Lambda^{(1)}$ is 1-complete. Hence the Auslander algebra $\Lambda^{(2)}$ of $\Lambda^{(1)}$ is 2-complete. Moreover, for any $n \geqslant 1$, we have an n-complete algebra $\Lambda^{(n)}$ which has an n-cluster tilting object $M^{(n)}$ such that $\Lambda^{(n+1)} = \operatorname{End}_{\Lambda^{(n)}}(M^{(n)})$. We give the presentation of $\Lambda^{(n)}$ by a quiver with relations. We apply our results to construct n-cluster tilting subcategories of derived categories of n-complete algebras.

Keywords: Cluster tilting; Higher Auslander algebra; Higher almost split sequence

Contents

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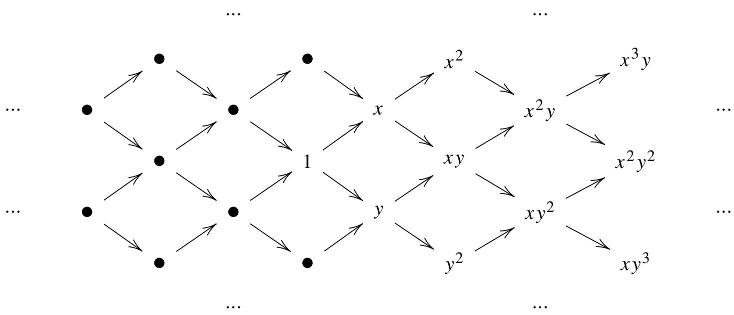
The concept of cluster tilting [24] is fundamental to categorify Fomin–Zelevinsky cluster algebras [28], and a fruitful theory has been developed in recent years (see survey papers [23,62,65, 54]). It also played an important role from the viewpoint of higher analogue of Auslander–Reiten theory [45–47] in the study of rigid Cohen–Macaulay modules, Calabi–Yau algebras and categories, and non-commutative crepant resolutions [21,22,26,31–33,50,51,55–58,67,68]. There are a lot of recent work on higher cluster tilting [3,17,25,27,37–39,52,53,59,66,69–71]. In this paper we shall present a systematic method to construct a series of finite-dimensional algebras Λ with n-cluster tilting objects.

In the representation theory of a representation-finite finite-dimensional algebra Λ with an additive generator M in $\operatorname{mod}\Lambda$, the endomorphism algebra $\Gamma := \operatorname{End}_\Lambda(M)$ called the Auslander algebra gives a prototype of the use of functor categories in Auslander–Reiten theory. The Auslander algebra Γ keeps all information of the category $\operatorname{mod}\Lambda$ in its algebraic structure, and it is a prominent result due to Auslander [5,13] that Auslander algebras are characterized by 'regularity of dimension two'

$$\operatorname{gl.dim}\Gamma \leqslant 2 \leqslant \operatorname{dom.dim}\Gamma.$$

Since almost split sequences in $\operatorname{mod}\Lambda$ correspond to minimal projective resolutions of simple Γ-modules, the Auslander–Reiten quiver of Λ coincides with the quiver of Γ^{op}. As a result the quiver of Γ has the structure of translation quivers. Moreover, the structure theory due to Riedtmann [64], Bongartz and Gabriel [20], Igusa and Todorov [40,41], Bautista, Gabriel, Roĭter and Salmeron [18], … realizes Auslander algebras as factor algebras of path algebras of translation quivers modulo mesh relations. They can be regarded as an analogue of the commutative relation $xy = yx$ in the formal power series ring $S_2 := k[[x, y]]$ of two variables since the mesh category of the translation quiver

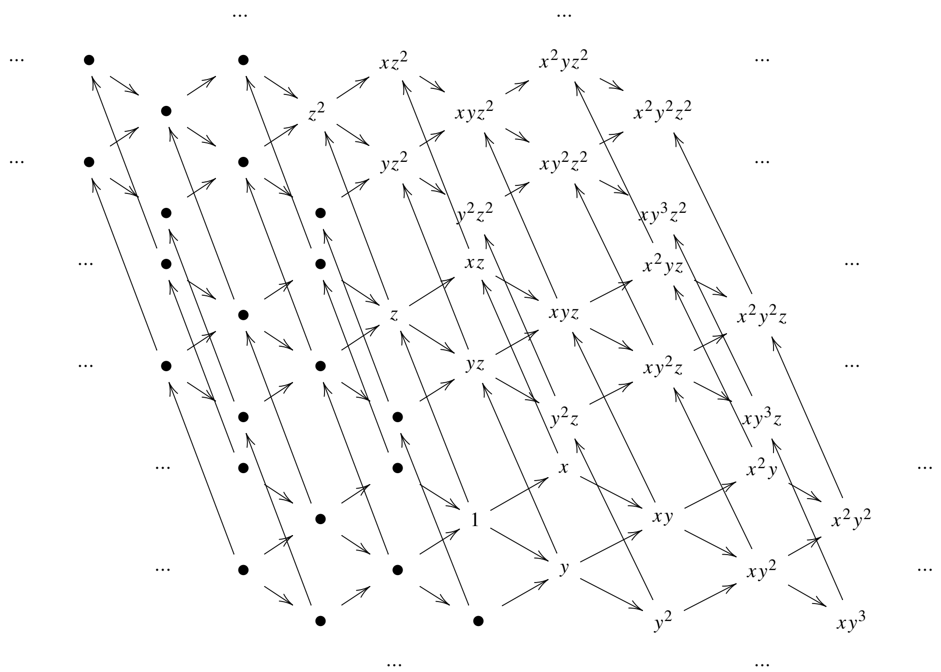


gives a universal Galois covering of S_2 in Gabriel's sense [29]. This is a basic pattern of Auslander–Reiten quivers, so it is suggestive in the representation theory to regard their module categories as a certain analogue of S_2. A typical example is given by the category $\mathrm{CM}(\Lambda)$ of Cohen–Macaulay modules over a quotient singularity $\Lambda := S_2^G$ corresponding to a finite subgroup G of $\mathrm{SL}(2,k)$ [7,11,63]. In this case $\mathrm{CM}(\Lambda)$ has an additive generator S_2, and the Auslander algebra $\Gamma := \mathrm{End}_\Lambda(S_2)$ is isomorphic to the skew group algebra $S_2 * G$ which is regular in the sense that $\mathrm{gl.dim}\,\Gamma = 2 = \mathrm{depth}\,\Gamma$. The Koszul complex $0 \to S \xrightarrow{\binom{x}{y}} S^2 \xrightarrow{(y,-x)} S \to k \to 0$ of S induces almost split sequences in $\mathrm{CM}(\Lambda)$. Hence the Auslander–Reiten quiver of Λ is given by the McKay quiver of G, and forms the translation quiver $\mathbf{Z}\Delta/\tau$ associated to an extended Dynkin diagram Δ.

It is natural to consider a higher-dimensional analogue of this classical theory, and n-cluster tilting (= maximal $(n-1)$-orthogonal) subcategories were introduced in [45,46] in this context. The endomorphism algebra $\Gamma := \mathrm{End}_\Lambda(M)$ of an n-cluster tilting object M in $\mathrm{mod}\,\Lambda$ is called an n-Auslander algebra, and characterized by 'regularity of dimension $n+1$'

$$\mathrm{gl.dim}\,\Gamma \leqslant n+1 \leqslant \mathrm{dom.dim}\,\Gamma.$$

It is known that the category $\mathrm{add}\,M$ has n-almost split sequences, which correspond to minimal projective resolutions of simple Γ-modules. It is natural to regard Γ as analogue of the formal power series ring $S_{n+1} := k[[x_1,\dots,x_{n+1}]]$ of $n+1$ variables. Actually a typical example of n-cluster tilting objects is given by a quotient singularity $\Lambda := S_{n+1}^G$ corresponding to a finite subgroup G of $\mathrm{SL}(n+1,k)$ acting on $k^{n+1}\backslash\{0\}$ freely. In this case $\mathrm{CM}(\Lambda)$ has an n-cluster tilting object S_{n+1}, and the n-Auslander algebra $\Gamma := \mathrm{End}_\Lambda(S_{n+1})$ is isomorphic to the skew group algebra $S_{n+1} * G$. Again the Koszul complex of S induces n-almost split sequences in $\mathrm{add}\,S_{n+1}$. So it is natural to hope in a certain generality that n-almost split sequences in $\mathrm{add}\,M$ can be constructed as a certain analogue of Koszul complexes of S_{n+1}, and that the basic pattern of quivers of n-Auslander algebras is given by the Galois covering



(the case $n = 2$),

of S_{n+1}, which has the set $\mathbf{Z}^{n+1}$ of vertices.

The aim of this paper is to give a class of finite-dimensional algebras with n-cluster tilting objects satisfying the desired properties above. Our construction is inductive in the following sense: We introduce a class of algebras Λ called n-complete algebras, which are algebras with n-cluster tilting objects M satisfying certain nice properties. Our main result asserts that the endomorphism algebra $\Gamma := \mathrm{End}_\Lambda(M)$ is $(n+1)$-complete, hence Γ has an $(n+1)$-cluster tilting object N. This procedure continues repeatedly, so $\mathrm{End}_\Gamma(N)$ is $(n+2)$-complete and has an $(n+2)$-cluster tilting object, and so on. We notice here that we consider not only n-cluster tilting objects in whole module categories $\mathrm{mod}\,\Lambda$ but also those in full subcategories

$$T^\perp := \{X \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^i_\Lambda(T, X) = 0\ (0 < i)\}$$

associated to tilting Λ-modules T. Such a generalization is natural from the viewpoint of study of Auslander-type conditions [46,36,44], and indispensable for our inductive construction to work. It is interesting that our inductive construction reminds us of a classical result due to Auslander and Reiten [10] which asserts that the category of coherent functors over a dualizing variety again forms a dualizing variety.

In forthcoming papers [35,48,49,61] n-complete algebras will be studied further.

Conventions. Throughout this paper, all *subcategories* are assumed to be full and closed under isomorphism, direct sums, and direct summands. We denote by $J_{\mathcal{C}}$ the Jacobson radical of an additive category $\mathcal{C}$ [13,4].

All modules are usually right modules, and the composition fg of morphisms means first g, then f. We denote by $\operatorname{mod}\Lambda$ the category of finitely generated Λ-modules, by J_Λ the Jacobson radical of Λ. For $M \in \operatorname{mod}\Lambda$, we denote by $\operatorname{add} M$ the subcategory of $\operatorname{mod}\Lambda$ consisting of direct summands of finite direct sums of copies of M. For example $\operatorname{add}\Lambda$ is the category $\operatorname{pr}\Lambda$ of finitely generated projective Λ-modules, and $\operatorname{add} D\Lambda$ is the category $\operatorname{in}\Lambda$ of finitely generated injective Λ-modules.

1. Our results

In this section, we shall present our results in this paper. Let Λ be a finite-dimensional algebra.

1.1. *n-Cluster tilting in module categories*

Let us recall a classical concept due to Auslander–Smalo [14]. A subcategory $\mathcal{C}$ of an additive category $\mathcal{X}$ is called *contravariantly finite* if for any $X \in \mathcal{X}$, there exists a morphism $f \in \operatorname{Hom}_{\mathcal{X}}(C, X)$ with $C \in \mathcal{C}$ such that $\operatorname{Hom}_{\mathcal{X}}(-, C) \xrightarrow{f} \operatorname{Hom}_{\mathcal{X}}(-, X) \to 0$ is exact on $\mathcal{C}$. Dually a *covariantly finite subcategory* is defined. A contravariantly and covariantly finite subcategory is called *functorially finite*.

Definition 1.1. Let $n \geqslant 1$. Let $\mathcal{C}$ be a subcategory of $\operatorname{mod}\Lambda$. We call $\mathcal{C}$ *n-rigid* if $\operatorname{Ext}^i_\Lambda(\mathcal{C}, \mathcal{C}) = 0$ for any $0 < i < n$. We call $\mathcal{C}$ *n-cluster tilting* if it is functorially finite and

$$\begin{aligned} \mathcal{C} &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X, \mathcal{C}) = 0\ (0 < i < n)\} \\ &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(\mathcal{C}, X) = 0\ (0 < i < n)\}. \end{aligned}$$

This equality can be understood such that the pair $(\mathcal{C}, \mathcal{C})$ forms a 'cotorsion pair' with respect to Ext^i for $0 < i < n$. We call an object $C \in \operatorname{mod}\Lambda$ *n-cluster tilting* (respectively, *n-rigid*) if so is $\operatorname{add} C$. Clearly $\operatorname{mod}\Lambda$ is a unique 1-cluster tilting subcategory, and 2-cluster tilting subcategories are often called *cluster tilting*.

Let us start with introducing basic terminologies. We have the duality

$$D := \operatorname{Hom}_k(-, k) : \operatorname{mod}\Lambda \leftrightarrow \operatorname{mod}\Lambda^{\mathrm{op}}.$$

We denote by

$$\begin{aligned} \nu = \nu_\Lambda &:= D\operatorname{Hom}_\Lambda(-, \Lambda) : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda \quad \text{and} \\ \nu^- = \nu^-_\Lambda &:= \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(D-, \Lambda) : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda \end{aligned}$$

the Nakayama functors of Λ. They induce mutually quasi-inverse equivalences $\nu : \operatorname{add}\Lambda \to \operatorname{add} D\Lambda$ and $\nu^- : \operatorname{add} D\Lambda \to \operatorname{add}\Lambda$. We denote by

$$\underline{\operatorname{mod}}\,\Lambda \quad \text{and} \quad \overline{\operatorname{mod}}\,\Lambda$$

the stable categories of $\operatorname{mod}\Lambda$ [13,4]. For a subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda$, we denote by $\underline{\mathcal{X}}$ (respectively, $\overline{\mathcal{X}}$) the corresponding subcategory of $\underline{\operatorname{mod}}\,\Lambda$ (respectively, $\overline{\operatorname{mod}}\,\Lambda$). We denote by

$$\mathrm{Tr} : \underline{\mathrm{mod}}\,\Lambda \leftrightarrow \underline{\mathrm{mod}}\,\Lambda^{\mathrm{op}}, \qquad \Omega : \underline{\mathrm{mod}}\,\Lambda \to \underline{\mathrm{mod}}\,\Lambda \quad \text{and} \quad \Omega^- : \overline{\mathrm{mod}}\,\Lambda \to \overline{\mathrm{mod}}\,\Lambda$$

Auslander–Bridger transpose duality, the syzygy functor and the cosyzygy functor [8].

For $n \geqslant 1$, we define *n-Auslander–Reiten translations* [45] by

$$\tau_n := D\,\mathrm{Tr}\,\Omega^{n-1} : \underline{\mathrm{mod}}\,\Lambda \to \overline{\mathrm{mod}}\,\Lambda,$$
$$\tau_n^- := \mathrm{Tr}\,D\Omega^{-(n-1)} : \overline{\mathrm{mod}}\,\Lambda \to \underline{\mathrm{mod}}\,\Lambda.$$

They are by definition given as follows: For $X \in \mathrm{mod}\,\Lambda$, take a minimal projective resolution and a minimal injective resolution

$$P_n \xrightarrow{f} P_{n-1} \to \cdots \to P_0 \to X \to 0 \quad \text{and} \quad 0 \to X \to I_0 \to \cdots \to I_{n-1} \xrightarrow{g} I_n.$$

Then we have

$$\tau_n X = \mathrm{Ker}\big(\nu P_n \xrightarrow{\nu f} \nu P_{n-1}\big) \quad \text{and} \quad \tau_n^- X = \mathrm{Cok}\big(\nu^- I_{n-1} \xrightarrow{\nu^- g} \nu^- I_n\big). \tag{1}$$

The functors $\tau = \tau_1 = D\,\mathrm{Tr}$ and $\tau^- = \tau_1^- = \mathrm{Tr}\,D$ are classical Auslander–Reiten translations, and we have $\tau_n = \tau\Omega^{n-1}$ and $\tau_n^- = \tau^-\Omega^{-(n-1)}$ by definition. Moreover $X \in \mathrm{mod}\,\Lambda$ satisfies $\tau_n X = 0$ (respectively, $\tau_n^- X = 0$) if and only if $\mathrm{pd}\,X_\Lambda < n$ (respectively, $\mathrm{id}\,X_\Lambda < n$).

For the case $\mathrm{gl.dim}\,\Lambda \leqslant n$, clearly τ_n and τ_n^- are induced by the functors $D\,\mathrm{Ext}^n_\Lambda(-,\Lambda) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda$ and $\mathrm{Ext}^n_{\Lambda^{\mathrm{op}}}(D-,\Lambda) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda$ respectively. In this case, we always lift τ_n and τ_n^- to endofunctors of $\mathrm{mod}\,\Lambda$ by putting

$$\tau_n := D\,\mathrm{Ext}^n_\Lambda(-,\Lambda) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda,$$
$$\tau_n^- := \mathrm{Ext}^n_{\Lambda^{\mathrm{op}}}(D-,\Lambda) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda.$$

Then τ_n (respectively, τ_n^-) clearly preserves monomorphisms (respectively, epimorphisms) in $\mathrm{mod}\,\Lambda$.

Let us consider the relationship between the functor τ_n and n-cluster tilting subcategories. The following results [45, Th. 2.3] show that the functor τ_n plays the role of Auslander–Reiten translation for n-cluster tilting subcategories.

Proposition 1.2.

(a) *For any n-cluster tilting subcategory $\mathcal{C}$ of* $\mathrm{mod}\,\Lambda$*, the functors τ_n and τ_n^- induce mutually quasi-inverse equivalences $\tau_n : \underline{\mathcal{C}} \to \overline{\mathcal{C}}$ and $\tau_n^- : \overline{\mathcal{C}} \to \underline{\mathcal{C}}$.*
(b) *τ_n gives a bijection from isoclasses of indecomposable non-projective objects in $\mathcal{C}$ to isoclasses of indecomposable non-injective objects in $\mathcal{C}$.*

Immediately we have the following results.

Proposition 1.3. *Let M be an n-cluster tilting object of* $\mathrm{mod}\,\Lambda$.

(a) *For any indecomposable object $X \in$* $\mathrm{add}\,M$*, precisely one of the following statement holds.*

(i) *X is τ_n-periodic, i.e. $\tau_n^{\ell} X \simeq X$ for some $\ell > 0$.*
(ii) *$X \simeq \tau_n^{\ell} I$ for some indecomposable injective Λ-module I and $\ell \geqslant 0$, and $X \simeq \tau_n^{-m} P$ for some indecomposable projective Λ-module P and $m \geqslant 0$.*

(b) *A bijection from isoclasses of indecomposable injective Λ-modules to isoclasses of indecomposable projective Λ-modules is given by $I \mapsto \tau_n^{\ell_I} I$, where ℓ_I is a maximal number ℓ satisfying $\tau_n^{\ell} I \neq 0$.*

(c) *If* $\mathrm{gl.dim}\,\Lambda \leqslant n$, *then the above* (i) *does not occur.*

Proof. (a)(b) Immediate from Proposition 1.2(b).

(c) Fix any indecomposable object $X \in \operatorname{add} M$. We consider two possibilities in (a). It is enough to show that $\tau_n^{\ell} X = 0$ holds for some $\ell \geqslant 0$. Take an injective hull $0 \to X \to I$. Since τ_n preserves monomorphisms because $\mathrm{gl.dim}\,\Lambda \leqslant n$, we have an exact sequence $0 \to \tau_n^i X \to \tau_n^i I$ for any $i \geqslant 0$. Since $\tau_n^{\ell} I = 0$ holds for sufficiently large ℓ by (b), we have $\tau_n^{\ell} X = 0$. Thus we have shown the assertion. □

These observation motivates to introduce the following analogue of preinjective modules, which was studied by Auslander–Solberg for $n = 1$ [15].

Definition 1.4. We define the *τ_n-closure* of $D\Lambda$ by

$$\mathcal{M} = \mathcal{M}_n(D\Lambda) := \operatorname{add}\{\tau_n^i(D\Lambda) \mid i \geqslant 0\} \subset \operatorname{mod} \Lambda.$$

Immediately from Proposition 1.2(a), we have the following result.

Proposition 1.5. *Any n-cluster tilting subcategory $\mathcal{C}$ of* $\operatorname{mod}\Lambda$ *contains $\mathcal{M}$.*

Summarizing Propositions 1.3(c) and 1.5, we have the uniqueness result of n-cluster tilting objects for algebras Λ with $\mathrm{gl.dim}\,\Lambda \leqslant n$, which is not valid if we drop the assumption $\mathrm{gl.dim}\,\Lambda \leqslant n$.

Theorem 1.6. *Assume* $\mathrm{gl.dim}\,\Lambda \leqslant n$ *and that Λ has an n-cluster tilting object M. Then $\mathcal{M} =$* $\operatorname{add} M$ *holds. In particular, $\mathcal{M}$ is a unique n-cluster tilting subcategory of* $\operatorname{mod}\Lambda$.

Thus the condition $\mathrm{gl.dim}\,\Lambda \leqslant n$ seems to be basic in the study of n-cluster tilting subcategories.

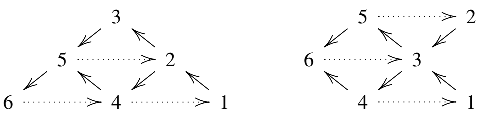
We note here that $\mathcal{M}$ enjoys nice properties below for the case $n = 2$, which we shall prove in Section 2. In particular, $\mathcal{M}$ provides us a rich source of 2-rigid objects.

Proposition 1.7. *Assume $n = 2$.*

(a) *$\mathcal{M}$ is 2-rigid.*
(b) *Assume* $\mathrm{gl.dim}\,\Lambda \leqslant 2$. *Then Λ has a 2-cluster tilting object if and only if $\Lambda \in \mathcal{M}$.*

Let us calculate $\mathcal{M}$ for a few examples.

Example 1.8. Let Λ and Λ' be Auslander algebras



Then one can calculate

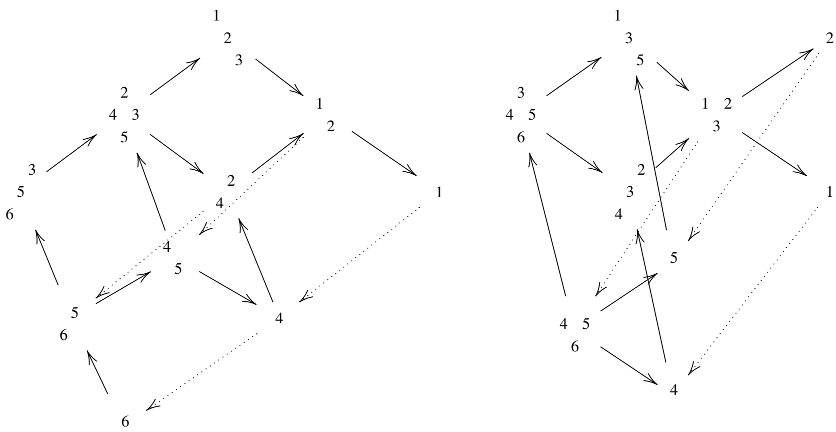
$$D\Lambda = \left(1 \oplus \begin{smallmatrix}1&\\&2\end{smallmatrix} \oplus \begin{smallmatrix}1&&\\&2&\\&&3\end{smallmatrix} \oplus \begin{smallmatrix}&2\\4&\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\4&&3\\&5&\end{smallmatrix} \oplus \begin{smallmatrix}&&3\\&5&\\6&&\end{smallmatrix}\right),$$

$$\tau_2(D\Lambda) = \left(4 \oplus \begin{smallmatrix}4&\\&5\end{smallmatrix} \oplus \begin{smallmatrix}&5\\6&\end{smallmatrix}\right), \qquad \tau_2^2(D\Lambda) = (6), \qquad \tau_2^3(D\Lambda) = 0,$$

$$D\Lambda' = \left(1 \oplus 2 \oplus \begin{smallmatrix}1&&2\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&&2\\&3&\\4&&\end{smallmatrix} \oplus \begin{smallmatrix}1&&\\&3&\\&&5\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\4&&5\\&6&\end{smallmatrix}\right),$$

$$\tau_2\big(D\Lambda'\big) = \left(4 \oplus 5 \oplus \begin{smallmatrix}4&&5\\&6&\end{smallmatrix}\right), \qquad \tau_2^2\big(D\Lambda'\big) = 0.$$

The quivers of $\mathcal{M}_2(D\Lambda)$ and $\mathcal{M}_2(D\Lambda')$ are the following, where dotted arrows indicate τ_2.



By Proposition 1.7(a), we have that $\mathcal{M}_2(D\Lambda)$ is a 2-cluster tilting subcategory of $\operatorname{mod}\Lambda$, while $\mathcal{M}_2(D\Lambda')$ is *not* a 2-cluster tilting subcategory of $\operatorname{mod}\Lambda'$. Nevertheless $\mathcal{M}_2(D\Lambda')$ can be regarded as a 2-cluster tilting subcategory of a certain subcategory of $\operatorname{mod}\Lambda$ defined as follows.

Definition 1.9 *(Relative version of Definition 1.1).* Let $n \geqslant 1$. Let $\mathcal{X}$ be an extension closed subcategory of $\operatorname{mod}\Lambda$. We call a subcategory $\mathcal{C}$ of $\mathcal{X}$ *n-cluster tilting* if it is functorially finite and

$$\begin{aligned}\mathcal{C} &= \big\{X \in \mathcal{X} \;\big|\; \operatorname{Ext}^i_\Lambda(X, \mathcal{C}) = 0 \ (0 < i < n)\big\}\\ &= \big\{X \in \mathcal{X} \;\big|\; \operatorname{Ext}^i_\Lambda(\mathcal{C}, X) = 0 \ (0 < i < n)\big\}.\end{aligned}$$

We call an object $C \in \mathcal{X}$ *n-cluster tilting* if so is $\operatorname{add} C$.

Especially we deal with subcategories $\mathcal{X}$ of $\operatorname{mod} \Lambda$ associated with tilting Λ-modules. Recall that a Λ-module T is called *tilting* [60,34] if there exists $m \geqslant 0$ such that

- $\operatorname{pd} T_\Lambda \leqslant m$,
- $\operatorname{Ext}^i_\Lambda(T, T) = 0$ for any $i > 0$,
- there exists an exact sequence $0 \to \Lambda \to T_0 \to \cdots \to T_m \to 0$ with $T_i \in \operatorname{add} T$.

In this case, we have an extension closed subcategory

$$T^\perp = \{X \in \operatorname{mod} \Lambda \mid \operatorname{Ext}^i_\Lambda(T, X) = 0 \ (0 < i)\}$$

of $\operatorname{mod} \Lambda$. This is a functorially finite subcategory of $\operatorname{mod} \Lambda$, and plays an important role in tilting theory [34] analogous to the category of Cohen–Macaulay modules over commutative rings [9, 12].

For tilting modules T with $\operatorname{pd} T_\Lambda \leqslant m$, we call n-cluster tilting subcategories of the category $T^\perp$ as *m-relative*. For the case $m = 0$ (i.e. $T^\perp = \operatorname{mod} \Lambda$), we use the terminology *absolute* instead of 0-relative. In Example 1.8, $\mathcal{M}_2(D\Lambda)$ is an absolute 2-cluster tilting subcategory of Λ, and $\mathcal{M}_2(D\Lambda')$ is a 1-relative 2-cluster tilting subcategory of Λ associated to a tilting Λ'-module $T = ({}_{4}3^{2} \oplus {}^{1}3_{5} \oplus 4\,{}^{3}_{6}\,5 \oplus 4 \oplus 5 \oplus {}^{4}{}_{6}{}^{5})$.

It was shown in [46, Th. 4.2.1] that n-cluster tilting objects are closely related to algebras of finite global dimension:

Theorem 1.10. *Let $n \geqslant 1$ and $n \geqslant m \geqslant 0$. For a finite-dimensional algebra Γ, the following conditions are equivalent.*

(a) *There exists a finite-dimensional algebra Λ and an m-relative n-cluster tilting object M of Λ such that $\Gamma \simeq \operatorname{End}_\Lambda(M)$.*
(b) *The following conditions are satisfied.*
   (i) $\operatorname{gl.dim} \Gamma \leqslant n + 1$.
   (ii) *The minimal injective resolution*

$$0 \to \Gamma \to I_0 \to \cdots \to I_n \to I_{n+1} \to 0$$

   of the Γ-module Γ satisfies $\operatorname{pd}(I_i)_\Gamma \leqslant m$ *for any* $0 \leqslant i \leqslant n$.
   (iii) *The opposite side version of* (ii).

In this case we call Γ an (m-relative) n-Auslander algebra (of Λ).

Now let us formalize Examples 1.8, where $\mathcal{M}$ give a relative n-cluster tilting subcategory.

Definition 1.11. Let Λ be a finite-dimensional algebra and $n \geqslant 1$. Let $\mathcal{M} = \mathcal{M}_n(D\Lambda)$ be the τ_n-closure of $D\Lambda$. We define subcategories of $\mathcal{M}$ by

$$\mathcal{I}(\mathcal{M}) := \operatorname{add} D\Lambda,$$

$$\mathcal{P}(\mathcal{M}) := \{X \in \mathcal{M} \mid \operatorname{pd} X_\Lambda < n\} = \{X \in \mathcal{M} \mid \tau_n X = 0\},$$

$$\mathcal{M}_I := \{X \in \mathcal{M} \mid X \text{ has no non-zero summands in } \mathcal{I}(\mathcal{M})\},$$
$$\mathcal{M}_P := \{X \in \mathcal{M} \mid X \text{ has no non-zero summands in } \mathcal{P}(\mathcal{M})\}.$$

- We call Λ *τ_n-finite* if $\mathrm{gl.dim}\,\Lambda \leqslant n$ and $\tau_n^\ell(D\Lambda) = 0$ holds for sufficiently large ℓ. In this case, it is easily shown that $\tau_n^\ell = 0$ holds (e.g. Proof of Proposition 1.3(c)).
- We call Λ *n-complete* if $\mathrm{gl.dim}\,\Lambda \leqslant n$ and the following conditions (A_n)–(C_n) are satisfied.
  (A_n) There exists a tilting Λ-module T satisfying $\mathcal{P}(\mathcal{M}) = \mathrm{add}\, T$,
  (B_n) $\mathcal{M}$ is an n-cluster tilting subcategory of $T^\perp$,
  (C_n) $\mathrm{Ext}^i_\Lambda(\mathcal{M}_P, \Lambda) = 0$ for any $0 < i < n$.
  We call Λ *absolutely n-complete* if $\mathcal{P}(\mathcal{M}) = \mathrm{add}\,\Lambda$.

We have the properties of n-complete algebras below, which we shall prove in Section 2. The statements (a)–(c) are similar to Propositions 1.2 and 1.3(b).

Proposition 1.12. *Let Λ be an n-complete algebra.*

(a) *We have mutually quasi-inverse equivalences $\tau_n : \mathcal{M}_P \to \mathcal{M}_I$ and $\tau_n^- : \mathcal{M}_I \to \mathcal{M}_P$.*
(b) *τ_n gives a bijection from isoclasses of indecomposable objects in $\mathcal{M}_P$ to those in $\mathcal{M}_I$.*
(c) *A bijection from isoclasses of indecomposable objects in $\mathcal{I}(\mathcal{M})$ to those in $\mathcal{P}(\mathcal{M})$ is given by $I \mapsto \tau_n^{\ell_I} I$, where ℓ_I is a maximal number ℓ satisfying $\tau_n^\ell I \neq 0$.*
(d) *Λ is τ_n-finite.*

In particular, if Λ is n-complete, then $\mathcal{M}$ has an additive generator M by (d) above. We call $\mathrm{End}_\Lambda(M)$ the *cone* of Λ. This is by definition an $(n-1)$-relative n-Auslander algebra of Λ.

For example, any finite-dimensional algebra Λ with $\mathrm{gl.dim}\,\Lambda < n$ is clearly n-complete since $\mathcal{M} = \mathcal{P}(\mathcal{M}) = \mathcal{I}(\mathcal{M}) = \mathrm{add}\, D\Lambda$ holds. It is interesting to know a characterization of n-complete algebras. For the case $n = 1$, we have a nice characterization (a) below. Also the following (b) gives a simple interpretation of absolute n-completeness.

Proposition 1.13.

(a) *A finite-dimensional algebra is 1-complete if and only if it is representation-finite and hereditary.*
(b) *A finite-dimensional algebra Λ is absolutely n-complete if and only if* $\mathrm{gl.dim}\,\Lambda \leqslant n$ *and Λ has an absolute n-cluster tilting object.*

Proof. (b) Since the ‘only if’ part is clear, we only have to show the ‘if’ part. By Theorem 1.6, we have that $\mathcal{M}$ is an n-cluster tilting subcategory of $\mathrm{mod}\,\Lambda$. Thus (C_n) holds. By Propositions 1.2(b) and 1.3(b), we have that Λ is τ_n-finite and $\mathcal{P}(\mathcal{M}) = \mathrm{add}\,\Lambda$ holds. Thus (A_n) and (B_n) also hold.

(a) Any 1-complete algebra is absolutely 1-complete by definition. Since absolute 1-cluster tilting objects are nothing but additive generators of $\mathrm{mod}\,\Lambda$, the assertion follows from (b). □

Now we state our main theorem in this paper. It gives an inductive construction of algebras with n-cluster tilting objects.

Theorem 1.14. *For any $n \geqslant 1$, the cone of an n-complete algebra is $(n+1)$-complete.*

For the case $n = 1$, we have the result below immediately from Proposition 1.13(a) and Theorem 1.14. This explains the reason why the Auslander algebras in Example 1.8 have 2-cluster tilting objects.

Corollary 1.15. *Let Λ be a representation-finite hereditary algebra and Γ an Auslander algebra of Λ. Then Γ has a* 1*-relative* 2*-cluster tilting object.*

Our Theorem 1.14 gives the following inductive construction of algebras with n-cluster tilting objects.

Corollary 1.16. *Let $\Lambda^{(1)}$ be a representation-finite hereditary algebra. Then there exists an algebra $\Lambda^{(n)}$ for any $n \geqslant 1$ such that $\Lambda^{(n)}$ is an n-complete algebra with the cone $\Lambda^{(n+1)}$.*

The quivers with relations of these algebras $\Lambda^{(n)}$ will be given in Theorem 6.12.

Now we apply Corollary 1.16 to a special case. We denote by $T_m(F)$ the $m \times m$ upper triangular matrix algebra over an algebra F. They form an important class of algebras by the following easy fact.

Proposition 1.17. *Let Λ be a ring-indecomposable finite-dimensional algebra. Then* $\mathrm{gl.dim}\,\Lambda \leqslant 1 \leqslant \mathrm{dom.dim}\,\Lambda$ *holds if and only if Λ is Morita equivalent to $T_m(F)$ for some division algebra F and $m \geqslant 1$.*

Proof. We provide a proof for the convenience of the reader.

Since $\mathrm{dom.dim}\,\Lambda \geqslant 1$, there exists an indecomposable projective-injective Λ-module P_1. Put $P_i := P_1 J_\Lambda^{i-1}$ for any $i > 0$. Then there exists $m \geqslant 1$ such that $P_{m+1} = 0$. Since $\mathrm{gl.dim}\,\Lambda \leqslant 1$, each P_i is a projective Λ-module. Since $\mathrm{soc}\,P_1$ is simple, each P_i is indecomposable, and so P_i has a unique maximal submodule P_{i+1}. Consequently P_1 has a unique composition series

$$P_1 \supset P_2 \supset \cdots \supset P_m \supset P_{m+1} = 0.$$

We often use the fact that any non-zero morphism between indecomposable projective modules is injective, which is a conclusion of $\mathrm{gl.dim}\,\Lambda \leqslant 1$. In particular $F := \mathrm{End}_\Lambda(P_1)$ is a division algebra. Put $P := \bigoplus_{i=1}^m P_i$.

(i) We shall show that $\mathrm{End}_\Lambda(P) \simeq T_m(F)$.

Since P_1 is injective, any non-zero morphism $P_i \to P_j$ extends to a morphism $P_1 \to P_1$, which is an isomorphism. Thus we have

$$\mathrm{Hom}_\Lambda(P_i, P_j) \simeq \begin{cases} 0 & (i < j), \\ F & (i \geqslant j). \end{cases}$$

This implies the assertion.

(ii) We shall show that P is a progenerator of Λ.

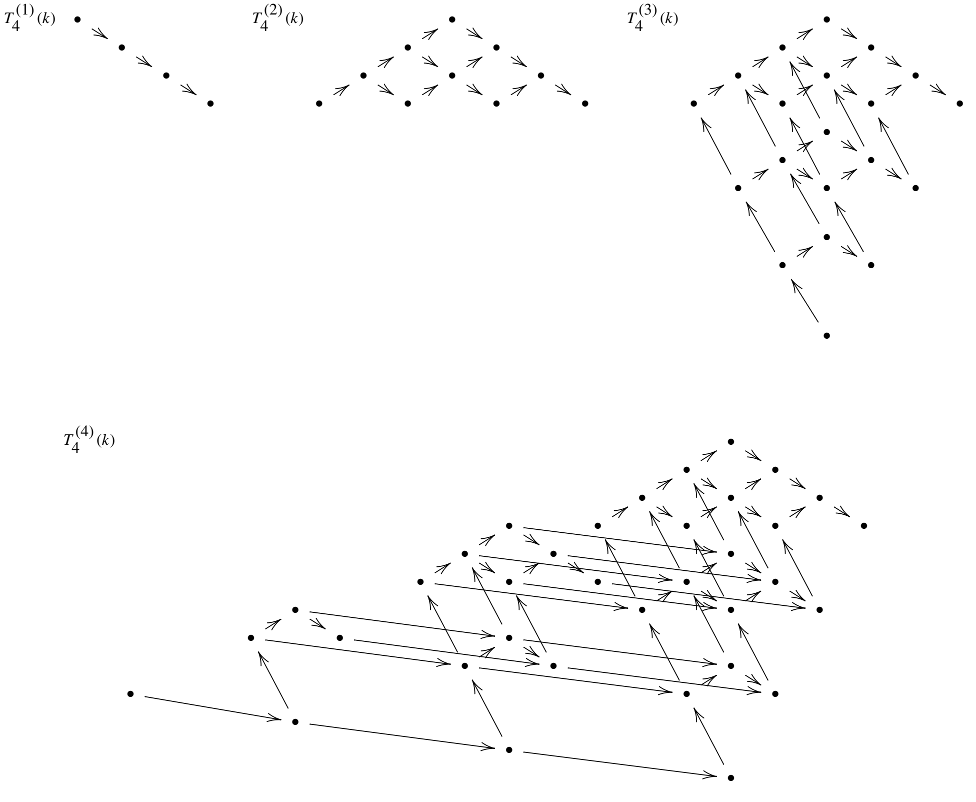
Since Λ is ring-indecomposable, we only have to show that, for any indecomposable projective Λ-module Q such that $\mathrm{Hom}_\Lambda(Q, P) \neq 0$ or $\mathrm{Hom}_\Lambda(P, Q) \neq 0$, we have $Q \in \mathrm{add}\,P$. If $\mathrm{Hom}_\Lambda(Q, P) \neq 0$, then $\mathrm{Hom}_\Lambda(Q, P_1) \neq 0$. Thus Q is a submodule of P_1, and we have $Q \simeq P_i$

for some i. If $\mathrm{Hom}_\Lambda(P, Q) \neq 0$, then $\mathrm{Hom}_\Lambda(Q, P_1) \neq 0$ since P_1 is an injective hull of each P_i. Thus $Q \in \mathrm{add}\, P$ by the previous observation. □

Applying Corollary 1.16 to $T_m(F)$, we have a family of algebras with n-cluster tilting objects. Moreover, they are absolute by the following result.

Theorem 1.18. *For any division algebra F and $m \geqslant 1$, there exists an algebra $T_m^{(n)}(F)$ for any $n \geqslant 1$ such that $T_m^{(1)}(F) = T_m(F)$ and $T_m^{(n)}(F)$ is an absolutely n-complete algebra with the cone $T_m^{(n+1)}(F)$.*

The quiver of $T_m^{(n)}(k)$ will be given in Theorem 6.12 as follows. It looks like an $(n+1)$-simplex. The relations are given by commutative relations for each small square, and zero relations for each small half square.



While there are a lot of algebras with relative n-cluster tilting objects, algebras with absolute n-cluster tilting objects are rather rare. In fact the following result shows a certain converse of Theorem 1.18.

Theorem 1.19. *Let Λ be a ring-indecomposable finite-dimensional algebra satisfying* $\mathrm{gl.dim}\, \Lambda \leqslant n \leqslant \mathrm{dom.dim}\, \Lambda$ *for some $n \geqslant 1$. Then Λ has an absolute n-cluster tilting subcategory if and only if Λ is Morita equivalent to $T_m^{(n)}(F)$ for some division algebra F and $m \geqslant 1$.*

For example an Auslander algebra Γ of a ring-indecomposable representation-finite algebra Λ has an absolute 2-cluster tilting object if and only if Λ is Morita equivalent to $T_m^{(2)}(F)$ for some division algebra F and $m \geqslant 1$.

A key step to prove Theorem 1.19 is the following more explicit version of Auslander correspondence (Theorem 1.10), which will be shown in Proposition 4.2.

Theorem 1.20. *Let $n \geqslant 1$. For a finite-dimensional algebra Γ, the following conditions are equivalent.*

(a) *There exists a finite-dimensional algebra Λ with* $\operatorname{gl.dim}\Lambda \leqslant n$ *and an absolute n-cluster tilting object M of Λ such that $\Gamma \simeq \operatorname{End}_\Lambda(M)$.*
(b) $\operatorname{gl.dim}\Gamma \leqslant n+1 \leqslant \operatorname{dom.dim}\Gamma$ *and* $\operatorname{Ext}^i_\Gamma(D\Gamma, \Gamma) = 0$ *for any* $0 < i \leqslant n$.

In forthcoming papers [35,48,49], absolutely n-complete algebras will be called *n-representation-finite algebras* and a lot of examples will be constructed. Also combinatorial aspects of $T_m^{(n)}(F)$ will be studied in [61].

1.2. n-Cluster tilting in derived categories

In Section 5, we construct n-cluster tilting subcategories in triangulated categories. Let Λ be a finite-dimensional algebra with $\operatorname{id}_\Lambda \Lambda = \operatorname{id} \Lambda_\Lambda < \infty$. We denote by

$$\mathcal{D} := \mathcal{K}^{\mathrm{b}}(\operatorname{pr}\Lambda)$$

the homotopy category of bounded complexes of finitely generated projective Λ-modules, and we identify it with $\mathcal{K}^{\mathrm{b}}(\operatorname{in}\Lambda)$ in the derived category of Λ. As in Definition 1.1, we call a functorially finite subcategory $\mathcal{C}$ of $\mathcal{D}$ *n-cluster tilting* if

$$\begin{aligned}\mathcal{C} &= \{X \in \mathcal{D} \mid \operatorname{Hom}_{\mathcal{D}}(X, \mathcal{C}[i]) = 0 \ (0 < i < n)\}\\ &= \{X \in \mathcal{D} \mid \operatorname{Hom}_{\mathcal{D}}(\mathcal{C}, X[i]) = 0 \ (0 < i < n)\}.\end{aligned}$$

If $\operatorname{mod}\Lambda$ has an absolute n-cluster tilting subcategory and $\operatorname{gl.dim}\Lambda \leqslant n$, then $\mathcal{D}$ also has an n-cluster tilting subcategory by the following result.

Theorem 1.21. *Let Λ be a finite-dimensional algebra with* $\operatorname{gl.dim}\Lambda \leqslant n$. *If $\mathcal{C}$ is an absolute n-cluster tilting subcategory of* $\operatorname{mod}\Lambda$, *then*

$$\mathcal{C}[n\mathbf{Z}] := \operatorname{add}\{X[\ell n] \mid X \in \mathcal{C},\ \ell \in \mathbf{Z}\} \tag{2}$$

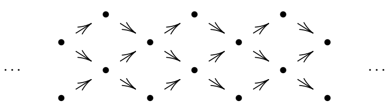
is an n-cluster tilting subcategory of $\mathcal{D}$.

Notice that we cannot drop the assumption that $\mathcal{C}$ is absolute.

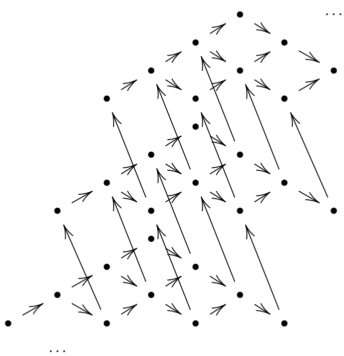
For the case $n = 1$, we have $\mathcal{C}[\mathbf{Z}] = \mathcal{D}$, which means the well-known fact that any object in $\mathcal{D}$ is a direct sum of stalk complexes if Λ is hereditary. It is natural to hope that $\mathcal{C}[n\mathbf{Z}]$ forms an $(n+2)$-angulated category under a certain proper definition [30].

In Theorem 6.12, we draw Auslander–Reiten quivers of $\mathcal{C}[n\mathbf{Z}]$ for $\Lambda = T_m^{(n)}(k)$ as follows. The relations are again given by commutative relations for each small square, and zero relations for each small half square.

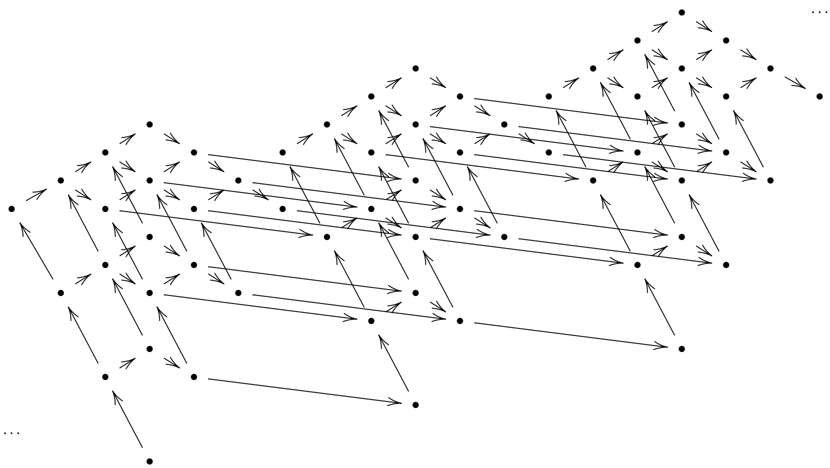
$T_4^{(1)}(k)$



$T_4^{(2)}(k)$



$T_4^{(3)}(k)$



We also give another construction of an n-cluster tilting subcategory of $\mathcal{D}$ by using derived analogue of n-Auslander–Reiten translations. Recall that there exists an autoequivalence

$$\mathbf{S} := D \circ \mathbf{R}\operatorname{Hom}_\Lambda(-, \Lambda) \simeq - \overset{\mathbf{L}}{\otimes}_\Lambda (D\Lambda) : \mathcal{D} \to \mathcal{D},$$

which gives the Serre functor of $\mathcal{D}$, i.e. there exists a functorial isomorphism

$$\operatorname{Hom}_{\mathcal{D}}(X, Y) \simeq D\operatorname{Hom}_{\mathcal{D}}(Y, \mathbf{S}X)$$

for any $X, Y \in \mathcal{D}$ [34,19]. We define an autoequivalence of $\mathcal{D}$ by

$$\mathbf{S}_n := \mathbf{S} \circ [-n] : \mathcal{D} \to \mathcal{D}.$$

Any n-cluster tilting subcategory $\mathcal{C}$ of $\mathcal{D}$ satisfies $\mathcal{C} = \mathbf{S}_n\mathcal{C} = \mathbf{S}_n^{-1}\mathcal{C}$ [51, Prop. 3.4]. Therefore $\mathbf{S}_n$ plays the role of n-Auslander–Reiten translations, and it is natural to introduce the following subcategory.

Definition 1.22. Define the $\mathbf{S}_n$*-closure* of an object $X \in \mathcal{D}$ by

$$\mathcal{U}_n(X) = \operatorname{add}\{\mathbf{S}_n^{\ell}(X) \mid \ell \in \mathbf{Z}\}.$$

The categories $\mathcal{M}_n(D\Lambda)$ and $\mathcal{U}_n(D\Lambda)$ are closely related since $\tau_n^{\ell} \simeq H^0(\mathbf{S}_n^{\ell}-)$ holds on $\operatorname{mod}\Lambda$ for any $\ell \geqslant 0$ if $\operatorname{gl.dim}\Lambda \leqslant n$ by Lemma 5.5. In particular, $\mathcal{C}[n\mathbf{Z}] = \mathcal{U}_n(\Lambda)$ holds in Theorem 1.21 if $\mathcal{C}$ has an additive generator.

We shall study the problem whether $\mathcal{U}_n(\Lambda)$ is an n-cluster tilting subcategory of $\mathcal{D}$. For a hereditary algebra Λ, one can easily show that $\mathcal{U}_1(\Lambda)$ is a 1-cluster tilting subcategory of $\mathcal{D}$ if and only if Λ is representation-finite. This observation suggests that it is related to the n-complete property. In fact we have the following another main result in Section 5.

Theorem 1.23. *Let Λ be a τ_n-finite algebra. Then $\mathcal{U}_n(\Lambda)$ is an n-cluster tilting subcategory of $\mathcal{D}$. Moreover, $\mathcal{U}_n(T)$ is an n-cluster tilting subcategory of $\mathcal{D}$ for any tilting complex $T \in \mathcal{D}$ satisfying* $\operatorname{gl.dim}\operatorname{End}_{\mathcal{D}}(T) \leqslant n$.

As a special case, if $\operatorname{gl.dim}\Lambda < n$ and T is a tilting Λ-module with $\operatorname{pd} T_\Lambda \leqslant 1$, then $\operatorname{gl.dim}\operatorname{End}_{\mathcal{D}}(T) \leqslant n$ holds and $\mathcal{U}_n(T)$ is an n-cluster tilting subcategory of $\mathcal{D}$. This generalizes the construction of 2-cluster tilting objects in cluster categories using tilting modules given in [24] as well as recent work of Amiot [1, Prop. 5.4.2] (see also [2, Th. 4.10]) and Barot, Fernandez, Platzeck, Pratti and Trepode [16].

As a special case of Theorem 1.23, $\mathcal{U}_n(\Lambda^{(n)})$ forms an n-cluster tilting subcategory of $\mathcal{D}$ for algebras $\Lambda^{(n)}$ given in Corollary 1.16. The quivers with relations of these categories will be given in Theorem 6.12.

At the end of this section, we note the following left–right symmetry of τ_n-finite algebras, which will be shown in Section 5.

Proposition 1.24. *A finite-dimensional algebra Λ is τ_n-finite if and only if Λ^{op} is τ_n-finite.*

We notice that easy examples show that n-completeness is not left–right symmetric.

2. Preliminaries

In this section, we give some preliminary results. Let us start with some properties of n-cluster tilting objects.

Definition 2.1. Let $\mathcal{C}$ be a Krull–Schmidt category.

(a) For an object $X \in \mathcal{C}$, a morphism $f_0 \in J_{\mathcal{C}}(X, C_1)$ is called *left almost split* if $C_1 \in \mathcal{C}$ and

$$\operatorname{Hom}_{\mathcal{C}}(C_1, -) \xrightarrow{f_0} J_{\mathcal{C}}(X, -) \to 0$$

is exact on $\mathcal{C}$. A left minimal and left almost split morphism is called a *source morphism*.

(b) We call a complex

$$X \xrightarrow{f_0} C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} \cdots \tag{3}$$

a *source sequence* of X if the following conditions are satisfied.

(i) $C_i \in \mathcal{C}$ and $f_i \in J_{\mathcal{C}}$ for any i,

(ii) we have the following exact sequence on $\mathcal{C}$,

$$\cdots \xrightarrow{f_2} \mathrm{Hom}_{\mathcal{C}}(C_2, -) \xrightarrow{f_1} \mathrm{Hom}_{\mathcal{C}}(C_1, -) \xrightarrow{f_0} J_{\mathcal{C}}(X, -) \to 0. \tag{4}$$

A *sink morphism* and a *sink sequence* are defined dually.

(c) We call a complex

$$0 \to X \to C_1 \to C_2 \to \cdots \to C_n \to Y \to 0$$

an *n-almost split sequence* if this is a source sequence of $X \in \mathcal{C}$ and a sink sequence of $Y \in \mathcal{C}$.

A source sequence (3) corresponds to a minimal projective resolution (4) of a functor $J_{\mathcal{C}}(X, -)$ on $\mathcal{C}$. Thus any indecomposable object $X \in \mathcal{C}$ has a unique source sequence up to isomorphisms of complexes if it exists.

Let us recall basic results for n-cluster tilting subcategories.

Theorem 2.2. *Let Λ be a finite-dimensional algebra and $n \geqslant 1$. Let T be a tilting Λ-module with $\mathrm{pd}\, T_\Lambda \leqslant n$.*

(a) *Let $\mathcal{C}$ be an n-cluster tilting subcategory of $T^\perp$.*

(i) *Any indecomposable object $X \in \mathcal{C} \setminus \mathrm{add}\, D\Lambda$ (respectively, $Y \in \mathcal{C} \setminus \mathrm{add}\, T$) has an n-almost split sequence*

$$0 \to X \to C_1 \to C_2 \to \cdots \to C_n \to Y \to 0$$

such that $Y \simeq \tau_n^- X$ and $X \simeq \tau_n Y$.

(ii) *Any indecomposable object $X \in \mathrm{add}\, D\Lambda$ has a source sequence of the form*

$$X \to C_1 \to \cdots \to C_n \to 0.$$

(iii) *Any indecomposable object $X \in \mathrm{add}\, T$ has a sink sequence of the form*

$$0 \to C_n \to \cdots \to C_1 \to X.$$

(b) *Let $\mathcal{C} = \mathrm{add}\, M$ be an n-rigid subcategory of $T^\perp$ satisfying $T \oplus D\Lambda \in \mathcal{C}$. Then the following conditions are equivalent.*

(i) *$\mathcal{C}$ is an n-cluster tilting subcategory of $T^\perp$.*

(ii) *$\Gamma := \mathrm{End}_\Lambda(M)$ satisfies $\mathrm{gl.dim}\, \Gamma \leqslant n + 1$.*

(iii) *Any indecomposable object $X \in \mathcal{C}$ has a source sequence of the form*

$$X \to C_1 \to \cdots \to C_{n+1} \to 0.$$

Proof. In [46], n-cluster tilting subcategories of the category

$$^{\perp}U = \{X \in \operatorname{mod} \Lambda \mid \operatorname{Ext}^i_\Lambda(X, U) = 0\ (0 < i)\},$$

for a cotilting Λ-module U is treated instead of $T^{\perp}$. We can apply results in [46] since DT is a cotilting Λ^{op}-module with $\operatorname{id}_\Lambda(DT) \leqslant n+1$, and $\mathcal{C}$ is an n-cluster tilting subcategory of $T^{\perp}$ if and only if $D\mathcal{C}$ is an n-cluster tilting subcategory of $^{\perp}(DT)$.

(a)(i) This is shown in [46, Th. 2.5.3].

(ii) It is easily shown (cf. [46, Prop. 2.4.1(2-ℓ)]) that there exists an exact sequence $0 \to X/\operatorname{soc} X \to C_1 \to \cdots \to C_n \to 0$ such that $C_i \in \mathcal{C}$ and

$$0 \to \operatorname{Hom}_\Lambda(C_n, -) \to \cdots \to \operatorname{Hom}_\Lambda(C_1, -) \to \operatorname{Hom}_\Lambda(X/\operatorname{soc} X, -) \to 0$$

is exact on $\mathcal{C}$. Connecting with the natural surjection $X \to X/\operatorname{soc} X$, we have a source sequence $X \to C_1 \to \cdots \to C_n \to 0$ of the desired form.

(iii) Let $\Gamma := \operatorname{End}_\Lambda(T)$. Then DT is a cotilting Γ-module, and Tilting theorem [34,60] gives an equivalence

$$\mathbf{F} = \operatorname{Hom}_\Lambda(T, -) : T^{\perp}_\Lambda \to {}^{\perp}(DT)_\Gamma$$

which preserves Ext-groups. Thus we have an n-cluster tilting subcategory $\mathbf{F}\mathcal{C}$ of $^{\perp}(DT)_\Gamma$. For any indecomposable object $X \in \operatorname{add} T_\Lambda$, there exists a sink sequence $0 \to C_n \to \cdots \to C_0 \to \mathbf{F}X$ of the indecomposable projective Γ-module $\mathbf{F}X$ with $C_i \in \mathbf{F}\mathcal{C}$ by the dual of (ii). Applying the quasi-inverse of $\mathbf{F}$, we have the desired sink sequence of X.

(b)(i) $\Leftrightarrow$ (ii) Apply [46, Th. 5.1(3)] for $d := 0$ and $m := \operatorname{id}_\Lambda(DT) \leqslant n$.

(ii) $\Leftrightarrow$ (iii) $C_{n+2} = 0$ holds in the source sequence (3) if and only if the simple Γ^{op}-module $\operatorname{top} \operatorname{Hom}_\Lambda(X, M)$ has projective dimension at most $n+1$. Since $\operatorname{gl.dim} \Gamma \leqslant n+1$ if and only if any simple Γ^{op}-module has projective dimension at most $n+1$, we have the assertion. □

Put

$$\mathcal{G}_n := \{X \in \operatorname{mod} \Lambda \mid \operatorname{Ext}^i_\Lambda(X, \Lambda) = 0\ (0 \leqslant i < n)\},$$
$$\mathcal{H}_n := \{X \in \operatorname{mod} \Lambda \mid \operatorname{Ext}^i_\Lambda(D\Lambda, X) = 0\ (0 \leqslant i < n)\}.$$

Lemma 2.3. *Let Λ be a finite-dimensional algebra with* $\operatorname{gl.dim} \Lambda \leqslant n$ *and* $X \in \operatorname{mod} \Lambda$.

(a) *We have mutually quasi-inverse equivalences*

$$\tau_n = D \operatorname{Ext}^n_\Lambda(-, \Lambda) : \mathcal{G}_n \to \mathcal{H}_n \quad \text{and} \quad \tau_n^- = \operatorname{Ext}^n_{\Lambda^{\mathrm{op}}}(D-, \Lambda) : \mathcal{H}_n \to \mathcal{G}_n.$$

(b) *If X has no non-zero projective summands and* $\operatorname{Ext}^i_\Lambda(X, \Lambda) = 0$ *for any $0 < i < n$, then $X \in \mathcal{G}_n$.*

(c) *If X has no non-zero injective summands and* $\mathrm{Ext}^i_\Lambda(D\Lambda, X) = 0$ *for any* $0 < i < n$, *then* $X \in \mathcal{H}_n$.

Proof. Although the assertions are elementary, we provide a proof for the convenience of the reader.

(a) For $X \in \mathcal{G}_n$, take a projective resolution

$$0 \to P_n \to \cdots \to P_0 \to X \to 0. \tag{5}$$

Applying ν, we have an exact sequence

$$0 \to \tau_n X \to \nu P_n \to \cdots \to \nu P_0 \to 0 \tag{6}$$

where we use $\mathrm{Ext}^i_\Lambda(X, \Lambda) = 0$ for any $0 \leqslant i < n$. The sequence (6) gives an injective resolution of $\tau_n X$. Applying ν^- to (6), we have a complex

$$0 \to P_n \to \cdots \to P_0 \to 0$$

whose homology at P_i is $\mathrm{Ext}^{n-i}_\Lambda(D\Lambda, \tau_n X)$. Comparing with (5), we have $\tau_n X \in \mathcal{H}_n$ and $\tau_n^- \tau_n X \simeq X$.

(b) Take a projective resolution

$$0 \to P_n \to \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \to 0.$$

Applying $\mathrm{Hom}_\Lambda(-, \Lambda)$, we have an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_\Lambda(X, \Lambda) &\xrightarrow{f_0} \mathrm{Hom}_\Lambda(P_0, \Lambda) \xrightarrow{f_1} \mathrm{Hom}_\Lambda(P_1, \Lambda) \to \cdots \\ &\to \mathrm{Hom}_\Lambda(P_n, \Lambda) \to \mathrm{Ext}^n_\Lambda(X, \Lambda) \to 0. \end{aligned}$$

Since X has no non-zero projective summand, f_1 is left minimal. Since $\mathrm{Hom}_\Lambda(-, \Lambda) : \mathrm{add}\, \Lambda_\Lambda \to \mathrm{add}\, {}_\Lambda\Lambda$ is an equivalence, $f_1 : \mathrm{Hom}_\Lambda(P_0, \Lambda) \to \mathrm{Hom}_\Lambda(P_1, \Lambda)$ is right minimal. Since $f_0 : \mathrm{Hom}_\Lambda(X, \Lambda) \to \mathrm{Hom}_\Lambda(P_0, \Lambda)$ is a split monomorphism by $\mathrm{gl.dim}\, \Lambda \leqslant n$, we have $\mathrm{Hom}_\Lambda(X, \Lambda) = 0$ and $X \in \mathcal{G}_n$.

(c) This is dual of (b). □

Lemma 2.4. *Let Λ be a finite-dimensional algebra with* $\mathrm{gl.dim}\, \Lambda \leqslant n$ *and $\mathcal{M}$ the τ_n-closure of $D\Lambda$ satisfying the condition* (C_n) *in Definition* 1.11. *Then we have the following.*

(a) *We have full functors*

$$\tau_n = D\,\mathrm{Ext}^n_\Lambda(-, \Lambda) : \mathcal{M} \to \mathcal{M} \quad \textit{and} \quad \tau_n^- = \mathrm{Ext}^n_{\Lambda^{\mathrm{op}}}(D-, \Lambda) : \mathcal{M} \to \mathcal{M}$$

which give mutually quasi-inverse equivalences

$$\tau_n : \mathcal{M}_P \to \mathcal{M}_I \quad \textit{and} \quad \tau_n^- : \mathcal{M}_I \to \mathcal{M}_P.$$

(b) *τ_n gives a bijection from isoclasses of indecomposable objects in $\mathcal{M}_P$ to those in $\mathcal{M}_I$.*

(c) $\mathcal{M}_P \subset \mathcal{G}_n$ *and* $\mathcal{M}_I \subset \mathcal{H}_n$.
(d) $\mathrm{Hom}_\Lambda(\mathcal{M}_P, \mathcal{P}(\mathcal{M})) = 0$.
(e) $\mathrm{Hom}_\Lambda(\tau_n^i(D\Lambda), \tau_n^j(D\Lambda)) = 0$ *for any* $i < j$.

Proof. (a)(c) We have $\mathcal{M}_P \subset \mathcal{G}_n$ by the condition (C_n) and Lemma 2.3(b). By Lemma 2.3(a), we have a fully faithful functor $\tau_n : \mathcal{M}_P \to \mathcal{H}_n$. By our construction of $\mathcal{M}$, we have $\tau_n(\mathcal{M}_P) = \mathcal{M}_I$. Thus we have an equivalence $\tau_n : \mathcal{M}_P \to \mathcal{M}_I$ with quasi-inverse τ_n^-. Since $\tau_n(\mathcal{P}(\mathcal{M})) = 0$ and $\tau_n^-(\mathcal{I}(\mathcal{M})) = 0$, we have full functors $\tau_n : \mathcal{M} \to \mathcal{M}$ and $\tau_n^- : \mathcal{M} \to \mathcal{M}$.

(b) Immediate from (a).

(d) For any $X \in \mathcal{P}(\mathcal{M})$, take a projective resolution

$$0 \to P_{n-1} \to \cdots \to P_0 \to X \to 0.$$

Applying $\mathrm{Hom}_\Lambda(\mathcal{M}_P, -)$ and using $\mathrm{Ext}^i_\Lambda(\mathcal{M}_P, \Lambda) = 0$ for any $0 \leqslant i < n$, we have $\mathrm{Hom}_\Lambda(\mathcal{M}_P, X) = 0$.

(e) We have $\mathcal{M}_I \subset \mathcal{H}_n$ by (c), so $\mathrm{Hom}_\Lambda(D\Lambda, \tau_n^{j-i}(D\Lambda)) = 0$. Since $\tau_n : \mathcal{M} \to \mathcal{M}_I$ is a full functor by (a), we have $\mathrm{Hom}_\Lambda(\tau_n^i(D\Lambda), \tau_n^j(D\Lambda)) = 0$. □

Now we shall prove Proposition 1.12.

By Lemma 2.4, we have (a)–(c) immediately. The assertion (d) follows immediately from (c). □

We give the following general property of τ_n-closures.

Proposition 2.5. *Let* Λ *be a finite-dimensional algebra and* $\mathcal{M}$ *the* τ_n*-closure of* $D\Lambda$.

(a) *If* $\mathrm{Ext}^i_\Lambda(\mathcal{M}_P, \Lambda) = 0$ *holds for any* $1 < i < n$, *then* $\mathcal{M}$ *is* n*-rigid.*
(b) *If* $n = 2$, *then* $\mathcal{M}$ *is* 2*-rigid.*

For the proof, we need the following general observation.

Lemma 2.6. *Let* Λ *be a finite-dimensional algebra,* $X, Y \in \mathrm{mod}\,\Lambda$ *and* $0 < i < n$. *If* $\mathrm{Ext}^j_\Lambda(X, \Lambda) = 0$ *for any* $n - i < j < n$, *then we have a surjection* $\mathrm{Ext}^{n-i}_\Lambda(X, Y) \to D\,\mathrm{Ext}^i_\Lambda(Y, \tau_n X)$.

Proof. By Auslander–Reiten duality, we have

$$D\,\mathrm{Ext}^i_\Lambda(Y, \tau_n X) \simeq D\,\mathrm{Ext}^1_\Lambda\big(\Omega^{i-1}Y, \tau\Omega^{n-1}X\big) \simeq \underline{\mathrm{Hom}}_\Lambda\big(\Omega^{n-1}X, \Omega^{i-1}Y\big).$$

Since we assumed $\mathrm{Ext}^j_\Lambda(X, \Lambda) = 0$ for $n - i < j < n$, it is easily checked (e.g. [6, 7.4]) that

$$\underline{\mathrm{Hom}}_\Lambda\big(\Omega^{n-1}X, \Omega^{i-1}Y\big) \simeq \underline{\mathrm{Hom}}_\Lambda\big(\Omega^{n-i}X, Y\big).$$

Since in general we have a surjection

$$\mathrm{Ext}^{n-i}_\Lambda(X, Y) \to \underline{\mathrm{Hom}}_\Lambda\big(\Omega^{n-i}X, Y\big),$$

we have the desired surjection. □

Now we shall prove Proposition 2.5. We only have to show (a).

(i) Let $X, Y \in \mathcal{M}$. We shall show that, if $\mathrm{Ext}^i_\Lambda(X, Y) = 0$ for any $0 < i < n$, then $\mathrm{Ext}^i_\Lambda(Y, \tau_n X) = 0$ and $\mathrm{Ext}^i_\Lambda(\tau_n X, \tau_n Y) = 0$ for any $0 < i < n$.

We can assume that $X \in \mathcal{M}_P$. Then we have $\mathrm{Ext}^j_\Lambda(X, \Lambda) = 0$ for any $1 < j < n$ by our assumption. Thus we have $\mathrm{Ext}^i_\Lambda(Y, \tau_n X) = 0$ for any $0 < i < n$ by Lemma 2.6. Replacing (X, Y) by $(Y, \tau_n X)$, we have $\mathrm{Ext}^i_\Lambda(\tau_n X, \tau_n Y) = 0$ for any $0 < i < n$.

(ii) Let $0 \leqslant j, k$. Since $\mathrm{Ext}^i_\Lambda(\tau_n^j(D\Lambda), D\Lambda) = 0$ for any $0 < i < n$, we have $\mathrm{Ext}^i_\Lambda(\tau_n^{j+k}(D\Lambda), \tau_n^k(D\Lambda)) = 0$ and $\mathrm{Ext}^i_\Lambda(\tau_n^k(D\Lambda), \tau_n^{j+k+1}(D\Lambda)) = 0$ for any $0 < i < n$ by (i). Thus we have $\mathrm{Ext}^i_\Lambda(\mathcal{M}, \mathcal{M}) = 0$ for any $0 < i < n$. □

Now we shall prove Proposition 1.7. It follows from Proposition 2.5(b) and the following result.

Lemma 2.7. *Assume* gl.dim $\Lambda \leqslant n$. *Then there exists an n-cluster tilting object in* mod Λ *if and only if $\mathcal{M}$ is n-rigid and $\Lambda \in \mathcal{M}$.*

Proof. 'Only if' part is clear from Theorem 1.6. We shall show 'if' part.

(i) Since $\Lambda \in \mathcal{M}$, we have that $\mathcal{M}$ satisfies the condition (C_n) in Definition 1.11. Thus $\tau_n : \mathcal{M}_P \to \mathcal{M}_I$ is an equivalence by Lemma 2.4(a). A bijection from isoclasses of indecomposable objects in $\mathcal{I}(\mathcal{M})$ to those in $\mathcal{P}(\mathcal{M})$ is given by $I \mapsto \tau_n^{\ell_I} I$, where ℓ_I is a maximal number ℓ satisfying $\tau_n^\ell I \neq 0$. In particular, $\tau_n^\ell(D\Lambda) = 0$ holds for sufficiently large ℓ, and $\mathcal{M}$ has an additive generator M. Moreover, we have that $\tau_n^\ell X = 0$ holds for any $X \in \mathrm{mod}\,\Lambda$ by a similar argument as in the proof of Proposition 1.3(c).

(ii) We shall show that $\mathrm{Ext}^i_\Lambda(X, \mathcal{M}) = 0$ for any $0 < i < n$ implies $X \in \mathcal{M}$.

We know $\mathrm{Ext}^i_\Lambda(\tau_n X, \tau_n \mathcal{M}) = 0$ for any $0 < i < n$ by Lemma 2.6. By our construction of $\mathcal{M}$, we have $\mathrm{Ext}^i_\Lambda(\tau_n X, \mathcal{M}) = 0$ for any $0 < i < n$. Consequently we have $\mathrm{Ext}^i_\Lambda(\tau_n^\ell X, \mathcal{M}) = 0$ for any $\ell \geqslant 0$ and $0 < i < n$. By (i), we can take a maximal number $\ell \geqslant 0$ satisfying $Y := \tau_n^\ell X \neq 0$. Since $\tau_n Y = 0$, we have $\mathrm{pd}\, Y_\Lambda < n$. Since we have $\mathrm{Ext}^i_\Lambda(Y, \Lambda) = 0$ for any $0 < i < n$ by $\Lambda \in \mathcal{M}$, we have that Y is a projective Λ-module. Thus we have $Y \in \mathcal{M}$. By Lemma 2.4(a) again, we have $X \simeq \tau_n^{-\ell} Y \in \mathcal{M}$.

(iii) By a dual argument, we have that $\mathrm{Ext}^i_\Lambda(\mathcal{M}, X) = 0$ for any $0 < i < n$ implies $X \in \mathcal{M}$. Consequently M given in (i) is an n-cluster tilting object of mod Λ. □

3. Proof of Theorems 1.14 and 1.18

Throughout this section we assume that Λ is n-complete with the τ_n-closure $\mathcal{M} = \mathrm{add}\, M$ of $D\Lambda$ and $\mathcal{P}(\mathcal{M}) = \mathrm{add}\, T$ unless stated otherwise. We put

$$\Gamma := \mathrm{End}_\Lambda(M).$$

We have the following result immediately from Theorem 1.10(a) $\Rightarrow$ (b).

Lemma 3.1. gl.dim $\Gamma \leqslant n + 1$.

This allows us to put

$$\tau_{n+1} := D\,\mathrm{Ext}^{n+1}_{\Gamma}(-,\Gamma) : \mathrm{mod}\,\Gamma \to \mathrm{mod}\,\Gamma,$$
$$\tau^{-}_{n+1} := \mathrm{Ext}^{n+1}_{\Gamma^{\mathrm{op}}}(D-,\Gamma) : \mathrm{mod}\,\Gamma \to \mathrm{mod}\,\Gamma.$$

Clearly τ_{n+1} (respectively, τ^{-}_{n+1}) preserves monomorphisms (respectively, epimorphisms) in $\mathrm{mod}\,\Gamma$. Moreover $X \in \mathrm{mod}\,\Gamma$ satisfies $\tau_{n+1}X = 0$ (respectively, $\tau^{-}_{n+1}X = 0$) if and only if $\mathrm{pd}\,X_\Gamma < n+1$ (respectively, $\mathrm{id}\,X_\Gamma < n+1$). We denote by

$$\mathcal{N} := \mathcal{M}_{n+1}(D\Gamma) = \mathrm{add}\bigl\{\tau^{i}_{n+1}(D\Gamma) \bigm| i \geqslant 0\bigr\}$$

the τ_{n+1}-closure of $D\Gamma$.

3.1. $(n+1)$-Rigidity of $\mathcal{N}$

The aim of this subsection is to show that $\mathcal{N}$ is $(n+1)$-rigid. As usual, we put

$$\mathcal{I}(\mathcal{N}) := \mathrm{add}\,D\Gamma,$$
$$\mathcal{P}(\mathcal{N}) := \{X \in \mathcal{N} \mid \mathrm{pd}\,X_\Gamma < n+1\} = \{X \in \mathcal{N} \mid \tau_{n+1}X = 0\},$$
$$\mathcal{N}_I := \bigl\{X \in \mathcal{N} \bigm| X \text{ has no non-zero summands in } \mathcal{I}(\mathcal{N})\bigr\},$$
$$\mathcal{N}_P := \bigl\{X \in \mathcal{N} \bigm| X \text{ has no non-zero summands in } \mathcal{P}(\mathcal{N})\bigr\}.$$

For $X \in \mathcal{M}$, put

$$\ell_X := \sup\bigl\{i \geqslant 0 \bigm| \tau^{i}_{n}X \neq 0\bigr\}.$$

Let us start with the following easy observation.

Proposition 3.2.

(a) $\ell_X < \infty$ *for any* $X \in \mathcal{M}$.
(b) *A bijection from isoclasses of indecomposable objects in* $\mathcal{M}$ *to pairs* (I,i) *of isoclasses of indecomposable objects* $I \in \mathcal{I}(\mathcal{M})$ *and* $0 \leqslant i \leqslant \ell_I$ *is given by* $(I,i) \mapsto \tau^{i}_{n}I$.

Proof. (b) By Lemma 2.4(b) and the definition of $\mathcal{M}$, any indecomposable object in $\mathcal{M}$ can be written uniquely as $\tau^{i}_{n}I$ for indecomposable $I \in \mathcal{I}(\mathcal{M})$ and $0 \leqslant i \leqslant \ell_I$.

(a) Since $\mathcal{M}$ contains only finitely many isoclasses of indecomposable objects, we have $\ell_I < \infty$ for any indecomposable $I \in \mathcal{I}(\mathcal{M})$. Since $\ell_{\tau^{i}_{n}I} = \ell_I - i$, we have the assertion. □

We put

$$\mathcal{G}_{n+1} := \bigl\{X \in \mathrm{mod}\,\Gamma \bigm| \mathrm{Ext}^{i}_{\Gamma}(X,\Gamma) = 0\ (0 \leqslant i < n+1)\bigr\},$$
$$\mathcal{H}_{n+1} := \bigl\{X \in \mathrm{mod}\,\Gamma \bigm| \mathrm{Ext}^{i}_{\Gamma}(D\Gamma,X) = 0\ (0 \leqslant i < n+1)\bigr\}.$$

We need the following observation.

Lemma 3.3.

(a) *We have functors*

$$\tau_{n+1} = D\,\mathrm{Ext}^{n+1}_{\Gamma}(-,\Gamma) : \mathrm{mod}\,\Gamma \to \mathcal{H}_{n+1} \quad \textit{and}$$
$$\tau^{-}_{n+1} = \mathrm{Ext}^{n+1}_{\Gamma^{\mathrm{op}}}(D-,\Gamma) : \mathrm{mod}\,\Gamma \to \mathcal{G}_{n+1}$$

which give mutually quasi-inverse equivalences

$$\tau_{n+1} : \mathcal{G}_{n+1} \to \mathcal{H}_{n+1} \quad \textit{and} \quad \tau^{-}_{n+1} : \mathcal{H}_{n+1} \to \mathcal{G}_{n+1}.$$

(b) $\mathcal{G}_{n+1}$ *and* $\mathcal{H}_{n+1}$ *are Serre subcategories of* $\mathrm{mod}\,\Gamma$.

Proof. We use the properties Theorem 1.10(b)(ii) and (iii).

(a) We have the desired functors since $\tau_{n+1}X \in \mathcal{H}_{n+1}$ and $\tau^{-}_{n+1}X \in \mathcal{G}_{n+1}$ for any $X \in \mathrm{mod}\,\Gamma$ by [43, 6.1(1)]. They give equivalences between $\mathcal{G}_{n+1}$ and $\mathcal{H}_{n+1}$ by Lemma 2.3.

(b) This follows from [42, Prop. 2.4]. □

Define functors

$$\mathbf{F}^i := \mathrm{Ext}^i_{\Lambda}(M,-) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma,$$
$$\mathbf{G}^i := D\,\mathrm{Ext}^i_{\Lambda}(-,M) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma.$$

Put $\mathbf{F} := \mathbf{F}^0$ and $\mathbf{G} := \mathbf{G}^0$ for simplicity. They induce equivalences

$$\mathbf{F} : \mathcal{M} \to \mathrm{add}\,\Gamma \quad \text{and} \quad \mathbf{G} : \mathcal{M} \to \mathrm{add}\,D\Gamma \subset \mathcal{N}.$$

A crucial role is played by a monomorphism

$$\alpha : \tau_{n+1}\mathbf{G} \to \mathbf{G}\tau_n$$

of functors on $\mathcal{M}$ given by the following proposition.

Proposition 3.4.

(a) *There exists an isomorphism* $\mathbf{F}\nu_{\Lambda} \simeq \mathbf{G}$ *of functors on* $\mathrm{add}\,\Lambda$.
(b) *There exists an isomorphism* $\nu_{\Gamma}\mathbf{F} \simeq \mathbf{G}$ *of functors on* $\mathcal{M}$.
(c) *There exist isomorphisms* $\mathbf{F}\tau_n \simeq \mathbf{G}^n$ *and* $\mathbf{G}\tau^{-}_n \simeq \mathbf{F}^n$ *of functors on* $\mathcal{M}$.
(d) *There exists a monomorphism* $\alpha : \tau_{n+1}\mathbf{G} \to \mathbf{G}\tau_n$ *of functors on* $\mathcal{M}$.
(e) $\alpha_X : \tau_{n+1}\mathbf{G}X \to \mathbf{G}\tau_n X$ *is a minimal right* $\mathcal{H}_{n+1}$*-approximation for any* $X \in \mathcal{M}$.
(f) *We have a functorial monomorphism*

$$\alpha^{\ell} : \tau^{\ell}_{n+1}\mathbf{G} \xrightarrow{\tau^{\ell-1}_{n+1}\alpha} \tau^{\ell-1}_{n+1}\mathbf{G}\tau_n \xrightarrow{\tau^{\ell-2}_{n+1}\alpha_{\tau_n}} \tau^{\ell-2}_{n+1}\mathbf{G}\tau^2_n \to \cdots \to \tau_{n+1}\mathbf{G}\tau^{\ell-1}_n \xrightarrow{\alpha_{\tau^{\ell-1}_n}} \mathbf{G}\tau^{\ell}_n.$$

Proof. (a)(b) Immediate.

(c) We only show $\mathbf{F}\tau_n \simeq \mathbf{G}^n$. Since gl.dim $\Lambda \leqslant n$, any $X \in \mathcal{M}$ has a projective resolution

$$0 \to P_n \to \cdots \to P_0 \to X \to 0. \tag{7}$$

Applying $\mathbf{G}$, we have an exact sequence

$$0 \to \mathbf{G}^n X \to \mathbf{G}P_n \to \cdots \to \mathbf{G}P_0 \to \mathbf{G}X \to 0, \tag{8}$$

where we use n-rigidity of $\mathcal{M}$. On the other hand, applying ν_Λ to (7), we have an exact sequence

$$0 \to \tau_n X \to \nu_\Lambda P_n \to \nu_\Lambda P_{n-1}.$$

Applying $\mathbf{F}$, we have an exact sequence

$$0 \to \mathbf{F}\tau_n X \to \mathbf{F}\nu_\Lambda P_n \to \mathbf{F}\nu_\Lambda P_{n-1}. \tag{9}$$

Comparing (8) and (9) by using (a), we have a commutative diagram

$$\begin{array}{ccccccl} 0 \to & \mathbf{G}^n X & \to & \mathbf{G}P_n & \to & \mathbf{G}P_{n-1} & \to \cdots \to \mathbf{G}P_0 \to \mathbf{G}X \to 0, \\ & & & \wr\| & & \wr\| & \\ 0 \to & \mathbf{F}\tau_n X & \to & \mathbf{F}\nu_\Lambda P_n & \to & \mathbf{F}\nu_\Lambda P_{n-1} & \end{array}$$

of exact sequences. Thus we have $\mathbf{F}\tau_n X \simeq \mathbf{G}^n X$.

(d) Since $\mathbf{G}P_i \simeq \mathbf{F}\nu_\Lambda P_i$ and gl.dim $\Lambda \leqslant n$, the sequence (8) gives a projective resolution of a Γ-module $\mathbf{G}X$. Applying ν_Γ, we have an exact sequence

$$0 \to \tau_{n+1}\mathbf{G}X \to \nu_\Gamma \mathbf{G}^n X \to \nu_\Gamma \mathbf{G}P_n. \tag{10}$$

Since $\nu_\Gamma \mathbf{G}^n \simeq \nu_\Gamma \mathbf{F}\tau_n \simeq \mathbf{G}\tau_n$ by (c) and (b) respectively, we have the assertion.

(e) We have $\tau_{n+1}\mathbf{G}X \in \mathcal{H}_{n+1}$ by Lemma 3.3(a). By (10), we have that $(\mathbf{G}\tau_n X)/(\tau_{n+1}\mathbf{G}X)$ is a submodule of $\nu_\Gamma \mathbf{G}P_n$. By Lemma 3.3(b), we only have to show that $\operatorname{soc} \nu_\Gamma \mathbf{G}P_n$ does not belong to $\mathcal{H}_{n+1}$. Since

$$\operatorname{soc} \nu_\Gamma \mathbf{G}P_n = \operatorname{soc} \nu_\Gamma \mathbf{F}\nu_\Lambda P_n = \operatorname{soc} \mathbf{G}\nu_\Lambda P_n$$

has injective dimension at most n by Theorem 2.2(a)(ii), we have that $\operatorname{soc} \nu_\Gamma \mathbf{G}P_n$ does not belong to $\mathcal{H}_{n+1}$.

(f) Since the functor τ_{n+1} preserves monomorphisms, we have the assertion by (d). □

We need the following general observation, which is valid for arbitrary Λ with gl.dim $\Lambda \leqslant n$.

Lemma 3.5. *Let $n \geqslant 1$ and Λ a finite-dimensional algebra with* gl.dim $\Lambda \leqslant n$. *Let $X \in$* mod Λ *and*

$$0 \to X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \to 0$$

an exact sequence in mod Λ *with $X_i \in$* add X.

(a) *If* $W \in \operatorname{mod} \Lambda$ *satisfies* $\operatorname{Ext}^i_\Lambda(W, X) = 0$ *for any* $0 < i < n$*, then we have an exact sequence*

$$0 \to \operatorname{Hom}_\Lambda(W, X_0) \xrightarrow{f_0} \cdots \xrightarrow{f_n} \operatorname{Hom}_\Lambda(W, X_{n+1})$$
$$\to \operatorname{Ext}^n_\Lambda(W, X_0) \xrightarrow{f_0} \cdots \xrightarrow{f_n} \operatorname{Ext}^n_\Lambda(W, X_{n+1}) \to 0.$$

(b) *If* $Y \in \operatorname{mod} \Lambda$ *satisfies* $\operatorname{Ext}^i_\Lambda(X, Y) = 0$ *for any* $0 < i < n$*, then we have an exact sequence*

$$0 \to \operatorname{Hom}_\Lambda(X_{n+1}, Y) \xrightarrow{f_n} \cdots \xrightarrow{f_0} \operatorname{Hom}_\Lambda(X_0, Y)$$
$$\to \operatorname{Ext}^n_\Lambda(X_{n+1}, Y) \xrightarrow{f_n} \cdots \xrightarrow{f_0} \operatorname{Ext}^n_\Lambda(X_0, Y) \to 0.$$

Proof. We only prove (a). We can assume $n > 1$. Put $L_i := \operatorname{Im} f_{i-1}$. Then we have an exact sequence

$$0 \to L_i \to X_i \to L_{i+1} \to 0 \quad (1 \leqslant i \leqslant n).$$

Applying $\operatorname{Hom}_\Lambda(W, -)$, we have exact sequences

$$0 \to \operatorname{Hom}_\Lambda(W, L_i) \to \operatorname{Hom}_\Lambda(W, X_i) \to \operatorname{Hom}_\Lambda(W, L_{i+1}) \to \operatorname{Ext}^1_\Lambda(W, L_i) \to 0,$$
$$0 \to \operatorname{Ext}^{n-1}_\Lambda(W, L_{i+1}) \to \operatorname{Ext}^n_\Lambda(W, L_i) \to \operatorname{Ext}^n_\Lambda(W, X_i) \to \operatorname{Ext}^n_\Lambda(W, L_{i+1}) \to 0,$$

and an isomorphism

$$\operatorname{Ext}^j_\Lambda(W, L_{i+1}) \simeq \operatorname{Ext}^{j+1}_\Lambda(W, L_i) \quad (0 < j < n-1).$$

Since $L_1 = X_0$ and $L_{n+1} = X_{n+1}$ belong to $\operatorname{add} X$, we have

$$\operatorname{Ext}^1_\Lambda(W, L_i) \simeq \operatorname{Ext}^2_\Lambda(W, L_{i-1}) \simeq \cdots \simeq \begin{cases} \operatorname{Ext}^i_\Lambda(W, L_1) = 0 & (1 \leqslant i < n), \\ \operatorname{Ext}^{n-1}_\Lambda(W, L_2) & (i = n), \end{cases}$$
$$\operatorname{Ext}^{n-1}_\Lambda(W, L_{i+1}) \simeq \operatorname{Ext}^{n-2}_\Lambda(W, L_{i+2}) \simeq \cdots \simeq \operatorname{Ext}^{i-1}_\Lambda(W, L_{n+1}) = 0 \quad (1 < i \leqslant n).$$

Thus we have exact sequences

$$0 \to \operatorname{Hom}_\Lambda(W, L_i) \to \operatorname{Hom}_\Lambda(W, X_i) \to \operatorname{Hom}_\Lambda(W, L_{i+1}) \to 0 \quad (1 \leqslant i < n),$$
$$0 \to \operatorname{Ext}^n_\Lambda(W, L_i) \to \operatorname{Ext}^n_\Lambda(W, X_i) \to \operatorname{Ext}^n_\Lambda(W, L_{i+1}) \to 0 \quad (1 < i \leqslant n),$$
$$0 \to \operatorname{Hom}_\Lambda(W, L_n) \to \operatorname{Hom}_\Lambda(W, X_n) \to \operatorname{Hom}_\Lambda(W, L_{n+1}) \to \operatorname{Ext}^n_\Lambda(W, L_1)$$
$$\to \operatorname{Ext}^n_\Lambda(W, X_1) \to \operatorname{Ext}^n_\Lambda(W, L_2) \to 0.$$

Connecting them, we have the desired exact sequence. □

The following result is crucial to study properties of Γ.

Lemma 3.6. *Let*

$$0 \to M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \to 0$$

be an exact sequence in $\operatorname{mod} \Lambda$ *with* $M_i \in \mathcal{M}$. *Put* $X := \operatorname{Cok} \mathbf{F} f_n$ *and* $Y := \operatorname{Ker} \mathbf{G} f_0$.

(a) *We have exact sequences*

$$0 \to \mathbf{F}M_0 \xrightarrow{\mathbf{F}f_0} \mathbf{F}M_1 \xrightarrow{\mathbf{F}f_1} \cdots \xrightarrow{\mathbf{F}f_{n-1}} \mathbf{F}M_n \xrightarrow{\mathbf{F}f_n} \mathbf{F}M_{n+1} \to X \to 0,$$
$$0 \to Y \to \mathbf{G}M_0 \xrightarrow{\mathbf{G}f_0} \mathbf{G}M_1 \xrightarrow{\mathbf{G}f_1} \cdots \xrightarrow{\mathbf{G}f_{n-1}} \mathbf{G}M_n \xrightarrow{\mathbf{G}f_n} \mathbf{G}M_{n+1} \to 0.$$

(b) *We have* $X \in \mathcal{G}_{n+1}$, $Y \in \mathcal{H}_{n+1}$, $\tau_{n+1} X \simeq Y$ *and* $\tau_{n+1}^- Y \simeq X$.

(c) *We have exact sequences*

$$0 \to \mathbf{G}^n M_0 \xrightarrow{\mathbf{G}^n f_0} \mathbf{G}^n M_1 \xrightarrow{\mathbf{G}^n f_1} \cdots \xrightarrow{\mathbf{G}^n f_{n-1}} \mathbf{G}^n M_n \xrightarrow{\mathbf{G}^n f_n} \mathbf{G}^n M_{n+1} \to Y \to 0,$$
$$0 \to \mathbf{F}\tau_n M_0 \xrightarrow{\mathbf{F}\tau_n f_0} \mathbf{F}\tau_n M_1 \xrightarrow{\mathbf{F}\tau_n f_1} \cdots \xrightarrow{\mathbf{F}\tau_n f_{n-1}} \mathbf{F}\tau_n M_n \xrightarrow{\mathbf{F}\tau_n f_n} \mathbf{F}\tau_n M_{n+1} \to Y \to 0.$$

(d) *If* $M_i \in \mathcal{M}_P$ *for any* i, *then we have an exact sequence*

$$0 \to \tau_n M_0 \xrightarrow{\tau_n f_0} \tau_n M_1 \xrightarrow{\tau_n f_1} \cdots \xrightarrow{\tau_n f_{n-1}} \tau_n M_n \xrightarrow{\tau_n f_n} \tau_n M_{n+1} \to 0.$$

(e) *We have exact sequences*

$$0 \to X \to \mathbf{F}^n M_0 \xrightarrow{\mathbf{F}^n f_0} \mathbf{F}^n M_1 \xrightarrow{\mathbf{F}^n f_1} \cdots \xrightarrow{\mathbf{F}^n f_{n-1}} \mathbf{F}^n M_n \xrightarrow{\mathbf{F}^n f_{n+1}} \mathbf{F}^n M_{n+1} \to 0,$$
$$0 \to X \to \mathbf{G}\tau_n^- M_0 \xrightarrow{\mathbf{G}\tau_n^- f_0} \mathbf{G}\tau_n^- M_1 \xrightarrow{\mathbf{G}\tau_n^- f_1} \cdots \xrightarrow{\mathbf{G}\tau_n^- f_{n-1}} \mathbf{G}\tau_n^- M_n \xrightarrow{\mathbf{G}\tau_n^- f_n} \mathbf{G}\tau_n^- M_{n+1} \to 0.$$

(f) *If* $M_i \in \mathcal{M}_I$ *for any* i, *then we have an exact sequence*

$$0 \to \tau_n^- M_0 \xrightarrow{\tau_n^- f_0} \tau_n^- M_1 \xrightarrow{\tau_n^- f_1} \cdots \xrightarrow{\tau_n^- f_{n-1}} \tau_n^- M_n \xrightarrow{\tau_n^- f_n} \tau_n^- M_{n+1} \to 0.$$

Proof. (a) Since M is n-rigid, we have desired exact sequences by Lemma 3.5.

(b) We apply ν_Γ to the upper sequence in (a) and compare with the lower sequence in (a) by using Proposition 3.4(b). Then we have $X \in \mathcal{G}_{n+1}$ and $\tau_{n+1} X \simeq Y$. We have $Y \in \mathcal{H}_{n+1}$ and $\tau_{n+1}^- Y \simeq X$ by Lemma 2.3(a).

(c) The upper sequence is exact by Lemma 3.5(b). Since $\mathbf{F}\tau_n \simeq \mathbf{G}^n$ holds by Proposition 3.4(c), the lower sequence is exact.

(d) We have $\operatorname{Ext}^i_\Lambda(\mathcal{M}_P, \Lambda) = 0$ for any $0 \leqslant i < n$ by Lemma 2.4(c). Thus the sequence is exact by Lemma 3.5(b).

(e)(f) These are shown dually. □

We need the following easy observation.

Lemma 3.7. *Let*

$$M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots$$

be a complex with $M_i \in \mathcal{M}$. Assume $M_0 \in \mathcal{M}_P$ and that each f_i is left minimal. Then $M_i \in \mathcal{M}_P$ for any i and we have a complex

$$\tau_n M_0 \xrightarrow{\tau_n f_0} \tau_n M_1 \xrightarrow{\tau_n f_1} \cdots$$

such that each $\tau_n f_i$ is left minimal.

Proof. We have $\mathrm{Hom}_\Lambda(\mathcal{M}_P, \mathcal{P}(\mathcal{M})) = 0$ by Lemma 2.4(d). Since any f_i is left minimal, we have $M_i \in \mathcal{M}_P$ inductively. Since $\tau_n : \mathcal{M}_P \to \mathcal{M}_I$ is an equivalence by Lemma 2.4(a), each $\tau_n f_i$ is also left minimal. □

We shall use the following special case in inductive argument.

Lemma 3.8. *Let*

$$0 \to M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \to 0$$

be an exact sequence in $\mathrm{mod}\, \Lambda$ *with $M_i \in \mathcal{M}$. Assume $M_0 \in \mathcal{M}_P$ and that each f_i is left minimal. Then $M_i \in \mathcal{M}_P$ for any i and we have an exact sequence*

$$0 \to \tau_n M_0 \xrightarrow{\tau_n f_0} \tau_n M_1 \xrightarrow{\tau_n f_1} \cdots \xrightarrow{\tau_n f_{n-1}} \tau_n M_n \xrightarrow{\tau_n f_n} \tau_n M_{n+1} \to 0$$

such that each $\tau_n f_i$ is left minimal.

Proof. Immediate from Lemma 3.7 and Lemma 3.6(d). □

We are ready to show the following key observation.

Proposition 3.9. *Let $X \in \mathcal{M}$ be indecomposable and $\ell \geqslant 0$. Put $I := \mathbf{G}X$.*

(a) *If $\ell > \ell_X$, then $\tau_{n+1}^\ell I = 0$.*
(b) *If $\ell = \ell_X$, then $\tau_{n+1}^\ell I$ is an indecomposable object in $\mathcal{P}(\mathcal{N})$.*
(c) *If $0 \leqslant \ell < \ell_X$, then $\tau_{n+1}^\ell I \in \mathcal{G}_{n+1}$ and* $\mathrm{pd}(\tau_{n+1}^\ell I)_\Gamma = n + 1$.

Proof. (a) Since $\tau_{n+1}^\ell I$ is a submodule of $\mathbf{G}\tau_n^\ell X = 0$ by Proposition 3.4(f), we have $\tau_{n+1}^\ell I = 0$.

(c) $\tau_n^\ell X \in \mathcal{M}_P$ holds for any $0 \leqslant \ell < \ell_X$. Take a minimal injective resolution

$$0 \to \tau_n X \xrightarrow{f_{-1}} I_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{n-1}} I_n \to 0. \tag{11}$$

Since each f_i is left minimal, we have an exact sequence

$$0 \to \tau_n^{\ell+1} X \xrightarrow{\tau_n^\ell f_{-1}} \tau_n^\ell I_0 \xrightarrow{\tau_n^\ell f_0} \cdots \xrightarrow{\tau_n^\ell f_{n-1}} \tau_n^\ell I_n \to 0$$

for any $0 \leqslant \ell < \ell_X$ by applying Lemma 3.8 to (11) repeatedly. Applying $\mathbf{F}$, we have a Γ-module X_ℓ with an exact sequence

$$0 \to \mathbf{F}\tau_n^{\ell+1} X \xrightarrow{\mathbf{F}\tau_n^\ell f_{-1}} \mathbf{F}\tau_n^\ell I_0 \xrightarrow{\mathbf{F}\tau_n^\ell f_0} \cdots \xrightarrow{\mathbf{F}\tau_n^\ell f_{n-1}} \mathbf{F}\tau_n^\ell I_n \to X_\ell \to 0$$

by Lemma 3.6(a). We have $X_0 = \mathbf{F}^n \tau_n X \simeq \mathbf{G}X = I$ by Proposition 3.4(c). Using Lemma 3.6(b) and (c) repeatedly, we have $\tau_{n+1}^\ell I \simeq \tau_{n+1}^\ell X_0 \simeq \tau_{n+1}^{\ell-1} X_1 \simeq \cdots \simeq X_\ell \in \mathcal{G}_{n+1}$ for any $0 \leqslant \ell < \ell_X$.

(b) Since $\tau_{n+1}^{\ell_X+1} I = 0$ by (a), we have $\tau_{n+1}^{\ell_X} I \in \mathcal{P}(\mathcal{N})$. We have that $\tau_{n+1}^{\ell_X} I$ is indecomposable by (c) and Lemma 3.3(a). □

Summarizing with Lemma 3.1, we have the following conclusion.

Lemma 3.10. *Γ is τ_{n+1}-finite and satisfies the condition* (C_{n+1}) *in Definition* 1.11.

We have the main results in this section.

Theorem 3.11.

(a) *$\mathcal{N}$ is $(n+1)$-rigid.*
(b) *We have full functors*

$$\tau_{n+1} = D\,\mathrm{Ext}_\Gamma^{n+1}(-, \Gamma) : \mathcal{N} \to \mathcal{N} \quad \text{and} \quad \tau_{n+1}^- = \mathrm{Ext}_{\Gamma^{\mathrm{op}}}^{n+1}(D-, \Gamma) : \mathcal{N} \to \mathcal{N}$$

which give mutually quasi-inverse equivalences

$$\tau_{n+1} : \mathcal{N}_P \to \mathcal{N}_I \quad \text{and} \quad \tau_{n+1}^- : \mathcal{N}_I \to \mathcal{N}_P.$$

(c) *τ_{n+1} gives a bijection from isoclasses of indecomposable objects in $\mathcal{N}_P$ to those in $\mathcal{N}_I$.*
(d) *$\mathcal{N}_P \subset \mathcal{G}_{n+1}$ and $\mathcal{N}_I \subset \mathcal{H}_{n+1}$.*
(e) $\mathrm{Hom}_\Gamma(\tau_{n+1}^i(D\Gamma), \tau_{n+1}^j(D\Gamma)) = 0$ *for any $i < j$.*
(f) $\mathrm{Ext}_\Gamma^i(\mathcal{P}(\mathcal{N}), \mathcal{N}) = 0$ *for any $i > 0$.*

Proof. Γ is τ_{n+1}-finite and $\mathcal{N}$ satisfies the condition (C_{n+1}) by Lemmas 3.1 and 3.10. Thus (a) follows from Lemma 2.5(a), and (b)–(e) follow from Lemma 2.4. By (a) and $\mathrm{pd}\, X_\Gamma < n+1$ for any $X \in \mathcal{P}(\mathcal{N})$, we have (f). □

We also have the following description of indecomposable objects in $\mathcal{N}$.

Corollary 3.12.

(a) *There exist bijections among*
- *isoclasses of indecomposable objects in $\mathcal{N}$,*
- *pairs (X, i) of isoclasses of indecomposable objects $X \in \mathcal{M}$ and $0 \leqslant i \leqslant \ell_X$,*

- *triples (I, i, j) of isoclasses of indecomposable objects $I \in \mathcal{I}(\mathcal{M})$ and $0 \leqslant i, j$ satisfying $i + j \leqslant \ell_I$.*

 They are given by $\tau_{n+1}^{j}\mathbf{G}\tau_n^i I \leftrightarrow (\tau_n^i I, j) \leftrightarrow (I, i, j)$.

(b) *Under the bijection in* (a)*, $\tau_{n+1}^{j}\mathbf{G}\tau_n^i I$ belongs to $\mathcal{P}(\mathcal{N})$ if and only if $i + j = \ell_I$.*

(c) *$\mathcal{N}$ has an additive generator.*

Proof. (a) By Theorem 3.11(c), any indecomposable object in $\mathcal{N}$ can be written uniquely as $\tau_{n+1}^{j}\mathbf{G}X$ for an indecomposable object $X \in \mathcal{M}$ and $0 \leqslant j$. By Proposition 3.9, $\tau_{n+1}^{j}\mathbf{G}X$ is non-zero if and only if $0 \leqslant j \leqslant \ell_X$. By Proposition 3.2, X can be written uniquely as $\tau_n^i I$ for an indecomposable object $I \in \mathcal{I}(\mathcal{M})$ and $0 \leqslant i \leqslant \ell_I$. Since $\ell_X = \ell_I - i$, we have the assertion.

(b)(c) Immediate from (a). □

3.2. Tilting Γ-module in $\mathcal{P}(\mathcal{N})$

By Corollary 3.12, there exists a Γ-module U such that $\mathcal{P}(\mathcal{N}) = \operatorname{add} U$. In this subsection, we shall show that U is a tilting Γ-module.

We have $\operatorname{pd} U_\Gamma < n + 1$ by definition of $\mathcal{P}(\mathcal{N})$. By Theorem 3.11(f), we have $\operatorname{Ext}^i_\Gamma(U, U) = 0$ for any $0 < i$. By Theorem 3.11(c), we have that U and Γ have the same number of non-isomorphic indecomposable direct summands. For the case $n = 1$, these imply that U is a tilting Γ-module [4,34]. But we need more argument for arbitrary n.

Lemma 3.13.

(a) *$\tau_{n+1}^{\ell}\mathbf{G}\tau_n^{-\ell} X \in \mathcal{P}(\mathcal{N})$ for any $0 \leqslant \ell$ and $X \in \mathcal{P}(\mathcal{M})$.*

(b) *$\tau_{n+1}^{i}\mathbf{G}\tau_n^{-\ell} X \in \mathcal{N}_P$ for any $0 \leqslant i < \ell$ and $X \in \mathcal{M}$.*

Proof. (a) We can assume $\tau_n^{-\ell} X \neq 0$. Since $\ell_{\tau_n^{-\ell} X} = \ell$, the assertion follows from Corollary 3.12(b).

(b) Any indecomposable summand Y of $\tau_n^{-\ell} X$ satisfies $\ell_Y \geqslant \ell$. Thus the assertion follows from Corollary 3.12(b). □

We shall often use the following result.

Lemma 3.14. *Let*

$$0 \to X \xrightarrow{f_0} N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} N_{n+1} \xrightarrow{f_{n+1}} N_{n+2} \to 0$$

be an exact sequence in $\operatorname{mod} \Gamma$ *with $N_i \in \mathcal{N}_P$ and $X \in \mathcal{G}_{n+1}$. Then the sequence*

$$0 \to \tau_{n+1} X \xrightarrow{\tau_{n+1} f_0} \tau_{n+1} N_1 \xrightarrow{\tau_{n+1} f_1} \tau_{n+1} N_2 \xrightarrow{\tau_{n+1} f_2} \cdots \xrightarrow{\tau_{n+1} f_n} \tau_{n+1} N_{n+1} \xrightarrow{\tau_{n+1} f_{n+1}} \tau_{n+1} N_{n+2} \to 0$$

is exact.

Proof. We have $\operatorname{Ext}^i_\Gamma(\mathcal{N}_P, \Gamma) = 0$ for any $0 \leqslant i < n + 1$ by Theorem 3.11(d). We have the desired exact sequence by applying Lemma 3.5(b), where we replace n there by $n + 1$. □

Lemma 3.15. *Let $I \in \mathcal{I}(\mathcal{M})$ be indecomposable.*

(a) *There exist exact sequences*

$$0 \to \nu_\Lambda^- I \to T_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{n-2}} T_{n-1} \to 0,$$
$$0 \to \mathbf{F}I \to \mathbf{G}T_0 \xrightarrow{\mathbf{G}f_0} \cdots \xrightarrow{\mathbf{G}f_{n-2}} \mathbf{G}T_{n-1} \to 0$$

with $T_i \in \mathcal{P}(\mathcal{M})$.

(b) *For any $0 < \ell \leqslant \ell_I$, there exist exact sequences*

$$0 \to T_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} T_0 \xrightarrow{f_0} \tau_n^{\ell-1} I \to 0, \tag{12}$$
$$0 \to \mathbf{F}\tau_n^\ell I \to \tau_{n+1}^\ell \mathbf{G}\tau_n^{-\ell} T_n \xrightarrow{\tau_{n+1}^\ell \mathbf{G}\tau_n^{-\ell} f_n} \cdots \xrightarrow{\tau_{n+1}^\ell \mathbf{G}\tau_n^{-\ell} f_1} \tau_{n+1}^\ell \mathbf{G}\tau_n^{-\ell} T_0 \to 0$$

with $T_i \in \mathcal{P}(\mathcal{M})$.

Proof. (a) Since $\nu_\Lambda^- I$ is a projective Λ-module and T is a tilting Λ-module with $\operatorname{pd} T_\Lambda < n$, we have the upper sequence. Applying $\mathbf{G}$, we have an exact sequence

$$0 \to \mathbf{G}\nu_\Lambda^- I \longrightarrow \mathbf{G}T_0 \xrightarrow{\mathbf{G}f_0} \cdots \xrightarrow{\mathbf{G}f_{n-2}} \mathbf{G}T_{n-1} \to 0$$

by Lemma 3.5 since $\operatorname{Ext}^i_\Lambda(\Lambda \oplus T, M) = 0$ for any $i > 0$. Since $\mathbf{G}\nu_\Lambda^- I = \mathbf{F}I$ by Proposition 3.4(a), we have the lower sequence.

(b) Since $\tau_n^{\ell-1} I \in \mathcal{M} \subseteq T^\perp$, there exists an exact sequence

$$\cdots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} \tau_n^{\ell-1} I \to 0$$

with $T_i \in \mathcal{P}(\mathcal{M})$ and $\operatorname{Im} f_i \in T^\perp$. Since

$$\operatorname{Ext}^1_\Lambda(\operatorname{Im} f_n, \operatorname{Im} f_{n+1}) \simeq \operatorname{Ext}^2_\Lambda(\operatorname{Im} f_{n-1}, \operatorname{Im} f_{n+1}) \simeq \cdots \simeq \operatorname{Ext}^{n+1}_\Lambda(\operatorname{Im} f_0, \operatorname{Im} f_{n+1}) = 0$$

by $\operatorname{gl.dim} \Lambda \leqslant n$, we have that $\operatorname{Im} f_n \in \mathcal{P}(\mathcal{M})$. Thus we have the sequence (12).

Clearly we can assume that each f_i is right minimal. Applying τ_n^- to (12) repeatedly, we have an exact sequence

$$0 \to \tau_n^{-i} T_n \xrightarrow{\tau_n^{-i} f_n} \cdots \xrightarrow{\tau_n^{-i} f_1} \tau_n^{-i} T_0 \xrightarrow{\tau_n^{-i} f_0} \tau_n^{\ell-1-i} I \to 0 \tag{13}$$

for any $0 \leqslant i \leqslant \ell - 1$ by the dual of Lemma 3.8 since $\tau_n^i I \in \mathcal{M}_I$ for any $0 < i$.

Applying $\mathbf{F}$ to (13), we have a Γ-module X_i with an exact sequence

$$0 \to \mathbf{F}\tau_n^{-i} T_n \xrightarrow{\mathbf{F}\tau_n^{-i} f_n} \cdots \xrightarrow{\mathbf{F}\tau_n^{-i} f_1} \mathbf{F}\tau_n^{-i} T_0 \xrightarrow{\mathbf{F}\tau_n^{-i} f_0} \mathbf{F}\tau_n^{\ell-1-i} I \to X_i \to 0$$

for any $0 \leqslant i \leqslant \ell - 1$ by Lemma 3.6(a), and for $i = -1$ by Lemma 3.6(c). In particular we have $X_{-1} \simeq \mathbf{F}\tau_n^\ell I$. By Lemma 3.6(b), we have $X_i \in \mathcal{G}_{n+1}$ and $\tau_{n+1} X_i \simeq X_{i-1}$ for any $0 \leqslant i \leqslant \ell - 1$. Especially we have

$$\tau_{n+1}^i X_{\ell-1} \in \mathcal{G}_{n+1} \quad (0 \leqslant i \leqslant \ell - 1) \quad \text{and} \quad \tau_{n+1}^\ell X_{\ell-1} \simeq X_{-1} \simeq \mathbf{F}\tau_n^\ell I. \tag{14}$$

On the other hand, applying Lemma 3.6(e) to the sequence (13) for $i = \ell - 1$, we have an exact sequence

$$0 \to X_{\ell-1} \to \mathbf{G}\tau_n^{-\ell} T_n \xrightarrow{\mathbf{G}\tau_n^{-\ell} f_n} \cdots \xrightarrow{\mathbf{G}\tau_n^{-\ell} f_1} \mathbf{G}\tau_n^{-\ell} T_0 \to 0. \tag{15}$$

We have $\tau_{n+1}^i \mathbf{G}\tau_n^{-\ell} T \in \mathcal{N}_P$ for any $0 \leqslant i \leqslant \ell - 1$ by Lemma 3.13(b). Applying τ_{n+1} to (15) repeatedly, we have an exact sequence

$$0 \to \tau_{n+1}^i X_{\ell-1} \to \tau_{n+1}^i \mathbf{G}\tau_n^{-\ell} T_n \xrightarrow{\tau_{n+1}^i \mathbf{G}\tau_n^{-\ell} f_n} \cdots \xrightarrow{\tau_{n+1}^i \mathbf{G}\tau_n^{-\ell} f_1} \tau_{n+1}^i \mathbf{G}\tau_n^{-\ell} T_0 \to 0$$

for any $0 \leqslant i \leqslant \ell$ by Lemma 3.14 since we have (14). Putting $i = \ell$, we have the desired sequence by (14). □

Summarizing above results, we have the following desired result.

Theorem 3.16. *There exists a tilting Γ-module U such that* $\operatorname{pd} U_\Gamma < n + 1$ *and* $\mathcal{P}(\mathcal{N}) = \operatorname{add} U$.

Proof. As we observed at the beginning of this subsection, we already have $\operatorname{pd} U_\Gamma < n + 1$ and $\operatorname{Ext}_\Gamma^i(U, U) = 0$ for any $i > 0$. By Lemmas 3.15 and 3.13(a), there exists an exact sequence

$$0 \to \Gamma \to U_0 \to \cdots \to U_n \to 0$$

with $U_i \in \operatorname{add} U$. Thus U is a tilting Γ-module. □

3.3. Mapping cone construction of $(n + 1)$-almost split sequences

In this section, we complete our proof of Theorem 1.14. Our method is to construct source sequences in $\mathcal{N}$ and apply Theorem 2.2(b)(iii) ⇒ (i). A crucial role is played by a monomorphism

$$\alpha : \tau_{n+1}\mathbf{G} \to \mathbf{G}\tau_n$$

of functors on $\mathcal{M}$ in Proposition 3.4(d).

Lemma 3.17. *Fix an indecomposable object $X \in \mathcal{M}$ and $\ell \geqslant 0$. Take a source morphism $f_0 : X \to M_1$ in $\mathcal{M}$.*

(a) *Any morphism $\tau_{n+1}^\ell \mathbf{G}X \to \tau_{n+1}^i \mathbf{G}M$ with $i > \ell$ is zero.*
(b) *Any morphism $\tau_{n+1}^\ell \mathbf{G}X \to \tau_{n+1}^\ell \mathbf{G}M$ which is not a split monomorphism factors through $\tau_{n+1}^\ell \mathbf{G} f_0 : \tau_{n+1}^\ell \mathbf{G}X \to \tau_{n+1}^\ell \mathbf{G}M_1$.*

(c) *Any morphism* $\tau_{n+1}^{\ell}\mathbf{G}X \to \tau_{n+1}^{i}\mathbf{G}M$ *with* $0 \leqslant i < \ell$ *factors through* $\tau_{n+1}^{\ell-1}\alpha_X : \tau_{n+1}^{\ell}\mathbf{G}X \to \tau_{n+1}^{\ell-1}\mathbf{G}\tau_n X$.

Proof. (a) Immediate from Theorem 3.11(e).

(b) By Theorem 3.11(b), any morphism $\tau_{n+1}^{\ell}\mathbf{G}X \to \tau_{n+1}^{\ell}\mathbf{G}M$ which is not a split monomorphism can be written as $\tau_{n+1}^{\ell}\mathbf{G}g$ with a morphism $g : X \to M$ which is not a split monomorphism. Since g factors through f_0, we have that $\tau_{n+1}^{\ell}\mathbf{G}g$ factors through $\tau_{n+1}^{\ell}\mathbf{G}f_0$.

(c) By Theorem 3.11(b), any morphism $\tau_{n+1}^{\ell}\mathbf{G}X \to \tau_{n+1}^{i}\mathbf{G}M$ can be written as $\tau_{n+1}^{i}g$ for a morphism $g : \tau_{n+1}^{\ell-i}\mathbf{G}X \to \mathbf{G}M$. Since τ_{n+1} preserves monomorphisms, we have that $\tau_{n+1}^{\ell-i-1}\alpha_X : \tau_{n+1}^{\ell-i}\mathbf{G}X \to \tau_{n+1}^{\ell-i-1}\mathbf{G}\tau_n X$ is a monomorphism. Since $\mathbf{G}M$ is an injective Γ-module, g factors through $\tau_{n+1}^{\ell-i-1}\alpha_X$. Thus $\tau_{n+1}^{i}g$ factors through $\tau_{n+1}^{\ell-1}\alpha_X$. □

Immediately we have the following conclusion.

Proposition 3.18. *Fix an indecomposable object* $X \in \mathcal{M}$ *and* $0 \leqslant \ell \leqslant \ell_X$. *Take a source morphism* $f_0 : X \to M_1$ *in* $\mathcal{M}$. *Then a left almost split morphism of* $\tau_{n+1}^{\ell}\mathbf{G}X$ *is given by*

$$\begin{pmatrix} \tau_{n+1}^{\ell}\mathbf{G}f_0 \\ \tau_{n+1}^{\ell-1}\alpha_X \end{pmatrix} : \tau_{n+1}^{\ell}\mathbf{G}X \to \left(\tau_{n+1}^{\ell}\mathbf{G}M_1\right) \oplus \left(\tau_{n+1}^{\ell-1}\mathbf{G}\tau_n X\right).$$

Recall that any indecomposable object $X \in \mathcal{M}$ has a source sequence by Theorem 2.2(a). By using it, we shall construct source sequences of an indecomposable object $\tau_{n+1}^{\ell}\mathbf{G}X$ in $\mathcal{N}$ for $0 \leqslant \ell \leqslant \ell_X$.

First we consider the case $\ell = 0$.

Proposition 3.19. *For an indecomposable object* $X \in \mathcal{M}$, *take a source sequence*

$$X \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \to 0$$

in $\mathcal{M}$. *Applying* $\mathbf{G}$, *we have an exact sequence*

$$\mathbf{G}X \xrightarrow{\mathbf{G}f_0} \mathbf{G}M_1 \xrightarrow{\mathbf{G}f_1} \cdots \xrightarrow{\mathbf{G}f_{n-1}} \mathbf{G}M_n \xrightarrow{\mathbf{G}f_n} \mathbf{G}M_{n+1} \to 0$$

which is a source sequence of $\mathbf{G}X$ *in* $\mathcal{N}$.

Proof. Clearly the sequence is exact. Since each f_i is left minimal and $\mathbf{G}$ is a fully faithful functor, each $\mathbf{G}f_i$ is left minimal. Moreover $\mathbf{G}f_0$ is a source morphism in $\mathcal{N}$ by Proposition 3.18. □

Next we consider the case $0 < \ell \leqslant \ell_X$. Recall that any indecomposable object in $\mathcal{M}_I$ is a left term of an n-almost split sequence by Theorem 2.2(a). Let us start with the following observation.

Lemma 3.20. *For an indecomposable object* $X \in \mathcal{M}_P$, *take the following n-almost split sequence,*

$$0 \to \tau_n X \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} X \to 0.$$

(a) *We have the following source sequence in $\mathcal{M}$,*

$$X \xrightarrow{\tau_n^- f_0} \tau_n^- M_1 \xrightarrow{\tau_n^- f_1} \cdots \xrightarrow{\tau_n^- f_{n-1}} \tau_n^- M_n \xrightarrow{\tau_n^- f_n} \tau_n^- X \to 0. \tag{16}$$

(b) $\operatorname{soc} \mathbf{G}X \in \mathcal{G}_{n+1}$ *and* $\tau_{n+1}(\operatorname{soc} \mathbf{G}X) \simeq \operatorname{soc}(\mathbf{G}\tau_n X)$.

Proof. (a) Any term in (16) belongs to $\mathcal{M}_P$. Since we have $\operatorname{Hom}_\Lambda(\mathcal{M}_P, \mathcal{P}(\mathcal{M})) = 0$ by Lemma 2.4(d) and we have an equivalence $\tau_n^- : \mathcal{M}_I \to \mathcal{M}_P$, we have the assertion.

(b) Applying $\mathbf{G}$ to source sequences of X and $\tau_n X$, we have exact sequences

$$0 \to \operatorname{soc} \mathbf{G}X \to \mathbf{G}X \xrightarrow{\mathbf{G}\tau_n^- f_0} \mathbf{G}\tau_n^- M_1 \xrightarrow{\mathbf{G}\tau_n^- f_1} \cdots \xrightarrow{\mathbf{G}\tau_n^- f_{n-1}} \mathbf{G}\tau_n^- M_n \xrightarrow{\mathbf{G}\tau_n^- f_n} \mathbf{G}\tau_n^- X \to 0,$$

$$0 \to \operatorname{soc}(\mathbf{G}\tau_n X) \to \mathbf{G}\tau_n X \xrightarrow{\mathbf{G}f_0} \mathbf{G}M_1 \xrightarrow{\mathbf{G}f_1} \cdots \xrightarrow{\mathbf{G}f_{n-1}} \mathbf{G}M_n \xrightarrow{\mathbf{G}f_n} \mathbf{G}X \to 0.$$

Comparing these sequences by Lemma 3.6(a)(e), we have the assertions. □

We also need the following information.

Proposition 3.21. *For an indecomposable object $X \in \mathcal{M}$ and $\ell \geqslant 0$, let $S := \tau_{n+1}^\ell(\operatorname{soc} \mathbf{G}X)$.*

(a) *We have $S \simeq \operatorname{soc}(\mathbf{G}\tau_n^\ell X)$.*
(b) *If $0 \leqslant \ell < \ell_X$, then $S \in \mathcal{G}_{n+1}$. If $\ell = \ell_X$, then $S \notin \mathcal{G}_{n+1}$. If $\ell > \ell_X$, then $S = 0$.*

Proof. (a) We only have to show the case $\ell = 1$. If $X \in \mathcal{M}_P$, then the assertion follows from Lemma 3.20(b). Assume $X \in \mathcal{P}(\mathcal{M})$. Since $\tau_n X = 0$, the right-hand side is zero. Since $\operatorname{soc} \mathbf{G}X = \operatorname{top} \mathbf{F}X$ has projective dimension at most n by Theorem 2.2(a)(iii), the left-hand side is also zero.

(b) Immediate from (a) and Lemma 3.20(b). □

For $X \in \mathcal{M}$, define a morphism $\iota_X : \tau_n \tau_n^- X \to X$ by taking an isomorphism $X \simeq Y \oplus I$ with $Y \in \mathcal{M}_I$ and $I \in \mathcal{I}(\mathcal{M})$ and putting

$$\iota_X : \tau_n \tau_n^- X \simeq Y \xrightarrow{\binom{1}{0}} Y \oplus I \simeq X.$$

We denote by β_X the morphism

$$\beta_X : \tau_{n+1}\mathbf{G}\tau_n^- X \xrightarrow{\alpha_{\tau_n^- X}} \mathbf{G}\tau_n \tau_n^- X \xrightarrow{\mathbf{G}\iota_X} \mathbf{G}X.$$

Although ι_X depends on the choice of decomposition of X, we have the following.

Lemma 3.22. *β_X is independent of the choice of decomposition of X. In particular, β gives a monomorphism $\beta : \tau_{n+1}\mathbf{G}\tau_n^- \to \mathbf{G}$ of functors on $\mathcal{M}$.*

Proof. By Theorem 2.2(a)(ii), we have $\operatorname{id}(\operatorname{soc} \mathbf{G}I)_\Gamma \leqslant n$. Since we have $\tau_{n+1}\mathbf{G}\tau_n^- X \in \mathcal{H}_{n+1}$ by Lemma 3.3(a) and $\mathcal{H}_{n+1}$ is a Serre subcategory of $\operatorname{mod} \Gamma$ by Lemma 3.3(b), we have $\operatorname{Hom}_\Gamma(\tau_{n+1}\mathbf{G}\tau_n^- X, \mathbf{G}I) = 0$. This implies the assertion. □

Proposition 3.23. *For an indecomposable object $X \in \mathcal{M}_P$ and $0 < \ell \leqslant \ell_X$, take the following n-almost split sequence in $\mathcal{M}$,*

$$0 \to \tau_n X \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} X \to 0.$$

(a) *We have the following commutative diagram of exact sequences.*

$$\begin{array}{ccccccccccc}
0 & \to & \tau_{n+1}^{\ell}(\operatorname{soc}\mathbf{G}X) & \to & \tau_{n+1}^{\ell}\mathbf{G}X & \xrightarrow{\tau_{n+1}^{\ell}\mathbf{G}\tau_n^- f_0} & \tau_{n+1}^{\ell}\mathbf{G}\tau_n^- M_1 & \to \cdots \xrightarrow{\tau_{n+1}^{\ell}\mathbf{G}\tau_n^- f_n} & \tau_{n+1}^{\ell}\mathbf{G}\tau_n^- X & \to & 0 \\
 & & \downarrow \wr & & \downarrow \tau_{n+1}^{\ell-1}\beta_{\tau_n X} & & \downarrow \tau_{n+1}^{\ell-1}\beta_{M_1} & & \downarrow \tau_{n+1}^{\ell-1}\beta_X & & \\
0 & \to & \tau_{n+1}^{\ell-1}(\operatorname{soc}\mathbf{G}\tau_n X) & \to & \tau_{n+1}^{\ell-1}\mathbf{G}\tau_n X & \xrightarrow{\tau_{n+1}^{\ell-1}\mathbf{G}f_0} & \tau_{n+1}^{\ell-1}\mathbf{G}M_1 & \to \cdots \xrightarrow{\tau_{n+1}^{\ell-1}\mathbf{G}f_n} & \tau_{n+1}^{\ell-1}\mathbf{G}X & \to & 0
\end{array}$$

(b) *Taking a mapping cone, we have the following $(n+1)$-almost split sequence in $\mathcal{N}$,*

$$0 \to \tau_{n+1}^{\ell}\mathbf{G}X \xrightarrow{\begin{pmatrix} \tau_{n+1}^{\ell}\mathbf{G}\tau_n^- f_0 \\ \tau_{n+1}^{\ell-1}\beta_{\tau_n X} \end{pmatrix}} \left(\tau_{n+1}^{\ell}\mathbf{G}\tau_n^- M_1\right) \oplus \left(\tau_{n+1}^{\ell-1}\mathbf{G}\tau_n X\right) \xrightarrow{\begin{pmatrix} \tau_{n+1}^{\ell}\mathbf{G}\tau_n^- f_1 & 0 \\ \tau_{n+1}^{\ell-1}\beta_{M_1} & -\tau_{n+1}^{\ell-1}\mathbf{G}f_0 \end{pmatrix}}$$
$$\cdots \xrightarrow{\begin{pmatrix} \tau_{n+1}^{\ell}\mathbf{G}\tau_n^- f_n & 0 \\ \tau_{n+1}^{\ell-1}\beta_{M_n} & -\tau_{n+1}^{\ell-1}\mathbf{G}f_{n-1} \end{pmatrix}} \left(\tau_{n+1}^{\ell}\mathbf{G}\tau_n^- X\right) \oplus \left(\tau_{n+1}^{\ell-1}\mathbf{G}M_n\right) \xrightarrow{(\tau_{n+1}^{\ell-1}\beta_X \ -\tau_{n+1}^{\ell-1}\mathbf{G}f_n)} \tau_{n+1}^{\ell-1}\mathbf{G}X \to 0.$$

Proof. (a) By Lemma 3.20(a), we have an exact sequence

$$0 \to \operatorname{soc}\mathbf{G}X \to \mathbf{G}X \xrightarrow{\mathbf{G}\tau_n^- f_0} \mathbf{G}\tau_n^- M_1 \xrightarrow{\mathbf{G}\tau_n^- f_1} \cdots \xrightarrow{\mathbf{G}\tau_n^- f_{n-1}} \mathbf{G}\tau_n^- M_n \xrightarrow{\mathbf{G}\tau_n^- f_n} \mathbf{G}\tau_n^- X \to 0. \quad (17)$$

Since $\ell \leqslant \ell_X$, we have $\tau_n^i X \in \mathcal{M}_P$ for any $0 \leqslant i < \ell$. By Proposition 3.21, we have

$$\tau_{n+1}^i(\operatorname{soc}\mathbf{G}X) \simeq \operatorname{soc}\left(\mathbf{G}\tau_n^i X\right) \quad (0 \leqslant i \leqslant \ell) \quad \text{and} \quad \tau_{n+1}^i(\operatorname{soc}\mathbf{G}X) \in \mathcal{G}_{n+1} \quad (0 \leqslant i < \ell).$$

Since $\tau_{n+1}^i\mathbf{G}X \in \mathcal{N}_P$ for any $0 \leqslant i < \ell$ by Proposition 3.9(c), we have an exact sequence

$$0 \to \tau_{n+1}^i(\operatorname{soc}\mathbf{G}X) \to \tau_{n+1}^i\mathbf{G}X \xrightarrow{\tau_{n+1}^i\mathbf{G}\tau_n^- f_0} \tau_{n+1}^i\mathbf{G}\tau_n^- M_1 \xrightarrow{\tau_{n+1}^i\mathbf{G}\tau_n^- f_1} \cdots \xrightarrow{\tau_{n+1}^i\mathbf{G}\tau_n^- f_n} \tau_{n+1}^i\mathbf{G}\tau_n^- X \to 0$$

for any $0 \leqslant i \leqslant \ell$ by applying Lemma 3.7 (replace $\mathcal{M}$ there by $\mathcal{N}$) and Lemma 3.14 to the sequence (17) repeatedly.

By a similar argument, we have an exact sequence

$$0 \to \tau_{n+1}^i(\operatorname{soc}\mathbf{G}\tau_n X) \to \tau_{n+1}^i\mathbf{G}\tau_n X \xrightarrow{\tau_{n+1}^i\mathbf{G}f_0} \tau_{n+1}^i\mathbf{G}M_1 \xrightarrow{\tau_{n+1}^i\mathbf{G}f_1} \cdots \xrightarrow{\tau_{n+1}^i\mathbf{G}f_n} \tau_{n+1}^i\mathbf{G}X \to 0$$

for any $0 \leqslant i < \ell$. Thus we have the desired exact sequences.

Using the morphism $\tau_{n+1}^{\ell-1}\beta : \tau_{n+1}^{\ell}\mathbf{G}\tau_n^- \to \tau_{n+1}^{\ell-1}\mathbf{G}$ of functors, we have the desired commutative diagram. The left-hand side morphism $\tau_{n+1}^{\ell}(\operatorname{soc}\mathbf{G}X) \to \tau_{n+1}^{\ell-1}(\operatorname{soc}\mathbf{G}\tau_n X)$ is an isomorphism since it is a monomorphism between simple Γ-modules.

(b) Clearly our sequence is exact. By Proposition 3.18, the morphism $\begin{pmatrix} \tau_{n+1}^{\ell}\mathbf{G}\tau_n^- f_0 \\ \tau_{n+1}^{\ell-1}\beta_{\tau_n X} \end{pmatrix}$: $\tau_{n+1}^{\ell}\mathbf{G}X \to (\tau_{n+1}^{\ell}\mathbf{G}\tau_n^- M_1) \oplus (\tau_{n+1}^{\ell-1}\mathbf{G}\tau_n X)$ is left almost split. Since all morphisms in our sequence are contained in the Jacobson radical of the category $\mathcal{N}$, they are left minimal. By Lemma 3.5(b) (replace n there by $n+1$), our sequence gives a source sequence of $\tau_{n+1}^{\ell}\mathbf{G}X$. Since source sequences are unique up to isomorphisms of complexes, it is an $(n+1)$-almost split sequence by Theorem 2.2(a)(i). □

Consequently we have the following.

Proposition 3.24. *Any indecomposable object $X \in \mathcal{N}$ has a source sequence in $\mathcal{N}$ of the form*

$$X \to N_1 \to \cdots \to N_{n+2} \to 0.$$

Proof. Immediate from Proposition 3.19 and Proposition 3.23. □

We have the following conclusion.

Theorem 3.25. *N is an $(n+1)$-cluster tilting object in $U^{\perp}$.*

Proof. This follows from Theorem 2.2(b)(iii) ⇒ (i) and Proposition 3.24. □

Now we complete the proof of Theorem 1.14. We have that Γ is τ_{n+1}-finite and satisfies (C_{n+1}) by Lemma 3.10, (A_{n+1}) by Theorem 3.16, and (B_{n+1}) by Theorem 3.25. □

3.4. Description of the cone of Γ

The aim of this subsection is to give a description of the cone of Γ and prove Theorem 1.18 as an application. For simplicity, we assume that our additive generator M of $\mathcal{M}$ is basic, so Γ is also basic.

For $\ell \geqslant 0$, we denote by

$$\operatorname{sim}^{\ell} \Gamma \quad \text{(respectively, } \operatorname{sim}_{\ell} \Gamma\text{)}$$

the set of simple Γ-modules S satisfying $\tau_{n+1}^{\ell} S \neq 0$ (respectively, $\tau_{n+1}^{-\ell} S \neq 0$). Define subcategories of $\operatorname{mod} \Gamma$ by

$$\mathcal{G}_{n+1}^{\ell} := \bigl\{X \in \operatorname{mod} \Gamma \bigm| \text{any composition factor } S \text{ of } X \text{ belongs to } \operatorname{sim}^{\ell} \Gamma\bigr\},$$
$$\mathcal{H}_{n+1}^{\ell} := \{X \in \operatorname{mod} \Gamma \mid \text{any composition factor } S \text{ of } X \text{ belongs to } \operatorname{sim}_{\ell} \Gamma\}.$$

We have the following preliminary properties.

Lemma 3.26.

(a) $\operatorname{sim}^{\ell} \Gamma$ *(respectively,* $\operatorname{sim}_{\ell} \Gamma$*) consists of* $\operatorname{soc} \mathbf{G}X$ *for any indecomposable $X \in \mathcal{M}$ satisfying $\tau_n^{\ell} X \neq 0$ (respectively, $\tau_n^{-\ell} X \neq 0$).*

(b) *We have* $\mathcal{G}^1_{n+1} = \mathcal{G}_{n+1}$ *and* $\mathcal{H}^1_{n+1} = \mathcal{H}_{n+1}$.
(c) *For any* $\ell > 0$, *we have equivalences*

$$\mathcal{G}^\ell_{n+1} \xrightarrow{\tau_{n+1}} \mathcal{G}^{\ell-1}_{n+1} \cap \mathcal{H}^1_{n+1} \xrightarrow{\tau_{n+1}} \mathcal{G}^{\ell-2}_{n+1} \cap \mathcal{H}^2_{n+1} \xrightarrow{\tau_{n+1}} \cdots \xrightarrow{\tau_{n+1}} \mathcal{G}^1_{n+1} \cap \mathcal{H}^{\ell-1}_{n+1} \xrightarrow{\tau_{n+1}} \mathcal{H}^\ell_{n+1}$$

whose quasi-inverses are given by τ^-_{n+1}.

Proof. (a) Immediate from Proposition 3.21(a).

(b) We only show $\mathcal{G}^1_{n+1} = \mathcal{G}_{n+1}$. By Proposition 3.21(b), we have that a simple Γ-module S belongs to $\mathcal{G}^1_{n+1}$ if and only if it belongs to $\mathcal{G}_{n+1}$. Since $\mathcal{G}^1_{n+1}$ and $\mathcal{G}_{n+1}$ are Serre subcategories by Lemma 3.3(b), we have the assertion.

(c) By (b) and Lemma 3.3(a), we have an equivalence $\tau_{n+1} : \mathcal{G}^1_{n+1} \to \mathcal{H}^1_{n+1}$. This restricts to the desired equivalences. □

There exist idempotents e^ℓ and e_ℓ of Γ such that the factor algebras $\Gamma^\ell := \Gamma/\langle e^\ell\rangle$ and $\Gamma_\ell := \Gamma/\langle e_\ell\rangle$ satisfy

$$\mathcal{G}^\ell_{n+1} = \operatorname{mod} \Gamma^\ell \quad \text{and} \quad \mathcal{H}^\ell_{n+1} = \operatorname{mod} \Gamma_\ell.$$

We have surjections

$$\Gamma = \Gamma^0 \to \Gamma^1 \to \Gamma^2 \to \cdots \quad \text{and} \quad \Gamma = \Gamma_0 \to \Gamma_1 \to \Gamma_2 \to \cdots$$

of algebras. We have the following description of e_ℓ.

Lemma 3.27. $\operatorname{add}(e_\ell M) = \operatorname{add}\{\tau^i_n(D\Lambda) \mid 0 \leqslant i < \ell\}$ *holds for any* $\ell \geqslant 0$.

Proof. For an indecomposable object $X \in \mathcal{M}$, we put $S := \operatorname{soc} \mathbf{G}X$. By definition of e_ℓ, we have that $X \in \operatorname{add}(e_\ell M)$ if and only if $S \notin \operatorname{sim}_\ell \Gamma$. By Lemma 3.26(a), we have that $S \notin \operatorname{sim}_\ell \Gamma$ if and only if $\tau^{-\ell}_n X = 0$ if and only if $X \in \operatorname{add}\{\tau^i_n(D\Lambda) \mid 0 \leqslant i < \ell\}$. □

Now we need the functorial monomorphism $\alpha^\ell : \tau^\ell_{n+1}\mathbf{G} \to \mathbf{G}\tau^\ell_n$ given in Proposition 3.4(f). The following result generalizes Proposition 3.4(e).

Lemma 3.28.

(a) $\alpha^\ell_X : \tau^\ell_{n+1}\mathbf{G}X \to \mathbf{G}\tau^\ell_n X$ *is a minimal right* $\mathcal{H}^\ell_{n+1}$*-approximation of* $\mathbf{G}\tau^\ell_n X$ *for any* $X \in \mathcal{M}$ *and* $\ell \geqslant 0$.
(b) *We have* $\tau^\ell_{n+1}\mathbf{G} = \operatorname{Hom}_\Gamma(\Gamma_\ell, \mathbf{G}\tau^\ell_n -)$.

Proof. (a) Fix $\ell > 0$ and assume that the assertion is true for $\ell - 1$. By Lemma 3.26(c), any object in $\mathcal{H}^\ell_{n+1}$ can be written as $\tau^\ell_{n+1} Y$ with $Y \in \mathcal{G}^\ell_{n+1}$. Take any morphism $f : \tau^\ell_{n+1} Y \to \mathbf{G}\tau^\ell_n X$. We write α^ℓ_X as a composition

$$\tau^\ell_{n+1}\mathbf{G}X \xrightarrow{\tau_{n+1}\alpha^{\ell-1}_X} \tau_{n+1}\mathbf{G}\tau^{\ell-1}_n X \xrightarrow{\alpha_{\tau^{\ell-1}_n X}} \mathbf{G}\tau^\ell_n X.$$

By Proposition 3.4(e), there exists $g : \tau_{n+1}^{\ell} Y \to \tau_{n+1}\mathbf{G}\tau_n^{\ell-1}X$ such that $f = (\alpha_{\tau_n^{\ell-1}X})g$. By Lemma 3.26(c), there exists $h : \tau_{n+1}^{\ell-1} Y \to \mathbf{G}\tau_n^{\ell-1}X$ such that $g = \tau_{n+1}h$. By the inductive hypothesis, we have $\operatorname{Im} h \subset \tau_{n+1}^{\ell-1}\mathbf{G}X$. Thus we have $\operatorname{Im} f \subset \tau_{n+1}^{\ell}\mathbf{G}X$.

(b) Since $\mathcal{H}_{n+1}^{\ell} = \operatorname{mod} \Gamma_\ell$, the assertion follows immediately from (a). $\square$

Lemma 3.29. *For $\ell \geqslant 0$, we have $\tau_{n+1}^{\ell}(D\Gamma) \simeq D\Gamma_\ell$ as Γ-modules.*

Proof. Clearly the inclusion $D\Gamma_\ell \to D\Gamma$ is a minimal right $\mathcal{H}_{n+1}^{\ell}$-approximation of the Γ-module $D\Gamma$.

On the other hand, $\tau_{n+1}^{\ell}\mathbf{G}M \subset \mathbf{G}\tau_n^{\ell}M$ is a right $\mathcal{H}_{n+1}^{\ell}$-approximation by Lemma 3.28. Since we assume that M is basic, we have $M \simeq X \oplus \tau_n^{\ell}M$ as Λ-modules, where X and $\tau_n^{\ell}M$ have no non-zero direct summands in common. Then $\tau_n^{-\ell}X = 0$ holds. Since $\operatorname{soc}\mathbf{G}X \notin \mathcal{H}_{n+1}^{\ell}$ holds by Lemma 3.26(a), we have $\operatorname{Hom}_\Gamma(\mathcal{H}_{n+1}^{\ell}, \mathbf{G}X) = 0$. Consequently, the minimal right $\mathcal{H}_{n+1}^{\ell}$-approximation of $D\Gamma \simeq (\mathbf{G}X) \oplus (\mathbf{G}\tau_n^{\ell}M)$ is given by $\tau_{n+1}^{\ell}\mathbf{G}M = \tau_{n+1}^{\ell}(D\Gamma)$. $\square$

We have the following description of the cone of Γ.

Proposition 3.30. *The cone of Γ is Morita equivalent to*

$$\begin{pmatrix} \Gamma^0 & 0 & 0 & \cdots \\ \Gamma^1 & \Gamma^1 & 0 & \cdots \\ \Gamma^2 & \Gamma^2 & \Gamma^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Gamma_0 & \Gamma_1 & \Gamma_2 & \cdots \\ 0 & \Gamma_1 & \Gamma_2 & \cdots \\ 0 & 0 & \Gamma_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proof. We only have to show that $\operatorname{End}_\Gamma(\bigoplus_{\ell \geqslant 0} \tau_{n+1}^{\ell}(D\Gamma))$ has the desired form. By Lemma 3.29, we have

$$\operatorname{Hom}_\Gamma\big(\tau_{n+1}^{i}(D\Gamma), \tau_{n+1}^{j}(D\Gamma)\big) = \operatorname{Hom}_\Gamma(D\Gamma_i, D\Gamma_j) = \operatorname{Hom}_{\Gamma^{\mathrm{op}}}(\Gamma_j, \Gamma_i) = \Gamma_i$$

for any $i \geqslant j$. By Theorem 3.11(e), we have $\operatorname{Hom}_\Gamma(\tau_{n+1}^{i}(D\Gamma), \tau_{n+1}^{j}(D\Gamma)) = 0$ for any $i < j$. Thus the assertion follows. $\square$

The following crucial result gives a sufficient condition for Λ such that Γ is absolutely $(n+1)$-complete.

Lemma 3.31. *Let Λ be an absolutely n-complete algebra. Assume that there exist surjections*

$$\Lambda = \Lambda_0 \to \Lambda_1 \to \Lambda_2 \to \cdots$$

of algebras such that $\tau_n^{\ell}(D\Lambda) \simeq D\Lambda_\ell$ as Λ-modules for any $\ell \geqslant 0$. Then the cone Γ of Λ is absolutely $(n+1)$-complete.

Proof. (i) First we shall show the following assertions.

- $(1-e_\ell)M$ is a Λ_ℓ-module,
- $\mathrm{Hom}_\Lambda(e_\ell M, D\Lambda_\ell)=0$,
- $e_\ell M=\langle e_\ell\rangle M$.

Since $\tau_n^i(D\Lambda)\simeq D\Lambda_i$, the first assertion follows from Lemma 3.27. By Lemmas 2.4(e) and 3.27 again, we have $\mathrm{Hom}_\Lambda(e_\ell M, D\Lambda_\ell)=0$ and $\mathrm{Hom}_\Lambda(e_\ell M,(1-e_\ell)M)=0$. This implies that $e_\ell M$ is a sub Γ^{op}-module of M, and we have $e_\ell M=\langle e_\ell\rangle M$.

(ii) Next we shall show the assertion.

We only have to show that $\mathcal{P}(\mathcal{N})=\mathrm{add}\,\Gamma$. By Corollary 3.12, any indecomposable object in $\mathcal{P}(\mathcal{N})$ can be written as $\tau_{n+1}^\ell\mathbf{G}X$ for some indecomposable object $X\in\mathcal{M}$ such that $P:=\tau_n^\ell X$ belongs to $\mathcal{P}(\mathcal{M})$. Since Λ is absolutely n-complete, P is a projective Λ-module. By Lemma 3.28(b), we have $\tau_{n+1}^\ell\mathbf{G}X=\mathrm{Hom}_\Gamma(\Gamma_\ell,\mathbf{G}P)$, which is a direct summand of

$$\mathrm{Hom}_\Gamma(\Gamma_\ell,\mathbf{G}\Lambda)=\mathrm{Hom}_\Gamma(\Gamma_\ell,DM)=D(\Gamma_\ell\otimes_\Gamma M)=D\big(M/\langle e_\ell\rangle M\big).$$

Using observations in (i), we have

$$\begin{aligned}D\big(M/\langle e_\ell\rangle M\big)&=D(M/e_\ell M)=D\big((1-e_\ell)M\big)=\mathrm{Hom}_{\Lambda_\ell}\big((1-e_\ell)M,D\Lambda_\ell\big)\\&=\mathrm{Hom}_\Lambda(M,D\Lambda_\ell).\end{aligned}$$

This is a projective Γ-module, and we have the assertion. □

Now we are ready to prove Theorem 1.18.

By Lemmas 3.29 and 3.31, we only have to show that the $m\times m$ triangular matrix ring $\Lambda:=T_m^{(1)}(F)$ satisfies the condition for $n=1$ in Lemma 3.31. Let $\{f_1,\dots,f_m\}$ be a complete set of orthogonal primitive idempotents of Λ such that

$$f_iJ_\Lambda\simeq f_{i+1}\Lambda$$

for any $1\leqslant i<m$ as Λ-modules. Put

$$\Lambda_\ell:=\Lambda/\langle f_1+\cdots+f_\ell\rangle.$$

Then one can easily check that

$$\tau^\ell(D\Lambda)\simeq T_{m-\ell}^{(1)}(F)\simeq D\Lambda_\ell$$

holds as Λ-modules. Thus Λ satisfies the condition for $n=1$ in Lemma 3.31. □

4. Absolute *n*-cluster tilting subcategories

In this section we shall study algebras with absolute n-cluster tilting objects and prove Theorem 1.19 as an application. Let us start with the following easy observations.

Lemma 4.1. *Let Λ be a finite-dimensional algebra with an absolute n-cluster tilting object M ($n \geqslant 1$).*

(a) $\mathrm{Hom}_\Lambda(M, I)$ *is a projective-injective* $\mathrm{End}_\Lambda(M)$*-module for any injective Λ-module I.*
(b) *We have* $\mathrm{gl.dim}\,\Lambda = \mathrm{id}\, M_\Lambda$.
(c) *For any indecomposable direct summand X of M, we have either X is projective or* $\mathrm{pd}\, X_\Lambda \geqslant n$.

Proof. (a) $\mathrm{Hom}_\Lambda(M, I)$ is projective by $I \in \mathrm{add}\, M$. It is injective by $\mathrm{Hom}_\Lambda(M, I) = D\,\mathrm{Hom}_\Lambda(\nu^- I, M)$ and $\nu^- I \in \mathrm{add}\, M$.

(b) Since any $X \in \mathrm{mod}\,\Lambda$ has an exact sequence

$$0 \to M_n \to \cdots \to M_1 \to X \to 0$$

with $M_i \in \mathrm{add}\, M$ by [46, Prop. 2.4.1(2-ℓ)], we have that $\mathrm{id}\, X_\Lambda \leqslant \mathrm{id}\, M_\Lambda$.

(c) Immediate from $\mathrm{Ext}^i_\Lambda(X, \Lambda) = 0$ for any $0 < i < n$. □

Now we prove the following key result.

Proposition 4.2. *Let Λ be a finite-dimensional algebra with an absolute n-cluster tilting object M ($n \geqslant 1$). Let $\Gamma = \mathrm{End}_\Lambda(M)$. Then $\mathrm{Ext}^i_\Gamma(D\Gamma, \Gamma) = 0$ for any $0 < i \leqslant n$ if and only if* $\mathrm{gl.dim}\,\Lambda \leqslant n$.

Proof. Take an injective resolution

$$0 \to M \to I_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{n-1}} I_n \xrightarrow{f_n} \Omega^{-n-1} M \to 0. \tag{18}$$

Applying $\mathbf{F} = \mathrm{Hom}_\Lambda(M, -)$, we have an exact sequence

$$0 \to \Gamma \to \mathbf{F} I_0 \to \cdots \to \mathbf{F} I_n \to C \to 0 \tag{19}$$

where we use $\mathrm{Ext}^i_\Lambda(M, M) = 0$ for any $0 < i < n$. By Lemma 4.1(a), each $\mathbf{F} I_i$ is a projective-injective Γ-module. Since we have $\mathrm{gl.dim}\,\Gamma \leqslant n + 1$ by Theorem 1.10, we have that C is injective. Conversely any indecomposable injective non-projective Γ-module I is isomorphic to a summand of C since any indecomposable injective Γ-module appears in the minimal injective resolution of the Γ-module Γ by $\mathrm{gl.dim}\,\Gamma < \infty$. Hence we only have to show that $\mathrm{Ext}^i_\Gamma(C, \Gamma) = 0$ holds for any $0 < i \leqslant n$ if and only if $\mathrm{gl.dim}\,\Lambda \leqslant n$.

Applying $\mathrm{Hom}_\Gamma(-, \Gamma)$ to the projective resolution (19) of C and using Yoneda's Lemma, we have an isomorphism

$$\begin{array}{ccccc}
\mathrm{Hom}_\Gamma(\mathbf{F} I_n, \Gamma) & \to \cdots \to & \mathrm{Hom}_\Gamma(\mathbf{F} I_0, \Gamma) & \to & \mathrm{Hom}_\Gamma(\Gamma, \Gamma) \\
\wr\uparrow & & \wr\uparrow & & \wr\uparrow \\
\mathrm{Hom}_\Lambda(I_n, M) & \to \cdots \to & \mathrm{Hom}_\Lambda(I_0, M) & \to & \mathrm{Hom}_\Lambda(M, M)
\end{array}$$

of complexes. Thus $\mathrm{Ext}^i_\Gamma(C, \Gamma) = 0$ for any $0 < i \leqslant n$ if and only if the complex

$$\mathrm{Hom}_\Lambda(I_n, M) \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_0} \mathrm{Hom}_\Lambda(I_0, M) \to \mathrm{Hom}_\Lambda(M, M) \tag{20}$$

obtained by applying $\mathrm{Hom}_\Lambda(-, M)$ to (18) is exact.

Using the condition $\mathrm{Ext}^i_\Lambda(D\Lambda, M) = 0$ for any $0 < i < n$, one can easily check that the following conditions are equivalent.

- (20) is exact at $\mathrm{Hom}_\Lambda(I_i, M)$ for any $0 < i < n$.
- $\mathrm{Ext}^i_\Lambda(\Omega^{-n-1}M, M) = 0$ for any $0 < i < n$.

Since M is an n-cluster tilting object in $\mathrm{mod}\,\Lambda$, this is equivalent to

- $\Omega^{-n-1}M \in \mathrm{add}\,M$.

Again using the condition $\mathrm{Ext}^i_\Lambda(D\Lambda, M) = 0$ for any $0 < i < n$, one can easily check that the following conditions are equivalent.

- (20) is exact at $\mathrm{Hom}_\Lambda(I_0, M)$.
- $\mathrm{Ext}^1_\Lambda(\mathrm{Im}\, f_1, M) = 0$.
- $\cdots$
- $\mathrm{Ext}^{n-1}_\Lambda(\mathrm{Im}\, f_{n-1}, M) = 0$.
- $\mathrm{Ext}^n_\Lambda(\Omega^{-n-1}M, M) \xrightarrow{f_n} \mathrm{Ext}^n_\Lambda(I_n, M)$ is injective.

Using Auslander–Reiten duality, this is equivalent to

- $\underline{\mathrm{Hom}}_\Lambda(\tau_n^- M, I_n) \xrightarrow{f_n} \underline{\mathrm{Hom}}_\Lambda(\tau_n^- M, \Omega^{-n-1}M)$ is surjective.

Since we have $\mathrm{add}\,M = \mathrm{add}(\Lambda \oplus \tau_n^- M)$ by Proposition 1.2(a), this is equivalent to

- $\mathrm{Hom}_\Lambda(M, I_n) \xrightarrow{f_n} \mathrm{Hom}(M, \Omega^{-n-1}M)$ is surjective.

Consequently (20) is exact if and only if $\Omega^{-n-1}M \in \mathrm{add}\,M$ and $\mathrm{Hom}_\Lambda(M, I_n) \xrightarrow{f_n} \mathrm{Hom}_\Lambda(M, \Omega^{-n-1}M)$ is surjective. This occurs if and only if f_n is a split epimorphism if and only if $\mathrm{id}\, M_\Lambda \leqslant n$ if and only if $\mathrm{gl.dim}\,\Lambda \leqslant n$ by Lemma 4.1(b). □

Now we prove the following crucial result, which gives the inductive step in our proof of Theorem 1.19.

Proposition 4.3. *Let Λ be a finite-dimensional algebra with an absolute n-cluster tilting object M ($n \geqslant 1$). If $\Gamma = \mathrm{End}_\Lambda(M)$ has an absolute $(n+1)$-cluster tilting subcategory, then* $\mathrm{gl.dim}\,\Lambda \leqslant n \leqslant \mathrm{dom.dim}\,\Lambda$.

Proof. We have shown $\mathrm{gl.dim}\,\Lambda \leqslant n$ in Proposition 4.2. Take a minimal injective resolution

$$0 \to \Lambda \to I_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{n-2}} I_{n-1} \xrightarrow{f_{n-1}} I_n \to 0 \tag{21}$$

of the Λ-module Λ. Applying $\mathbf{F} = \mathrm{Hom}_\Lambda(M, -)$, we have an exact sequence

$$0 \to \mathbf{F}\Lambda \to \mathbf{F}I_0 \xrightarrow{\mathbf{F}f_0} \cdots \xrightarrow{\mathbf{F}f_{n-2}} \mathbf{F}I_{n-1} \xrightarrow{\mathbf{F}f_{n-1}} \mathbf{F}I_n$$

of Γ-modules since we have $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for any $0 < i < n$.

Now we put $X := \mathrm{Cok}(\mathbf{F}f_{n-1})$ and $Y := \tau_{n+1}X$. We have $\mathrm{gl.dim}\,\Gamma \leqslant n+1$ by Theorem 1.10. Since each $\mathbf{F}I_i$ is a projective-injective Γ-module by Lemma 4.1(a), we have that X is an injective Γ-module. Hence both X and Y belong to our absolute $(n+1)$-cluster tilting subcategory by Proposition 1.2(a).

Applying Lemma 3.6(b) and (c) (replace n there by $n+1$) to the sequence (21), we have an exact sequence

$$0 = \mathbf{F}\tau_n\Lambda \to \mathbf{F}\tau_n I_0 \xrightarrow{\mathbf{F}\tau_n f_0} \cdots \xrightarrow{\mathbf{F}\tau_n f_{n-2}} \mathbf{F}\tau_n I_{n-1} \xrightarrow{\mathbf{F}\tau_n f_{n-1}} \mathbf{F}\tau_n I_n \to Y \to 0, \tag{22}$$

which gives a projective resolution of the Γ-module Y. Thus we have $\mathrm{pd}\,Y_\Gamma \leqslant n$. By Lemma 4.1(c) (replace n there by $n+1$), we have that Y is a projective Γ-module.

Since each f_i in (21) belongs to $J_{\mathrm{mod}\,\Lambda}$, each $\mathbf{F}\tau_n f_i$ in (22) belongs to $J_{\mathrm{mod}\,\Gamma}$. Hence (22) is a minimal projective resolution of Y. Consequently we have $\tau_n I_i = 0$ for any $0 \leqslant i < n$. Thus we have $\mathrm{pd}(I_i)_\Lambda < n$ for any $0 \leqslant i < n$. Again by Lemma 4.1(c), we have that I_i is a projective Λ-module for any $0 \leqslant i < n$. Thus $\mathrm{dom.dim}\,\Lambda \geqslant n$ holds. $\square$

Now we are ready to prove Theorem 1.19.

By Theorem 1.18, we only have to show 'only if' part. The assertion for $n = 1$ follows automatically from Proposition 1.17.

We assume $n \geqslant 2$ and put $\Lambda^{(n)} := \Lambda$. By Theorem 1.10, there exists a finite-dimensional algebra $\Lambda^{(n-1)}$ and an absolute $(n-1)$-cluster tilting object $M^{(n-1)}$ in $\mathrm{mod}\,\Lambda^{(n-1)}$ such that $\Lambda^{(n)}$ is isomorphic to $\mathrm{End}_{\Lambda^{(n-1)}}(M^{(n-1)})$. Clearly $\Lambda^{(n-1)}$ is also ring-indecomposable. Applying Proposition 4.3 to $(\Lambda, \Gamma) := (\Lambda^{(n-1)}, \Lambda^{(n)})$, we have $\mathrm{gl.dim}\,\Lambda^{(n-1)} \leqslant n-1 \leqslant \mathrm{dom.dim}\,\Lambda^{(n-1)}$.

Repeating similar argument, we have a ring-indecomposable finite-dimensional algebra $\Lambda^{(i)}$ $(1 \leqslant i \leqslant n)$ satisfying the following conditions:

(a) $\mathrm{gl.dim}\,\Lambda^{(i)} \leqslant i \leqslant \mathrm{dom.dim}\,\Lambda^{(i)}$,
(b) $\Lambda^{(i)}$ has an absolute i-cluster tilting object $M^{(i)}$,
(c) $\Lambda^{(i+1)}$ is isomorphic to $\mathrm{End}_{\Lambda^{(i)}}(M^{(i)})$.

Since $\mathrm{gl.dim}\,\Lambda^{(1)} \leqslant 1 \leqslant \mathrm{dom.dim}\,\Lambda^{(1)}$, we have that $\Lambda^{(1)}$ is Morita equivalent to $T_m^{(1)}(F)$ for some division algebra F and $m \geqslant 1$ by Proposition 1.17.

Using the conditions (b) and (c) inductively, we have that $\Lambda^{(i)}$ is Morita equivalent to our absolutely i-complete algebra $T_m^{(i)}(F)$ for any $1 \leqslant i \leqslant n$ since any $\mathrm{add}\,M^{(i)}$ is a unique absolute i-cluster tilting subcategory by Theorem 1.6. Thus we have the assertion. $\square$

5. n-Cluster tilting in derived categories

Throughout this section, let Λ be a finite-dimensional algebra with $\mathrm{id}_\Lambda \Lambda = \mathrm{id}\,\Lambda_\Lambda < \infty$. We denote by $\mathcal{D} := \mathcal{K}^{\mathrm{b}}(\mathrm{pr}\,\Lambda)$ the homotopy category of bounded complexes of finitely generated projective Λ-modules, and we identify it with $\mathcal{K}^{\mathrm{b}}(\mathrm{in}\,\Lambda)$. We consider the following subcategories of $\mathcal{D}$, which form a t-structure of $\mathcal{D}$ if $\mathrm{gl.dim}\,\Lambda < \infty$.

$$\mathcal{D}^{\leqslant 0} := \{X \in \mathcal{D} \mid H^i(X) = 0 \text{ for any } i > 0\},$$
$$\mathcal{D}^{\geqslant 0} := \{X \in \mathcal{D} \mid H^i(X) = 0 \text{ for any } i < 0\}.$$

Let us start with the following simple observation.

Lemma 5.1. *Let Λ be a finite-dimensional algebra with* $\mathrm{gl.dim}\,\Lambda < \infty$.

(a) *For any $X \in \mathcal{D}$, there exist only finitely many integers i satisfying* $\mathrm{Hom}_{\mathcal{D}}(\mathrm{mod}\,\Lambda, X[i]) \neq 0$.
(b) $\mathrm{mod}\,\Lambda$ *is a functorially finite subcategory of $\mathcal{D}$.*

Proof. (a) Take a sufficiently large integer k such that $H^i(X) = 0$ holds if $i < -k$ or $k < i$. Then we have $\mathrm{Hom}_{\mathcal{D}}(\mathrm{mod}\,\Lambda, X[i]) = 0$ if either $i < -k$ or $k + \mathrm{gl.dim}\,\Lambda < i$.

(b) We only show contravariant finiteness. Any $X \in \mathcal{D}$ is isomorphic to a bounded complex $(\cdots \to I^i \to I^{i+1} \to \cdots)$ of injective Λ-modules. It is easily checked that the natural map $Z^0 \to X$ is a right $(\mathrm{mod}\,\Lambda)$-approximation of X. □

The following easy observation is quite useful.

Lemma 5.2. *Let Λ be a finite-dimensional algebra with* $\mathrm{gl.dim}\,\Lambda \leqslant n$.

(a) *If $X \in \mathcal{D}$ satisfies $H^i(X) = 0$ for any $0 < i < n$, then $X \simeq Y \oplus Z$ for some $Y \in \mathcal{D}^{\leqslant 0}$ and $Z \in \mathcal{D}^{\geqslant n}$.*
(b) *If $X \in \mathcal{D}$ satisfies $H^i(X) = 0$ for any integer $i \notin n\mathbf{Z}$, then X is isomorphic to $\bigoplus_{\ell \in \mathbf{Z}} H^{\ell n}(X)[-\ell n]$.*

Proof. (a) We can assume that X is a bounded complex $(\cdots \to I^i \xrightarrow{d^i} I^{i+1} \to \cdots)$ of finitely generated injective Λ-modules. Put

$$Y := (\cdots \to I^{n-2} \to I^{n-1} \to \mathrm{Im}\,d^{n-1} \to 0 \to \cdots) \in \mathcal{D}^{\leqslant 0}.$$

By our assumption, we have an exact sequence

$$0 \to Z^0 \to I^0 \to I^1 \to \cdots \to I^{n-2} \to I^{n-1} \xrightarrow{d^{n-1}} I^n.$$

It follows from $\mathrm{gl.dim}\,\Lambda \leqslant n$ that $\mathrm{Im}\,d^{n-1}$ is an injective Λ-module. Thus the inclusion map $f : \mathrm{Im}\,d^{n-1} \to I^n$ splits, so there exists $g : I^n \to \mathrm{Im}\,d^{n-1}$ such that $gf = 1_{\mathrm{Im}\,d^{n-1}}$. We have the following chain homomorphism.

$$\begin{array}{ccccccccc}
Y = & (\cdots \to & I^{n-1} & \to & \mathrm{Im}\,d^{n-1} & \to & 0 & \to \cdots) \\
 & & \| & & \downarrow^{f} & & \downarrow & \\
X = & (\cdots \to & I^{n-1} & \xrightarrow{d^{n-1}} & I^n & \to & I^{n+1} & \to \cdots) \\
 & & \| & & \downarrow^{g} & & \downarrow & \\
Y = & (\cdots \to & I^{n-1} & \to & \mathrm{Im}\,d^{n-1} & \to & 0 & \to \cdots)
\end{array}$$

Thus Y is a direct summand of X, and we have the assertion.

(b) Immediate from (a). □

Now we are ready to prove Theorem 1.21. Put $\mathcal{B} := \mathcal{C}[n\mathbf{Z}]$.

(i) We shall show that $\mathcal{B}$ is a functorially finite subcategory of $\mathcal{D}$.

We only show contravariant finiteness. Fix $X \in \mathcal{D}$. By Lemma 5.1(a), there exist only finitely many integers ℓ such that $\mathrm{Hom}_{\mathcal{D}}(\mathcal{C}[\ell n], X) \neq 0$. Since $\mathcal{C}$ is a functorially finite subcategory of $\mathrm{mod}\,\Lambda$, we have that $\mathcal{C}[\ell n]$ is a contravariantly finite subcategory of $\mathcal{D}$ for any ℓ by Lemma 5.1(b). Thus there exists a right $\mathcal{B}$-approximation of X.

(ii) We shall show that $\mathcal{B}$ is n-rigid. We only have to show $\mathrm{Hom}_{\mathcal{D}}(\mathcal{C}[kn], \mathcal{C}[\ell n + i]) = 0$ for $k, \ell \in \mathbf{Z}$ and i $(0 < i < n)$. If $k > \ell$, then this is clearly zero. If $k < \ell$, then this is zero by $\mathrm{gl.dim}\,\Lambda = n$. If $k = \ell$, then this is again zero by n-rigidity of $\mathcal{C}$. Thus $\mathcal{B}$ is n-rigid.

(iii) Assume that $X \in \mathcal{D}$ satisfies $\mathrm{Hom}_{\mathcal{D}}(\mathcal{B}, X[i]) = 0$ for any $0 < i < n$. Since $\Lambda[-\ell n] \in \mathcal{B}$ for any $\ell \in \mathbf{Z}$, we have

$$H^{\ell n+i}(X) \simeq \mathrm{Hom}_{\mathcal{D}}\big(\Lambda[-\ell n], X[i]\big) = 0$$

for any $\ell \in \mathbf{Z}$ and $0 < i < n$. Thus $H^i(X) = 0$ holds for any integer $i \notin n\mathbf{Z}$. Applying Lemma 5.2(b), we have $X \simeq \bigoplus_{\ell \in \mathbf{Z}} H^{\ell n}(X)[-\ell n]$. Moreover we have

$$\mathrm{Ext}^i_{\Lambda}\big(\mathcal{C}, H^{\ell n}(X)\big) \simeq \mathrm{Hom}_{\mathcal{D}}\big(\mathcal{C}, H^{\ell n}(X)[i]\big) = 0$$

for any $0 < i < n$. Since $\mathcal{C}$ is an n-cluster tilting subcategory of $\mathrm{mod}\,\Lambda$, we have $H^{\ell n}(X) \in \mathcal{C}$ for any $\ell \in \mathbf{Z}$. Thus $X \in \mathcal{B}$.

(iv) Dually we have that if $X \in \mathcal{D}$ satisfies $\mathrm{Hom}_{\mathcal{D}}(X, \mathcal{B}[i]) = 0$ for any $0 < i < n$, then $X \in \mathcal{B}$. Thus we conclude that $\mathcal{B}$ is an n-cluster tilting subcategory of $\mathcal{D}$. □

In the rest of this section we shall prove Theorem 1.23.

Our question whether $\mathcal{U}_n(\Lambda)$ forms an n-cluster tilting subcategory of $\mathcal{D}$ is closely related to the following conditions for Λ.

Definition 5.3. Define the conditions (S_n) and (T_n) for Λ as follows:

(S_n) $\mathbf{S}_n\mathcal{D}^{\geqslant 0} \subset \mathcal{D}^{\geqslant 0}$.
(T_n) $\mathbf{S}_n^{\ell}\mathcal{D}^{\geqslant 0} \subset \mathcal{D}^{\geqslant 1}$ for sufficiently large ℓ.

It is easily shown that (S_n) is equivalent to $\mathbf{S}_n^{-1}\mathcal{D}^{\leqslant 0} \subset \mathcal{D}^{\leqslant 0}$, and (T_n) is equivalent to $\mathbf{S}_n^{-\ell}\mathcal{D}^{\leqslant 0} \subset \mathcal{D}^{\leqslant -1}$ for sufficiently large ℓ.

We have the following sufficient conditions for (S_n) and (T_n).

Proposition 5.4. *Let Λ be a finite-dimensional algebra.*

(a) $\mathrm{id}_{\Lambda}\,\Lambda = \mathrm{id}\,\Lambda_{\Lambda} \leqslant n$ *holds if and only if* (S_n) *holds.*
(b) *Λ is τ_n-finite if and only if* $\mathrm{gl.dim}\,\Lambda \leqslant n$ *and* (T_n) *hold.*

Proof. (a) To show 'only if' part, take any object $X \in \mathcal{D}^{\geqslant 0}$. Then X is isomorphic to a bounded complex

$$X \simeq \big(\cdots \to 0 \to 0 \to I^0 \to I^1 \to \cdots\big)$$

of injective Λ-modules. Since $\mathrm{pd}(D\Lambda)_\Lambda \leqslant n$, we have that X is isomorphic to a bounded complex

$$X \simeq \big(\cdots \to 0 \to 0 \to P^{-n} \to P^{1-n} \to \cdots\big)$$

of projective Λ-modules by taking a projective resolution. Applying $\mathbf{S}_n$, we have

$$\mathbf{S}_n X \simeq \big(\cdots \to 0 \to 0 \to \overset{0}{\nu\big(P^{-n}\big)} \to \overset{1}{\nu\big(P^{1-n}\big)} \to \cdots\big) \in \mathcal{D}^{\geqslant 0}.$$

Conversely, assume $\mathrm{id}\,\Lambda_\Lambda > n$. Take $X \in \mathrm{mod}\,\Lambda$ such that $\mathrm{Ext}^{n+1}_\Lambda(X, \Lambda) \neq 0$. Since $H^{-1}(\mathbf{S}_n X) \simeq D\,\mathrm{Ext}^{n+1}_\Lambda(X, \Lambda) \neq 0$, we have $\mathbf{S}_n \mathcal{D}^{\geqslant 0} \not\subset \mathcal{D}^{\geqslant 0}$.

To prove (b), we need the following relationship between two functors τ_n and $\mathbf{S}_n$.

Lemma 5.5. *Let Λ be a finite-dimensional algebra satisfying* $\mathrm{gl.dim}\,\Lambda \leqslant n$.

(a) *For any $\ell \geqslant 0$, we have an isomorphism $\tau_n^\ell H^0(-) \to H^0(\mathbf{S}_n^\ell -)$ of functors $\mathcal{D}^{\geqslant 0} \to \mathrm{mod}\,\Lambda$.*
(b) *For any $\ell \geqslant 0$, we have an isomorphism $\tau_n^{-\ell} H^0(-) \to H^0(\mathbf{S}_n^{-\ell} -)$ of functors $\mathcal{D}^{\leqslant 0} \to \mathrm{mod}\,\Lambda$.*

Proof. (a) We only have to show the case $\ell = 1$. We have a morphism $\gamma : H^0(-) \to \mathrm{id}$ of endofunctors $\mathcal{D}^{\geqslant 0} \to \mathcal{D}^{\geqslant 0}$. We shall show that $H^0(\mathbf{S}_n\gamma)$ gives the desired isomorphism. For any $X \in \mathcal{D}^{\geqslant 0}$, we have a triangle $Y[-1] \to H^0(X) \overset{\gamma_X}{\to} X \to Y$ with $Y \in \mathcal{D}^{\geqslant 1}$. Since ($\mathrm{S}_n$) holds by $\mathrm{gl.dim}\,\Lambda \leqslant n$ and Proposition 5.4(a), we have a triangle $\mathbf{S}_n Y[-1] \to \mathbf{S}_n H^0(X) \xrightarrow{\mathbf{S}_n\gamma_X} \mathbf{S}_n X \to \mathbf{S}_n Y$ with $\mathbf{S}_n Y \in \mathcal{D}^{\geqslant 1}$. Applying H^0, we have an isomorphism

$$\tau_n H^0(X) \simeq DH^n\big(\mathbf{R}\,\mathrm{Hom}_\Lambda\big(H^0(X), \Lambda\big)\big) \simeq H^0\big(\mathbf{S}_n H^0(X)\big) \xrightarrow{H^0(\mathbf{S}_n\gamma_X)} H^0(\mathbf{S}_n X).$$

(b) This is shown dually. □

Now we shall show Proposition 5.4(b). Both conditions imply $\mathrm{gl.dim}\,\Lambda \leqslant n$. By (a), we have that Λ satisfies (S_n). By Lemma 5.5(a), we have that $\tau_n^\ell = 0$ holds for sufficiently large ℓ if and only if (T_n) holds. Thus the assertion holds. □

Now we can prove Proposition 1.24.

This is a direct consequence of Proposition 5.4(b) since both global dimension and the condition (T_n) are left–right symmetric. □

We give easy properties of the condition (T_n).

Lemma 5.6. *Let Λ be a finite-dimensional algebra satisfying* (T_n).

(a) *For any $X, Y \in \mathcal{D}$, there exist only finitely many integers ℓ satisfying* $\mathrm{Hom}_{\mathcal{D}}(X, \mathbf{S}_n^\ell Y) \neq 0$.
(b) *If a finite-dimensional algebra Γ is derived equivalent to Λ, then it also satisfies* (T_n).

Proof. (a) Since we have $\mathrm{Hom}_{\mathcal{D}}(X, \mathcal{D}^{\leqslant -k}) = 0 = \mathrm{Hom}_{\mathcal{D}}(X, \mathcal{D}^{\geqslant k})$ for sufficiently large k, we have the assertion.

(b) We identify $\mathcal{D} = \mathcal{K}^{\mathrm{b}}(\mathrm{pr}\, \Lambda)$ with $\mathcal{K}^{\mathrm{b}}(\mathrm{pr}\, \Gamma)$. We denote by $\mathcal{D}_{\Lambda}^{\leqslant 0}$ and $\mathcal{D}_{\Gamma}^{\leqslant 0}$ the subcategories of $\mathcal{D}$ given by Λ and Γ respectively. We have $\Gamma \in \mathcal{D}_{\Lambda}^{\leqslant k}$ and $\Lambda \in \mathcal{D}_{\Gamma}^{\leqslant k}$ for sufficiently large k. Then we have $\mathcal{D}_{\Gamma}^{\leqslant 0} \subset \mathcal{D}_{\Lambda}^{\leqslant k}$ and $\mathcal{D}_{\Lambda}^{\leqslant 0} \in \mathcal{D}_{\Gamma}^{\leqslant k}$. Then we have

$$\mathbf{S}_n^{-\ell(2k+1)} \mathcal{D}_{\Gamma}^{\leqslant 0} \subset \mathbf{S}_n^{-\ell(2k+1)} \mathcal{D}_{\Lambda}^{\leqslant k} \subset \mathcal{D}_{\Lambda}^{\leqslant -k-1} \subset \mathcal{D}_{\Gamma}^{\leqslant -1}.$$

Thus Γ also satisfies (T_n). $\square$

We also need the result below. The assertion (b) was independently given by Amiot in [1, Prop. 5.4.2] (see also [2, Th. 4.10]) and Barot, Fernandez, Platzeck, Pratti and Trepode [16] for the case $n = 2$.

Proposition 5.7. *Let Λ be a finite-dimensional algebra.*

(a) *If Λ satisfies (S_n), then $\mathcal{U}_n(\Lambda)$ is n-rigid.*
(b) *If Λ satisfies $\mathrm{gl.dim}\, \Lambda \leqslant n$ and (T_n), then $\mathcal{U}_n(\Lambda)$ is an n-cluster tilting subcategory of $\mathcal{D}$.*

Proof. (a) Since $\Lambda \in \mathcal{D}^{\leqslant 0}$, we have $\mathbf{S}_n^{-\ell} \Lambda \in \mathcal{D}^{\leqslant 0}$ for any $\ell \geqslant 0$ by (S_n). This implies $\mathrm{Hom}_{\mathcal{D}}(\Lambda, \mathbf{S}_n^{-\ell} \Lambda[i]) = 0$ for any $\ell \geqslant 0$ and $0 < i < n$.

On the other hand, we have $\mathbf{S}_n^{\ell} \Lambda = \mathbf{S}_n^{\ell-1}(D\Lambda)[-n] \in \mathcal{D}^{\geqslant n}$ for any $\ell > 0$ by (S_n). This implies $\mathrm{Hom}_{\mathcal{D}}(\Lambda, \mathbf{S}_n^{\ell} \Lambda[i]) = 0$ for any $\ell > 0$ and $0 < i < n$.

(b) By (a) and Proposition 5.4(a), we have that $\mathcal{U}_n(\Lambda)$ is n-rigid. By Lemma 5.6(a), we have that $\mathcal{U}_n(\Lambda)$ is a functorially finite subcategory of $\mathcal{D}$.

Fix any indecomposable object $X \in \mathcal{D}$. Since $\mathcal{U}_n(\Lambda)$ is closed under $\mathbf{S}_n^{\pm 1}$, the following conditions are equivalent (e.g. [51, Prop. 3.5]).

- $\mathrm{Hom}_{\mathcal{D}}(\mathcal{U}_n(\Lambda), X[i]) = 0$ holds for any $0 < i < n$,
- $\mathrm{Hom}_{\mathcal{D}}(X, \mathcal{U}_n(\Lambda)[i]) = 0$ holds for any $0 < i < n$.

Thus it remains to show that, if these conditions are satisfied, then $X \in \mathcal{U}_n(\Lambda)$. By (T_n), there exists an integer ℓ such that $\mathbf{S}_n^{\ell} X \in \mathcal{D}^{\leqslant 0}$ and $\mathbf{S}_n^{\ell+1} X \notin \mathcal{D}^{\leqslant 0}$. Put $Y := \mathbf{S}_n^{\ell+1} X$. Then $\mathbf{S}_n^{-1} Y \in \mathcal{D}^{\leqslant 0}$ is isomorphic to a bounded complex

$$\mathbf{S}_n^{-1} Y \simeq \left(\cdots \to P^{-1} \to P^0 \to 0 \to 0 \to \cdots \right)$$

of projective Λ-modules, and Y is isomorphic to a bounded complex

$$Y \simeq \left(\cdots \to \overset{n-1}{\nu\left(P^{-1}\right)} \to \overset{n}{\nu\left(P^0\right)} \to 0 \to 0 \to \cdots \right) \tag{23}$$

of injective Λ-modules. On the other hand, since $\mathrm{Hom}_{\mathcal{D}}(\Lambda, Y[i]) = 0$ for any $0 < i < n$, we have $H^i(Y) = 0$ for any $0 < i < n$. Since Y is indecomposable and does not belong to $\mathcal{D}^{\leqslant 0}$, we have $Y \in \mathcal{D}^{\geqslant n}$ by Lemma 5.2(a). Thus the complex (23) is exact except $\nu(P^0)$. This implies $Y = H^n(Y)[-n]$ and that $H^n(Y)$ is an injective Λ-module. Consequently we have $Y \in \mathrm{add}(D\Lambda)[-n] \subset \mathcal{U}_n(\Lambda)$ and $X = \mathbf{S}_n^{-\ell-1} Y \in \mathcal{U}_n(\Lambda)$. $\square$

Now we are ready to prove Theorem 1.23.

We only have to show the latter assertion. By Proposition 5.4(b), we have that Λ satisfies (T_n). Let $T \in \mathcal{D}$ be a tilting complex of Λ such that $\Gamma = \mathrm{End}_{\mathcal{D}}(T)$ satisfies $\mathrm{gl.dim}\,\Gamma \leqslant n$. Since Γ is derived equivalent to Λ, we can identify $\mathcal{D}$ with $\mathcal{K}^{\mathrm{b}}(\mathrm{pr}\,\Gamma)$, and we have that Γ satisfies (T_n) by Lemma 5.6(b). Thus $\mathcal{U}_n(T) = \mathcal{U}_n(\Gamma)$ is an n-cluster tilting subcategory of $\mathcal{D}$ by Proposition 5.7(b). □

6. Auslander–Reiten quivers and relations

Throughout this section, we assume that the base field k is algebraically closed for simplicity. For an arrow or a path a in a quiver Q, we denote by $s(a)$ the start vertex and by $e(a)$ the end vertex.

Definition 6.1.

(a) A *weak translation quiver* $Q = (Q_0, Q_1, \tau)$ consists of a quiver (Q_0, Q_1) with a bijection

$$\tau : Q_P \to Q_I$$

for fixed subsets Q_P and Q_I of Q_0. Here we do not assume any relationship between τ and arrows in Q. We write $\tau x = 0$ symbolically for any $x \in Q_0 \backslash Q_P$.

(b) Let Λ be an n-complete algebra and $\mathcal{M} = \mathcal{M}_n(D\Lambda)$ the τ_n-closure of $D\Lambda$. Define a weak translation quiver $Q = (Q_0, Q_1, \tau_n)$ called the *Auslander–Reiten quiver* of $\mathcal{M}$ as follows:
- Q_0 (respectively, Q_P, Q_I) is the set of isoclasses of indecomposable objects in $\mathcal{M}$ (respectively, $\mathcal{M}_P$, $\mathcal{M}_I$).
- For $X, Y \in Q_0$, put $d_{XY} := \dim_k(J_{\mathcal{M}}(X, Y)/J^2_{\mathcal{M}}(X, Y))$ and draw d_{XY} arrows from X to Y.
- $\tau_n : Q_P \to Q_I$ is given by the equivalence $\tau_n : \mathcal{M}_P \to \mathcal{M}_I$.

(c) Again let Λ be an n-complete algebra and $\mathcal{U} = \mathcal{U}_n(D\Lambda)$ the $\mathbf{S}_n$-closure of $D\Lambda$. Define a weak translation quiver $Q = (Q_0, Q_1, \mathbf{S}_n)$ called the *Auslander–Reiten quiver* of $\mathcal{U}$ similarly, where we put $Q_P = Q_I := Q_0$ and we define $\mathbf{S}_n : Q_0 \to Q_0$ by the equivalence $\mathbf{S}_n : \mathcal{U} \to \mathcal{U}$.

For the case $n = 1$, the Auslander–Reiten quivers of $\mathcal{M} = \mathrm{mod}\,\Lambda$ and $\mathcal{U} = \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$ are usual one [13,4,34].

In the rest, let $\mathcal{C}$ be either $\mathcal{M}$ or $\mathcal{U}$ in Definition 6.1. By the following well-known fact, all source morphisms in $\mathcal{C}$ give the Auslander–Reiten quiver.

Lemma 6.2. *Let $X, Y \in \mathcal{C}$ be indecomposable objects and $f_0 : X \to M_1$ a source morphism. Then d_{XY} is equal to the number of Y appearing in the direct sum decomposition of M_1.*

Proof. The source morphism $f_0 : X \to M_1$ induces an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(M_1, Y)/J_{\mathcal{C}}(M_1, Y) \simeq J_{\mathcal{C}}(X, Y)/J^2_{\mathcal{C}}(X, Y).$$

Since we assumed that k is algebraically closed, we have that $\dim_k(\mathrm{Hom}_{\mathcal{C}}(M_1, Y)/J_{\mathcal{C}}(M_1, Y))$ is equal to the number of Y appearing in the direct sum decomposition of M_1. Thus we have the assertion. □

We consider a presentation of the category $\mathcal{C}$ by its Auslander–Reiten quiver with relations.

Definition 6.3. For a quiver Q, define an additive category $P(Q)$ called the *path category* of Q as follows.

- The set of indecomposable objects in $P(Q)$ is Q_0.
- For any $x, y \in Q_0$, $\mathrm{Hom}_{P(Q)}(x, y)$ is a k-vector space with the basis consisting of all paths from x to y in Q.

The presentation of $\mathcal{C}$ can be decided similarly to algebras. By (a) below, the category $\mathcal{C}$ is equivalent to some factor category $P(Q)/I$ of the path category $P(Q)$ of the Auslander–Reiten quiver Q of $\mathcal{C}$. By (b) below, the first two terms of source sequences in $\mathcal{C}$ describe generators of I.

Lemma 6.4. *Let Q be the Auslander–Reiten quiver of $\mathcal{C}$.*

(a) *Assume that we have a morphism $\mathbf{P}(a) \in J_{\mathcal{C}}(X, Y)$ for any arrow $a : X \to Y$ in Q, and that $\{\mathbf{P}(a) \mid s(a) = X,\ e(a) = Y\}$ forms a k-basis of $J_{\mathcal{C}}(X, Y)/J^2_{\mathcal{C}}(X, Y)$ for any $X, Y \in Q_0$. Then $\mathbf{P}$ extends uniquely to a full dense functor $\mathbf{P} : P(Q) \to \mathcal{C}$.*

(b) *Assume that we have a full dense functor $\mathbf{P} : P(Q) \to \mathcal{C}$, and that any $x \in Q_0$ has the source sequence with the first two terms*

$$\mathbf{P}(x) \xrightarrow{(a)} \bigoplus_{a\in Q_1, s(a)=x} \mathbf{P}\big(e(a)\big) \xrightarrow{(\mathbf{P}(r_{a,i}))} \bigoplus_{1\leqslant i\leqslant m_x} \mathbf{P}\big(e(r_{a,i})\big).$$

Then the kernel of $\mathbf{P}$ is generated by $\{\sum_{a\in Q_1, s(a)=x} r_{a,i}a \mid x \in Q_0,\ 1 \leqslant i \leqslant m_x\}$.

Proof. Since Q is locally finite and acyclic, the path category of Q coincides with its complete path category. We refer to [22, Prop. 3.1(b)] for (a), and to [22, Prop. 3.6] for (b). □

If Q is the Auslander–Reiten quiver of $\mathcal{C}$, then we often identify objects of $\mathcal{C}$ with those of $P(Q)$, and we denote the image $\mathbf{P}(a)$ of a morphism a in $P(Q)$ under $\mathbf{P}$ by the same letter a.

6.1. Cones and cylinders of weak translation quivers

Throughout this subsection, let Λ be an n-complete algebra with the τ_n-closure $\mathcal{M} = \mathcal{M}_n(D\Lambda) = \mathrm{add}\, M$ of $D\Lambda$. We denote by $Q = (Q_0, Q_1, \tau_n)$ the Auslander–Reiten quiver of $\mathcal{M}$. Then $\Gamma := \mathrm{End}_\Lambda(M)$ is $(n+1)$-complete by Theorem 1.14, so satisfies the conditions (A_{n+1})–(C_{n+1}), (S_{n+1}) and (T_{n+1}). We denote by

$$\mathcal{N} = \mathcal{M}_{n+1}(D\Gamma) \quad \text{and} \quad \mathcal{U} = \mathcal{U}_{n+1}(D\Gamma)$$

the τ_{n+1}-closure and the $\mathbf{S}_{n+1}$-closure of $D\Gamma$ respectively. They are $(n+1)$-cluster tilting subcategories of $\mathrm{mod}\,\Gamma$ and $\mathcal{K}^{\mathrm{b}}(\mathrm{pr}\,\Gamma)$ respectively by Theorem 1.23. The aim of this subsection is to draw the Auslander–Reiten quivers of $\mathcal{N}$ and $\mathcal{U}$ respectively by using Q. The key construction is the following.

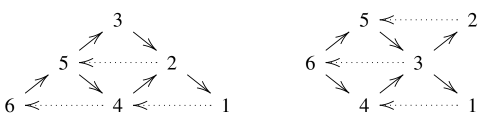
Definition 6.5. Let $Q = (Q_0, Q_1, \tau)$ be a weak translation quiver in general.

(a) We define a weak translation quiver $Q' = (Q'_0, Q'_1, \tau')$ called the *cone* of Q as follows:
- $Q'_0 := \{(x, \ell) \mid x \in Q_0,\ \ell \geqslant 0,\ \tau^\ell x \neq 0\}$.
- There are the following two kinds of arrows.
  - $(x, \ell)_1 : (x, \ell) \to (\tau x, \ell - 1)$ for any $(x, \ell) \in Q'_0$ satisfying $\ell > 0$.
  - $(a, \ell) : (x, \ell) \to (y, \ell)$ for any arrow $a : x \to y$ in Q satisfying $(x, \ell), (y, \ell) \in Q'_0$.
- $Q'_P := \{(x, \ell) \in Q'_0 \mid (x, \ell + 1) \in Q'_0\}$ and $Q'_I := \{(x, \ell) \in Q'_0 \mid \ell > 0\}$.
- Define a bijection $\tau' : Q'_P \to Q'_I$ by $\tau'(x, \ell) := (x, \ell + 1)$.

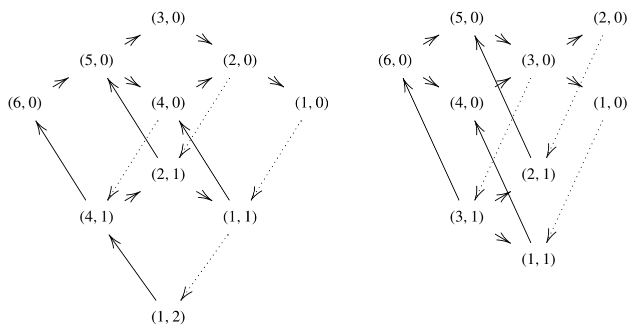
(b) We define a weak translation quiver $Q'' = (Q''_0, Q''_1, \tau'')$ called the *cylinder* of Q as follows:
- $Q''_0 = Q''_P = Q''_I := Q_0 \times \mathbf{Z} = \{(x, \ell) \mid x \in Q_0,\ \ell \in \mathbf{Z}\}$.
- There are the following two kinds of arrows.
  - $(x, \ell)_1 : (x, \ell) \to (\tau x, \ell - 1)$ for any $(x, \ell) \in Q''_0$ satisfying $x \in Q_P$.
  - $(a, \ell) : (x, \ell) \to (y, \ell)$ for any arrow $a : x \to y$ in Q and $\ell \in \mathbf{Z}$.
- Define a bijection $\tau'' : Q''_0 \to Q''_0$ by $\tau''(x, \ell) := (x, \ell + 1)$.

To simplify our description of relations below, we use the following convention: When we consider the path category $P(Q')$, we regard (x, ℓ) as a zero object if it does not belong to Q'_0, and regard (a, ℓ) as a zero morphism if it does not belong to Q'_1. We use the same convention for $P(Q'')$.

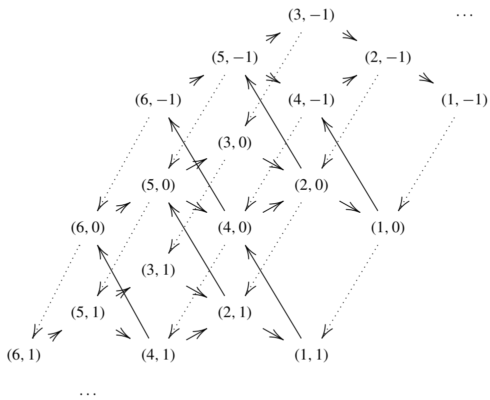
Example 6.6. Consider the following translation quivers, where dotted arrows indicate τ.

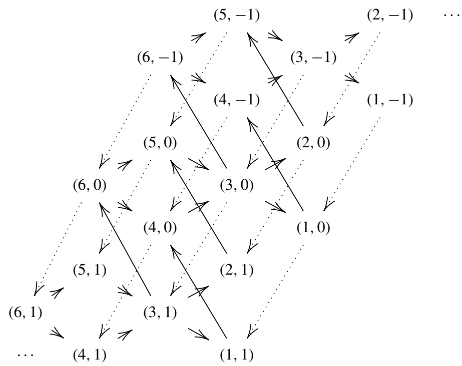


Their cones are given by the following, which coincide with the Auslander–Reiten quivers in Section 1.1.



Their cylinders are given by the following.





To draw the Auslander–Reiten quivers of $\mathcal{N}$ and $\mathcal{U}$, we introduce some notations. We use the functor

$$\mathbf{G} := D\,\mathrm{Hom}_\Lambda(-, M) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma.$$

For any path $p = a_1 \cdots a_m$ in Q and $\ell \in \mathbf{Z}$, we define a path in Q' and Q'' by

$$(p, \ell) := (a_1, \ell) \cdots (a_m, \ell).$$

For any morphism $r = \sum_p c_p p$ in $P(Q)$ with paths p in Q and $c_p \in k$, we define a morphism in $P(Q')$ and $P(Q'')$ by $(r, \ell) := \sum_p c_p (p, \ell)$. For any arrow $a : X \to Y$ in Q, we choose a morphism $\tau_n^- a : \tau_n^- X \to \tau_n^- Y$ in $P(Q)$ whose image under the functor $\mathbf{P} : P(Q) \to \mathcal{M}$ gives the image of a under the functor $P(Q) \xrightarrow{\mathbf{P}} \mathcal{M} \xrightarrow{\tau_n^-} \mathcal{M}$.

We have the following presentation of the category $\mathcal{N} = \mathcal{M}_{n+1}(D\Gamma)$.

Theorem 6.7. *Under the above circumstances, the Auslander–Reiten quiver of $\mathcal{N}$ is given by the cone $Q' = (Q'_0, Q'_1, \tau_{n+1})$ of the Auslander–Reiten quiver $Q = (Q_0, Q_1, \tau_n)$ of $\mathcal{M}$. Moreover, $\mathcal{N}$ is presented by the quiver Q' with relations*

- $(r, \ell) = 0$ *for any relation $r = 0$ for $\mathcal{M}$ and $\ell \geqslant 0$,*
- $(Y, \ell)_1 \cdot (\tau_n^- a, \ell) = (a, \ell - 1) \cdot (X, \ell)_1$ *for any arrow $a : \tau_n X \to Y$ in Q and $\ell > 0$.*

Proof. For any object $X \in \mathcal{M}$ and $\ell \geqslant 0$, we put

$$(X, \ell) := \tau_{n+1}^\ell \mathbf{G} X \in \mathcal{N}.$$

Under this notation, we have a bijection between Q'_0 and isoclasses of indecomposable objects in $\mathcal{N}$ by Corollary 3.12. Moreover Q'_P (respectively, Q'_I) corresponds to isoclasses of indecomposable objects in $\mathcal{N}_P$ (respectively, $\mathcal{N}_I$), and the equivalence $\tau_{n+1} : \mathcal{N}_P \to \mathcal{N}_I$ corresponds to the bijection $\tau' : Q'_P \to Q'_I$.

For any morphism $a : X \to Y$ in $\mathcal{M}$ (or an arrow $a : X \to Y$ in Q) and $\ell \geqslant 0$, we define a morphism

$$(a,\ell) := \tau_{n+1}^{\ell}\mathbf{G}a : (X,\ell) = \tau_{n+1}^{\ell}\mathbf{G}X \to (Y,\ell) = \tau_{n+1}^{\ell}\mathbf{G}Y$$

in $\mathcal{N}$. For any object $(X,\ell) \in \mathcal{N}$ with $\ell > 0$, we define a morphism

$$(X,\ell)_1 := \tau_{n+1}^{\ell-1}\alpha_X : (X,\ell) = \tau_{n+1}^{\ell}\mathbf{G}X \to (\tau_n X, \ell-1) = \tau_{n+1}^{\ell-1}\mathbf{G}\tau_n X$$

in $\mathcal{N}$. Under these notations, we shall describe all arrows and relations starting at each vertex $(X,\ell) \in Q'_0$. We divide into two cases.

(i) Consider the case (X,ℓ) with $\ell = 0$.

Let $X \overset{f_0}{\to} M_1 \overset{f_1}{\to} M_2$ be the first two terms of the source sequence of X in $\mathcal{M}$. By Proposition 3.19, we have the first two terms $(X,0) \overset{(f_0,0)}{\longrightarrow} (M_1,0) \overset{(f_1,0)}{\longrightarrow} (M_2,0)$ of the source sequence of $(X,0)$ in $\mathcal{N}$. By Lemma 6.2, all arrows starting at $(X,0)$ are given by $(a,0)$ for each arrow a in Q starting at X. By Lemma 6.4(b), all relations starting at $(X,0)$ are given by $(r,0) = 0$ for each relation $r = 0$ in $\mathcal{M}$ starting at X.

(ii) Consider the case (X,ℓ) with $\ell > 0$.

Let $\tau_n X \overset{f_0}{\to} M_1 \overset{f_1}{\to} M_2$ be the first two terms of the source sequence of $\tau_n X$ in $\mathcal{M}$. By Proposition 3.23, we have the first two terms

$$(X,\ell) \xrightarrow{\begin{pmatrix} (\tau_n^- f_0,\ell) \\ \tau_{n+1}^{\ell-1}\beta_{\tau_n X}\end{pmatrix}} \left(\tau_n^- M_1,\ell\right) \oplus (\tau_n X,\ell-1) \xrightarrow{\begin{pmatrix} (\tau_n^- f_1,\ell) & 0 \\ \tau_{n+1}^{\ell-1}\beta_{M_1} & -(f_0,\ell-1)\end{pmatrix}} \left(\tau_n^- M_2,\ell\right) \oplus (M_1,\ell-1)$$

of the $(n+1)$-almost split sequence of (X,ℓ) in $\mathcal{N}$. By our definition of β in Lemma 3.22, we have $\tau_{n+1}^{\ell-1}\beta_{\tau_n X} = (X,\ell)_1$ and

$$\begin{aligned}\tau_{n+1}^{\ell-1}\beta_{M_1} &\simeq \begin{pmatrix} \tau_{n+1}^{\ell-1}\alpha_{\tau_n^- M_1} \\ 0 \end{pmatrix} = \begin{pmatrix} (\tau_n^- M_1,\ell)_1 \\ 0 \end{pmatrix} : \left(\tau_n^- M_1,\ell\right) \to (M_1,\ell-1) \\ &\simeq \left(\tau_n\tau_n^- M_1,\ell-1\right) \oplus (I,\ell-1)\end{aligned}$$

for a decomposition $M_1 \simeq (\tau_n\tau_n^- M_1) \oplus I$ with $I \in \mathcal{I}(\mathcal{M})$. By Lemma 6.2, all arrows starting at $(X,0)$ are given by $(X,\ell)_1$ and $(a,0)$ for each arrow a in Q starting at X. By Lemma 6.4(b), all relations starting at (X,ℓ) appear in equalities

$$\left(\tau_n^- f_0,\ell\right)\cdot\left(\tau_n^- f_1,\ell\right) = 0 \quad \text{and} \quad \begin{pmatrix} (\tau_n^- M_1,\ell)_1 \\ 0 \end{pmatrix}\cdot\left(\tau_n^- f_0,\ell\right) = (f_0,\ell-1)\cdot(X,\ell)_1.$$

The former equality gives relations $(r,\ell) = 0$ for each relation $r = 0$ in $\mathcal{M}$ starting at X. The latter one gives relations $(Y,\ell)_1 \cdot (\tau_n^- a,\ell) = (a,\ell-1)\cdot(X,\ell)_1$ for each arrow $a : \tau_n X \to Y$ in Q. (Notice that $(Y,\ell)_1$ which does not belong to Q'_1 appears in the lower half of the morphism $\binom{(\tau_n^- M_1,\ell)_1}{0}$.)

Thus we have the desired assertions. □

Next we shall give a presentation of the category $\mathcal{U} = \mathcal{U}_{n+1}(D\Gamma)$. We need the following information, which is similar to Lemma 3.17.

Lemma 6.8. *Fix an indecomposable object $X \in \mathcal{M}$ and $\ell \in \mathbf{Z}$. Take a source morphism $f_0 : X \to M_1$ in $\mathcal{M}$.*

(a) *Any morphism $\mathbf{S}_{n+1}^{\ell}\mathbf{G}X \to \mathbf{S}_{n+1}^{i}\mathbf{G}M$ with $i > \ell$ is zero.*
(b) *Any morphism $\mathbf{S}_{n+1}^{\ell}\mathbf{G}X \to \mathbf{S}_{n+1}^{\ell}\mathbf{G}M$ which is not a split monomorphism factors through $\mathbf{S}_{n+1}^{\ell}\mathbf{G}f_0 : \mathbf{S}_{n+1}^{\ell}\mathbf{G}X \to \mathbf{S}_{n+1}^{\ell}\mathbf{G}M_1$.*
(c) *If $X \in \mathcal{M}_P$, then $\mathbf{S}_{n+1}\mathbf{G}X \simeq \tau_{n+1}\mathbf{G}X$ and any morphism $\mathbf{S}_{n+1}^{\ell}\mathbf{G}X \to \mathbf{S}_{n+1}^{i}\mathbf{G}M$ with $i < \ell$ factors through $\mathbf{S}_{n+1}^{\ell-1}\alpha_X : \mathbf{S}_{n+1}^{\ell}\mathbf{G}X \simeq \mathbf{S}_{n+1}^{\ell-1}\tau_{n+1}\mathbf{G}X \to \mathbf{S}_{n+1}^{\ell-1}\mathbf{G}\tau_n X$.*
(d) *If $X \in \mathcal{P}(\mathcal{M})$, then any morphism $\mathbf{S}_{n+1}^{\ell}\mathbf{G}X \to \mathbf{S}_{n+1}^{i}\mathbf{G}M$ with $i < \ell$ is zero.*

Proof. (a) Since $\mathbf{S}_{n+1}^{\ell-i}\mathbf{G}M = \mathbf{S}_{n+1}^{1+\ell-i}(\Gamma[n+1]) \in \mathcal{D}^{\leqslant -n-1}$ holds by (S_{n+1}), we have

$$\mathrm{Hom}_{\mathcal{D}}\big(\mathbf{S}_{n+1}^{\ell}\mathbf{G}X, \mathbf{S}_{n+1}^{i}\mathbf{G}M\big) \simeq \mathrm{Hom}_{\mathcal{D}}\big(\mathbf{S}_{n+1}^{\ell-i}\mathbf{G}X, D\Gamma\big) \simeq DH^0\big(\mathbf{S}_{n+1}^{\ell-i}\mathbf{G}X\big) = 0.$$

(b) This is clear since the functor $\mathbf{S}_{n+1}^{\ell}\mathbf{G} : \mathcal{M} \to \mathcal{U}$ is fully faithful.

(c) Since $\mathbf{G}X \in \mathcal{N}_P$ by Corollary 3.12(b), we have $\mathbf{S}_{n+1}\mathbf{G}X \simeq \tau_{n+1}\mathbf{G}X$ by (C_{n+1}). Applying $\mathbf{S}_{n+1}^{-i}$, we only have to show that

$$\begin{aligned} &DH^0\big(\mathbf{S}_{n+1}^{\ell-i-1}\alpha_X\big) : DH^0\big(\mathbf{S}_{n+1}^{\ell-i-1}\mathbf{G}\tau_n X\big) \simeq \mathrm{Hom}_{\mathcal{D}}\big(\mathbf{S}_{n+1}^{\ell-i-1}\mathbf{G}\tau_n X, D\Gamma\big) \\ &\quad \to DH^0\big(\mathbf{S}_{n+1}^{\ell-i}\mathbf{G}X\big) \simeq \mathrm{Hom}_{\mathcal{D}}\big(\mathbf{S}_{n+1}^{\ell-i}\mathbf{G}X, D\Gamma\big) \end{aligned}$$

is surjective. By Lemma 5.5 (replace n there by $n+1$), this is equal to the dual of

$$\tau_{n+1}^{\ell-i-1}\alpha_X : \tau_{n+1}^{\ell-i}\mathbf{G}X \to \tau_{n+1}^{\ell-i-1}\mathbf{G}\tau_n X.$$

This is injective since α_X is injective and the functor τ_{n+1} preserves monomorphisms.

(d) We have $\mathrm{pd}(\mathbf{G}X)_\Gamma \leqslant n$ by Corollary 3.12(b). Since $\mathbf{S}_{n+1}^{i-\ell}\mathbf{G}M = \mathbf{S}_{n+1}^{1+i-\ell}(\Gamma[n+1]) \in \mathcal{D}^{\leqslant -n-1}$ holds by (S_{n+1}), we have

$$\mathrm{Hom}_{\mathcal{D}}\big(\mathbf{S}_{n+1}^{\ell}\mathbf{G}X, \mathbf{S}_{n+1}^{i}\mathbf{G}M\big) \simeq \mathrm{Hom}_{\mathcal{D}}\big(\mathbf{G}X, \mathbf{S}_{n+1}^{i-\ell}\mathbf{G}M\big) = 0. \qquad \square$$

Consequently, we have the result (a) below which is an analogue of Proposition 3.18, and the result (b) below which is an analogue of Corollary 3.12.

Proposition 6.9.

(a) *For any object $X \in \mathcal{M}$ and $\ell \in \mathbf{Z}$, take a source morphism $f_0 : X \to M_1$ in $\mathcal{M}$.*
   (i) *If $X \in \mathcal{P}(\mathcal{M})$, then a left almost split morphism of $\mathbf{S}_{n+1}^{\ell}\mathbf{G}X$ in $\mathcal{U}$ is given by $\mathbf{S}_{n+1}^{\ell}\mathbf{G}f_0 : \mathbf{S}_{n+1}^{\ell}\mathbf{G}X \to \mathbf{S}_{n+1}^{\ell}\mathbf{G}M_1$.*

(ii) *If $X \in \mathcal{M}_P$, then a left almost split morphism of $\mathbf{S}_{n+1}^{\ell}\mathbf{G}X$ in $\mathcal{U}$ is given by*

$$\begin{pmatrix} \mathbf{S}_{n+1}^{\ell}\mathbf{G}f_0 \\ \mathbf{S}_{n+1}^{\ell-1}\alpha_X \end{pmatrix} : \mathbf{S}_{n+1}^{\ell}\mathbf{G}X \to \left(\mathbf{S}_{n+1}^{\ell}\mathbf{G}M_1\right) \oplus \left(\mathbf{S}_{n+1}^{\ell-1}\mathbf{G}\tau_n X\right).$$

(b) *A bijection from isoclasses of indecomposable objects in $\mathcal{U}$ to pairs (X, ℓ) of isoclasses of indecomposable objects $X \in \mathcal{M}$ and $\ell \in \mathbf{Z}$ is given by $\mathbf{S}_{n+1}^{\ell}\mathbf{G}X \leftrightarrow (X, \ell)$.*

Proof. (a) This is immediate from Lemma 6.8(a)–(d).

(b) Any indecomposable object in $\mathcal{U}$ is isomorphic to $\mathbf{S}_{n+1}^{\ell}\mathbf{G}X$ for some $X \in \mathcal{M}$ and $\ell \in \mathbf{Z}$. If $\mathbf{S}_{n+1}^{\ell}\mathbf{G}X \simeq \mathbf{S}_{n+1}^{\ell'}\mathbf{G}X'$, then $\ell = \ell'$ holds by Lemma 6.8(a), and $X \simeq X'$ holds since the functor $\mathbf{S}_{n+1}^{\ell}\mathbf{G} : \mathcal{M} \to \mathcal{U}$ is fully faithful. Thus we have the desired bijection. $\square$

We have the following presentation of the category $\mathcal{U} = \mathcal{U}_{n+1}(D\Gamma)$.

Theorem 6.10. *Under the above circumstances, the Auslander–Reiten quiver of $\mathcal{U}$ is given by the cylinder $Q'' = (Q_0'', Q_1'', \mathbf{S}_{n+1})$ of the Auslander–Reiten quiver $Q = (Q_0, Q_1, \tau_n)$ of $\mathcal{M}$. Moreover, $\mathcal{U}$ is presented by the quiver Q'' with relations*

- $(r, \ell) = 0$ *for any relation $r = 0$ for $\mathcal{M}$ and $\ell \in \mathbf{Z}$,*
- $(Y, \ell)_1 \cdot (a, \ell) = 0$ *for any arrow $a : X \to Y$ in Q with $X \in Q \backslash Q_P$ and $\ell \in \mathbf{Z}$,*
- $(Y, \ell)_1 \cdot (\tau_n^- a, \ell) = (a, \ell - 1) \cdot (X, \ell)_1$ *for any arrow $a : \tau_n X \to Y$ in Q and $\ell \in \mathbf{Z}$.*

Proof. The proof is similar to that of Theorem 6.7. For any object $X \in \mathcal{M}$, we put

$$(X, \ell) := \mathbf{S}_{n+1}^{\ell}\mathbf{G}X \in \mathcal{U}.$$

Under this notation, we have a bijection between Q_0'' and isoclasses of indecomposable objects in $\mathcal{U}$ by Proposition 6.9(b). For any morphism $a : X \to Y$ in $\mathcal{M}$ (or an arrow $a : X \to Y$ in Q) and $\ell \geqslant 0$, we define a morphism

$$(a, \ell) := \mathbf{S}_{n+1}^{\ell}\mathbf{G}a : (X, \ell) = \mathbf{S}_{n+1}^{\ell}\mathbf{G}X \to (Y, \ell) = \mathbf{S}_{n+1}^{\ell}\mathbf{G}Y.$$

For any object $(X, \ell) \in \mathcal{U}$, we define a morphism

$$(X, \ell)_1 := \mathbf{S}_{n+1}^{\ell-1}\alpha_X : (X, \ell) = \mathbf{S}_{n+1}^{\ell}\mathbf{G}X \to (\tau_n X, \ell - 1) = \mathbf{S}_{n+1}^{\ell-1}\mathbf{G}\tau_n X.$$

Under these notations, we shall describe all arrows and relations starting at each vertex $(X, \ell) \in Q_0''$. Since $\mathbf{S}_{n+1}$ is an autoequivalence of $\mathcal{U}$, we only have to consider two cases (i) and (ii) below.

(i) Consider the case $(X, 0)$ with $X \in \mathcal{P}(\mathcal{M})$.

Let $X \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2$ be the first two terms of the source sequence of X in $\mathcal{M}$. By Proposition 6.9(a) and Lemma 6.8(d), it is easily checked that the first two terms of the source sequence of $(X, 0)$ in $\mathcal{U}$ is given by

$$(X, 0) \xrightarrow{(f_0, 0)} (M_1, 0) \xrightarrow{\begin{pmatrix} (f_1, 0) \\ (M_1, 0)_1 \end{pmatrix}} (M_2, 0) \oplus (\tau_n M_1, 1).$$

By Lemma 6.4(b), all relations starting at $(X,0)$ are given by equalities

$$(f_1,0)\cdot(f_0,0)=0 \quad \text{and} \quad (M_1,0)_1\cdot(f_0,0)=0.$$

The former equality gives a relation $(r,0)=0$ for each relation $r=0$ in $\mathcal{M}$ starting at X. The latter equality gives a relation $(Y,0)_1\cdot(a,0)=0$ for each arrow $a:X\to Y$ in Q.

(ii) Consider the case $(X,1)$ with $X\in\mathcal{M}_P$.

Let $\tau_n X\xrightarrow{f_0}M_1\xrightarrow{f_1}M_2$ be the first two terms of the source sequence of $\tau_n X$ in $\mathcal{M}$. By Proposition 3.23, we have the first two terms

$$(X,1)\xrightarrow{\begin{pmatrix}(\tau_n^- f_0,1)\\ \beta_{\tau_n X}\end{pmatrix}}\left(\tau_n^- M_1,1\right)\oplus(\tau_n X,0)\xrightarrow{\begin{pmatrix}(\tau_n^- f_1,1) & 0\\ \beta_{M_1} & -(f_0,0)\end{pmatrix}}\left(\tau_n^- M_2,1\right)\oplus(M_1,0) \tag{24}$$

of the $(n+1)$-almost split sequence of $(X,1)$ in $\mathcal{N}$. We have $\beta_{\tau_n X}=\alpha_X=(X,1)_1$ and

$$\beta_{M_1}\simeq\begin{pmatrix}\alpha_{\tau_n^- M_1}\\ 0\end{pmatrix}=\begin{pmatrix}(\tau_n^- M_1,1)_1\\ 0\end{pmatrix}:\left(\tau_n^- M_1,1\right)\to(M_1,0)\simeq\left(\tau_n\tau_n^- M_1,0\right)\oplus(I,0)$$

for a decomposition $M_1\simeq(\tau_n\tau_n^- M_1)\oplus I$ with $I\in\mathcal{I}(\mathcal{M})$.

By Proposition 6.9 and [51, Prop. 3.9], we have that (24) is the first two terms of a source sequence of $(X,1)$ also in $\mathcal{U}$. By Lemma 6.4(b), all relations starting at $(X,1)$ are given by equalities

$$\left(\tau_n^- f_0,1\right)\cdot\left(\tau_n^- f_1,1\right)=0 \quad \text{and} \quad \begin{pmatrix}(\tau_n^- M_1,1)_1\\ 0\end{pmatrix}\cdot\left(\tau_n^- f_0,1\right)=(f_0,0)\cdot(X,1)_1.$$

The former equality gives a relation $(r,1)=0$ for each relation $r=0$ in $\mathcal{M}$ starting at X. The latter one gives a relation $(Y,1)_1\cdot(\tau_n^- a,1)=(a,0)\cdot(X,1)_1$ for each arrow $a:\tau_n X\to Y$ in Q. □

6.2. Examples

Throughout this subsection, let $Q=(Q_0,Q_1)$ be a Dynkin quiver and $\Lambda^{(1)}:=kQ$ the path algebra of Q. Let $n\geqslant 1$. By Corollary 1.16, we have an n-complete algebra $\Lambda^{(n)}$ with the cone $\Lambda^{(n+1)}$ for any $n\geqslant 1$. We denote by

$$\mathcal{M}^{(n)}:=\mathcal{M}_n\left(D\Lambda^{(n)}\right) \quad \text{and} \quad \mathcal{U}^{(n)}:=\mathcal{U}_n\left(D\Lambda^{(n)}\right)$$

the τ_n-closure and the $\mathbf{S}_n$-closure of $D\Lambda^{(n)}$ respectively. They are n-cluster tilting subcategories of $\operatorname{mod}\Lambda^{(n)}$ and $\mathcal{K}^{\mathrm{b}}(\operatorname{pr}\Lambda^{(n)})$ respectively. Let us draw the Auslander–Reiten quivers of $\mathcal{M}^{(n)}$ and $\mathcal{U}^{(n)}$. As usual, we denote by $\tau=\tau_1:\operatorname{mod}\Lambda^{(1)}\to\operatorname{mod}\Lambda^{(1)}$ the Auslander–Reiten translation of $\Lambda^{(1)}$. Let I_x be the indecomposable injective $\Lambda^{(1)}$-module corresponding to the vertex $x\in Q_0$ and

$$\ell_x:=\sup\left\{\ell\geqslant 0 \mid \tau^\ell I_x\neq 0\right\}.$$

Since Q is a Dynkin quiver, we have $\ell_x<\infty$ for any $x\in Q_0$. For $\ell\in\mathbf{Z}$, we put

$$\Delta_\ell^{(n)} := \{(\ell_1, \dots, \ell_n) \in \mathbf{Z}^n \mid \ell_1, \dots, \ell_n \geqslant 0,\ \ell_1 + \cdots + \ell_n \leqslant \ell\}.$$

For $1 \leqslant i \leqslant n$, we put

$$\boldsymbol{e}_i := (\overset{1}{0}, \dots, \overset{i-1}{0}, \overset{i}{1}, \overset{i+1}{0}, \dots, \overset{n}{0}) \in \mathbf{Z}^n \quad \text{and} \quad \boldsymbol{v}_i := \begin{cases} -\boldsymbol{e}_i & i = 1, \\ \boldsymbol{e}_{i-1} - \boldsymbol{e}_i & 1 < i \leqslant n. \end{cases}$$

Definition 6.11. Let Q be a Dynkin quiver and $n \geqslant 1$.

(a) We define a weak translation quiver $Q^{(n)} = (Q_0^{(n)}, Q_1^{(n)}, \tau_n)$ as follows:
- $Q_0^{(n)} := \{(x, \boldsymbol{\ell}) \mid x \in Q_0,\ \boldsymbol{\ell} \in \Delta_{\ell_x}^{(n)}\}$.
- There are the following $(n+1)$ kinds of arrows if their start and end vertices belong to $Q_0^{(n)}$.
  * $(a^*, \boldsymbol{\ell}) : (x, \boldsymbol{\ell}) \to (w, \boldsymbol{\ell})$ for any arrow $a : w \to x$ in Q.
  * $(b, \boldsymbol{\ell}) : (x, \boldsymbol{\ell}) \to (y, \boldsymbol{\ell} + \boldsymbol{v}_1)$ for any arrow $b : x \to y$ in Q.
  * $(x, \boldsymbol{\ell})_i : (x, \boldsymbol{\ell}) \to (x, \boldsymbol{\ell} + \boldsymbol{v}_i)$ for any $1 < i \leqslant n$.
- $Q_P^{(n)} := \{(x, \boldsymbol{\ell}) \in Q_0^{(n)} \mid (x, \boldsymbol{\ell} + \boldsymbol{e}_n) \in Q_0^{(n)}\}$ and $Q_I^{(n)} := \{(x, \boldsymbol{\ell}) \in Q_0^{(n)} \mid (x, \boldsymbol{\ell} - \boldsymbol{e}_n) \in Q_0^{(n)}\}$.
- Define a bijection $\tau_n : Q_P^{(n)} \to Q_I^{(n)}$ by $\tau_n(x, \boldsymbol{\ell}) := (x, \boldsymbol{\ell} + \boldsymbol{e}_n)$.

(b) We define a weak translation quiver $\widetilde{Q}^{(n)} = (\widetilde{Q}_0^{(n)}, \widetilde{Q}_1^{(n)}, \mathbf{S}_n)$ as follows:
- $\widetilde{Q}_0^{(n)} = \widetilde{Q}_P^{(n)} = \widetilde{Q}_I^{(n)} := \{(x, \ell_1, \dots, \ell_n) \mid x \in Q_0,\ (\ell_1, \dots, \ell_{n-1}) \in \Delta_{\ell_x}^{(n-1)},\ \ell_n \in \mathbf{Z}\}$.
- There are the following $(n+1)$ kinds of arrows if their start and end vertices belong to $\widetilde{Q}_0^{(n)}$.
  * $(a^*, \boldsymbol{\ell}) : (x, \boldsymbol{\ell}) \to (w, \boldsymbol{\ell})$ for any arrow $a : w \to x$ in Q.
  * $(b, \boldsymbol{\ell}) : (x, \boldsymbol{\ell}) \to (y, \boldsymbol{\ell} + \boldsymbol{v}_1)$ for any arrow $b : x \to y$ in Q.
  * $(x, \boldsymbol{\ell})_i : (x, \boldsymbol{\ell}) \to (x, \boldsymbol{\ell} + \boldsymbol{v}_i)$ for any $1 < i \leqslant n$.
- Define a bijection $\mathbf{S}_n : \widetilde{Q}_0^{(n)} \to \widetilde{Q}_0^{(n)}$ by $\mathbf{S}_n(x, \boldsymbol{\ell}) := (x, \boldsymbol{\ell} + \boldsymbol{e}_n)$.

To simplify our description of relations below, we use the following convention: When we consider the path category $P(Q^{(n)})$, we regard $(x, \boldsymbol{\ell})$ as a zero object if it does not belong to $Q_0^{(n)}$, and regard $(a^*, \boldsymbol{\ell})$, etc., as a zero morphism if it does not belong to $Q_1^{(n)}$. We use the same convention for $P(\widetilde{Q}^{(n)})$.

Now we have the presentations of $\mathcal{M}^{(n)}$ and $\mathcal{U}^{(n)}$ as follows.

Theorem 6.12. *Under the above circumstances, we have the following assertions.*

(a) *The Auslander–Reiten quivers of $\mathcal{M}^{(n)}$ and $\mathcal{U}^{(n)}$ are given by $Q^{(n)}$ and $\widetilde{Q}^{(n)}$ respectively.*
(b) *The categories $\mathcal{M}^{(n)}$ and $\mathcal{U}^{(n)}$ are presented by quivers $Q^{(n)}$ and $\widetilde{Q}^{(n)}$ with the following relations respectively: For any $\boldsymbol{\ell} \in \mathbf{Z}^n$ and $1 < i, j \leqslant n$,*

$$(w, \boldsymbol{\ell})_i \cdot (a^*, \boldsymbol{\ell}) = (a^*, \boldsymbol{\ell} + \boldsymbol{v}_i) \cdot (x, \boldsymbol{\ell})_i \quad \text{for any arrow } a : w \to x \text{ in } Q,$$

$$(y, \boldsymbol{\ell} + \boldsymbol{v}_1)_i \cdot (b, \boldsymbol{\ell}) = (b, \boldsymbol{\ell} + \boldsymbol{v}_i) \cdot (x, \boldsymbol{\ell})_i \quad \text{for any arrow } b : x \to y \text{ in } Q,$$

$$(x, \boldsymbol{\ell} + \boldsymbol{v}_j)_i \cdot (x, \boldsymbol{\ell})_j = (x, \boldsymbol{\ell} + \boldsymbol{v}_i)_j \cdot (x, \boldsymbol{\ell})_i \quad \text{for any } x \in Q_0,$$

$$\sum_{a\in Q_1,\ e(a)=x} (a,\boldsymbol{\ell})\cdot\big(a^*,\boldsymbol{\ell}\big) = \sum_{b\in Q_1,\ s(b)=x} \big(b^*,\boldsymbol{\ell}+\boldsymbol{v}_1\big)\cdot(b,\boldsymbol{\ell}) \quad \textit{for any } x\in Q_0.$$

We have the quiver with relations of $\Lambda^{(n+1)}$ by taking the opposite of those of $\mathcal{M}^{(n)}$ for any $n \geqslant 1$.

Proof. It is well known that the assertions are valid for $n=1$ [34].

Clearly $Q^{(n)}$ and $\widetilde{Q}^{(n)}$ are the cone and the cylinder of $Q^{(n-1)}$ respectively under the following identifications for $x\in Q_0$, $a\in Q_1$ and $\boldsymbol{\ell}\in\mathbf{Z}^{n-1}$ and

$$(x,\boldsymbol{\ell},\ell_n)\longleftrightarrow\big((x,\boldsymbol{\ell}),\ell_n\big),\qquad (a,\boldsymbol{\ell},\ell_n)\longleftrightarrow\big((a,\boldsymbol{\ell}),\ell_n\big),$$

$$\big(a^*,\boldsymbol{\ell},\ell_n\big)\longleftrightarrow\big(\big(a^*,\boldsymbol{\ell}\big),\ell_n\big),\qquad (x,\boldsymbol{\ell},\ell_n)_i\longleftrightarrow\begin{cases}((x,\boldsymbol{\ell})_i,\ell_n) & 1\leqslant i<n,\\ ((x,\boldsymbol{\ell}),\ell_n)_1 & i=n.\end{cases}$$

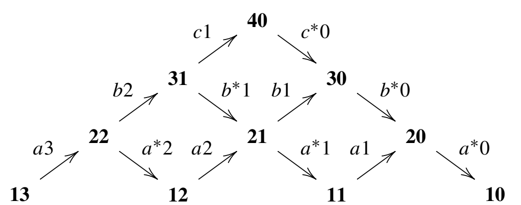
It is easily checked that our relations for $Q^{(n)}$ and $\widetilde{Q}^{(n)}$ are obtained from our relations of $Q^{(n-1)}$ by applying Theorems 6.7 and 6.10 respectively. Thus the assertion follows inductively. □

Example 6.13. For simplicity, we denote by

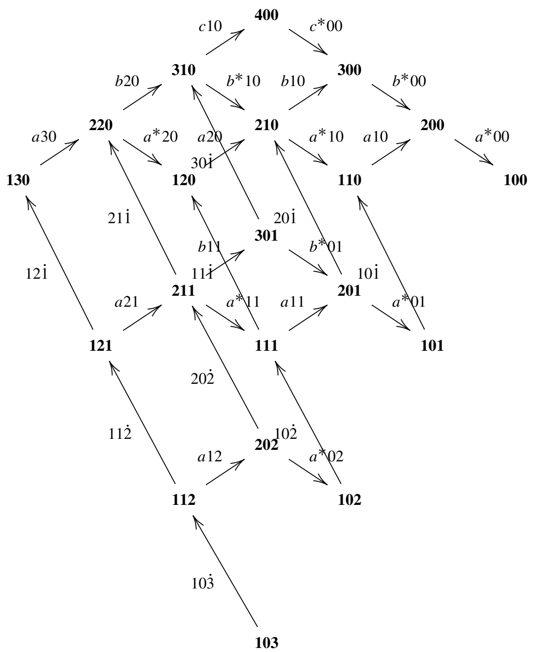
$$x\ell_1\cdots\ell_n\quad\big(\text{respectively, } x\ell_1\cdots\dot{\ell}_i\cdots\ell_n,\ a^*\ell_1\cdots\ell_n,\ b\ell_1\cdots\ell_n\big)$$

the vertex $(x,\boldsymbol{\ell})\in Q_0^{(n)}$ (respectively, the arrow $(x,\boldsymbol{\ell})_i$, $(a^*,\boldsymbol{\ell})$, $(b,\boldsymbol{\ell})\in Q_1^{(n)}$) for $\boldsymbol{\ell}=(\ell_1,\dots,\ell_n)$.

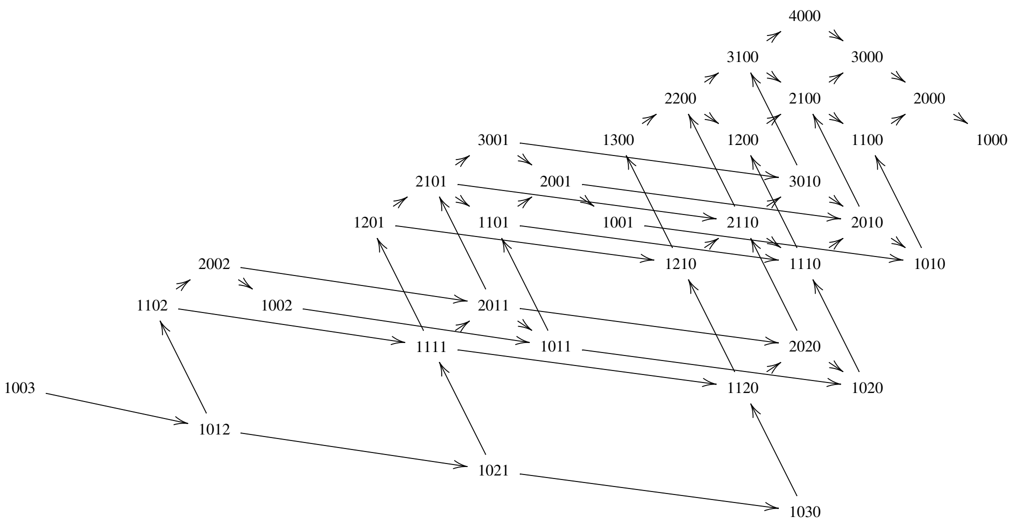
(a) Let Q be the quiver $\mathbf{1}\xrightarrow{a}\mathbf{2}\xrightarrow{b}\mathbf{3}\xrightarrow{c}\mathbf{4}$ of type A_4. In this case we have $\Lambda^{(n)}=T_4^{(n)}(k)$ in Theorem 1.18. Then the Auslander–Reiten quiver of $\mathcal{M}^{(1)}$ is the following.



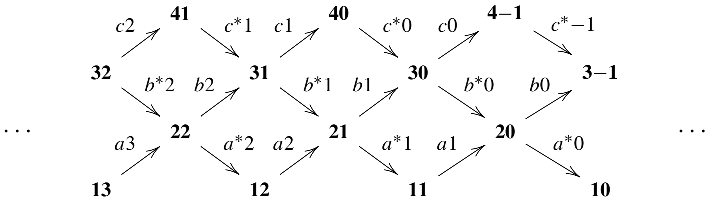
The Auslander–Reiten quiver of $\mathcal{M}^{(2)}$ is the following.



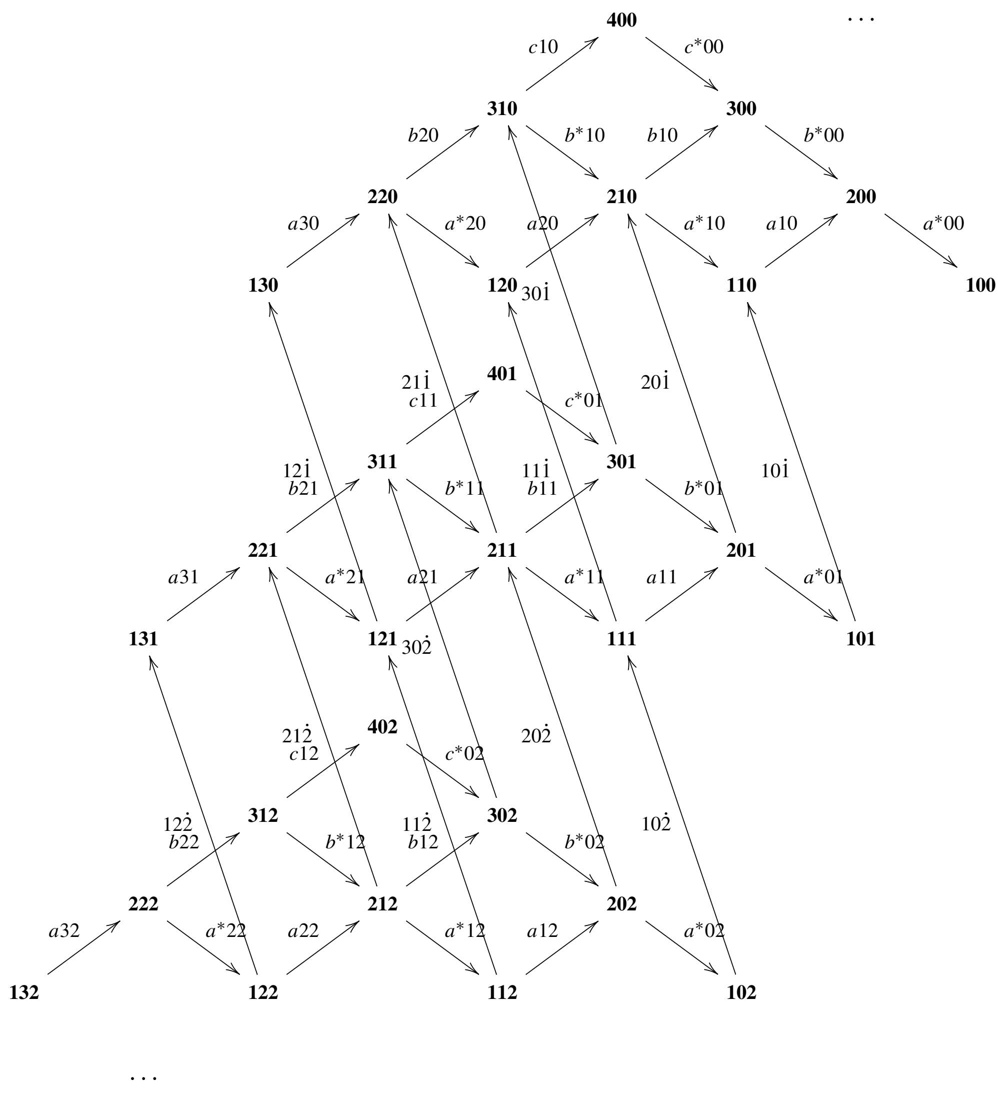
The Auslander–Reiten quiver of $\mathcal{M}^{(3)}$ is the following.



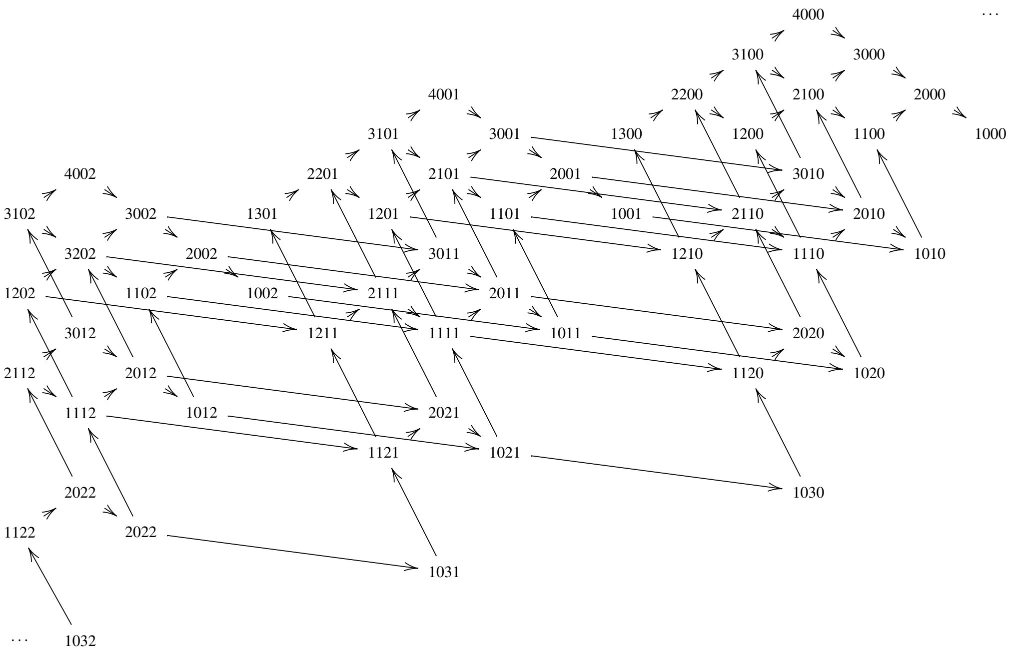
On the other hand, the Auslander–Reiten quiver of $\mathcal{U}^{(1)}$ is the following.



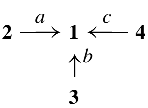
The Auslander–Reiten quiver of $\mathcal{U}^{(2)}$ is the following.



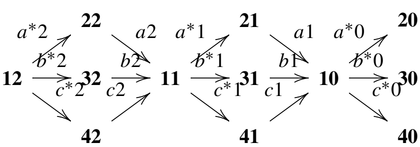
The Auslander–Reiten quiver of $\mathcal{U}^{(3)}$ is the following.



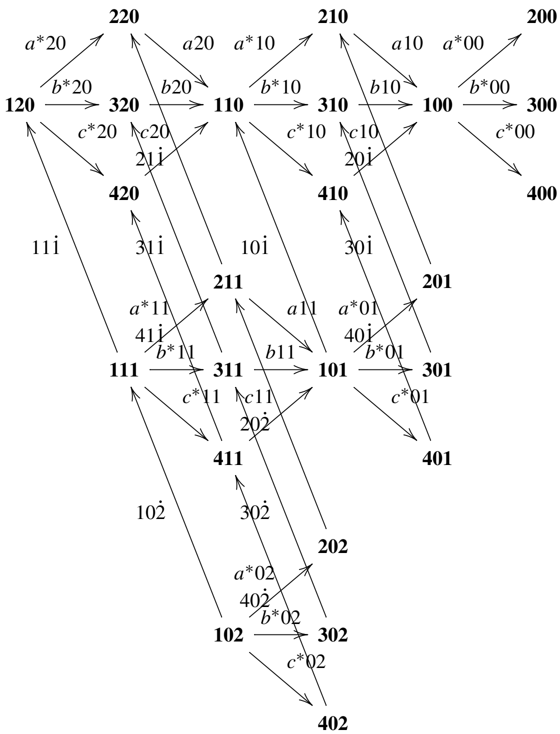
(b) Let Q be the quiver



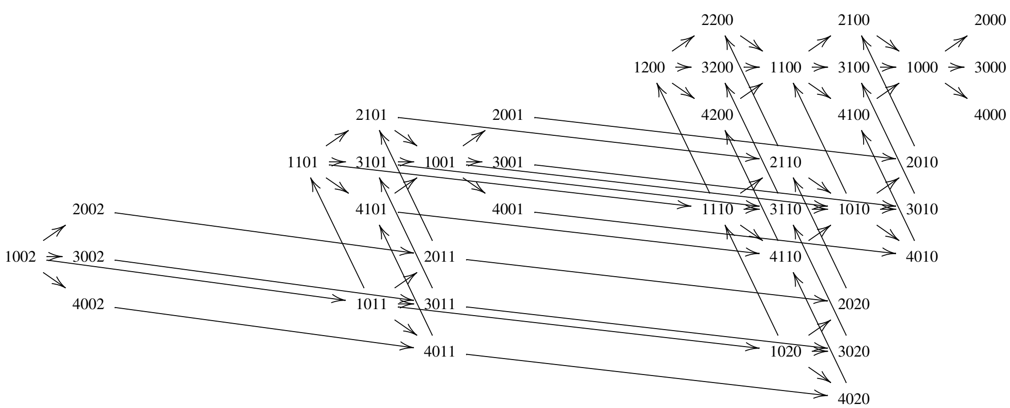
of type D_4. The Auslander–Reiten quiver of $\mathcal{M}^{(1)}$ is the following.



The Auslander–Reiten quiver of $\mathcal{M}^{(2)}$ is the following.



The Auslander–Reiten quiver of $\mathcal{M}^{(3)}$ is the following.



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