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Maximal unramified 3-extensions of imaginary quadratic fields and $\mathrm{SL}_2(\mathbb{Z}_3)$

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Abstract

The structure of the Galois group of the maximal unramified p-extension of an imaginary quadratic field is restricted in various ways. In this paper we construct a family of finite 3-groups satisfying these restrictions. We prove several results about this family and characterize them as finite extensions of certain quotients of a Sylow pro-3 subgroup of $\mathrm{SL}_2(\mathbb{Z}_3)$. We verify that the first group in the family does indeed arise as such a Galois group and provide a small amount of evidence that this may hold for the other members. If this was the case then it would imply that there is no upper bound on the possible lengths of a finite p-class tower.

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1. Maximal unramified p-extensions and Schur-σ groups

Let k be an imaginary quadratic number field and p be a prime. The p-class tower of k is the sequence of fields

$$k = k_1 \subseteq k_2 \subseteq k_3 \subseteq \cdots$$

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where k_{n+1} is the maximal unramified abelian p-extension of k_n. By Galois theory the fields k_n correspond to the subgroups in the derived series of $G = \mathrm{Gal}(k^{nr,p}/k)$ where $k^{nr,p} = \bigcup_{n\geqslant 1} k_n$ is the maximal unramified p-extension of k. If we let $Cl_p(F)$ denote the p-Sylow subgroup of the class group of a number field F then by class field theory $\mathrm{Gal}(k_{n+1}/k_n) \cong Cl_p(k_n)$ for $n \geqslant 1$. In particular $G/[G,G] \cong \mathrm{Gal}(k_2/k_1) \cong Cl_p(k)$ and so is finite.

Now assume also that $p \neq 2$. In [11] the notion of a Schur-σ group is introduced. It encapsulates various properties that the Galois group G is known to satisfy in this case. These are:

1. The generator rank and relation rank of G (as a pro-p group) are equal;
2. $G/[G,G]$ is finite;
3. There exists an automorphism σ of order 2 on G which induces the inverse automorphism $a \mapsto a^{-1}$ on $G/[G,G]$.

Several structural results are proved there about the presentations of such groups. One consequence of their work is that if $d(G) \geqslant 3$ then the extension $k^{nr,p}/k$ is infinite. It follows that all such extensions which are finite and non-abelian must have $d(G) = 2$.

In general it is exceedingly difficult to compute the Galois group G. For those examples in which the group is known to be finite the length of the derived series is usually small. Indeed to date the largest length observed is 3 and in all these examples $p = 2$, see [5]. In the next section we will define a family of finite Schur-σ groups with $p = 3$ and then show that the derived length for groups in this family is unbounded. In the last section we show that the first group in the family is isomorphic to $\mathrm{Gal}(k^{nr,p}/k)$ for several different choices of k.

2. A family of Schur σ-groups of unbounded derived length

Let F be the free pro-3 group on two generators x and y. Let G_n be the Schur-σ group defined by the pro-3 presentation

$$G_n = \langle x, y \mid r_n^{-1}\sigma(r_n), t^{-1}\sigma(t)\rangle$$

where $r_n = x^3 y^{-3^n}$, $t = yxyx^{-1}y$ and $\sigma : F \to F$ is the automorphism defined by $x \mapsto x^{-1}$ and $y \mapsto y^{-1}$. We will prove the following result.

Theorem 2.1. *For $n \geqslant 1$ the following hold*:

(i) *G_n is a finite 3-group of order 3^{3n+2}*;
(ii) *G_n is nilpotent of class $2n+1$*;
(iii) *G_n has derived length $\lfloor \log_2(3n+3) \rfloor$*.

The remainder of this section is devoted to the proof. We first define some auxiliary groups which are easier to study than G_n. Let H_n be given by the pro-3 presentation

$$H_n = \langle x, y \mid x^3, y^{3^n}, t^{-1}\sigma(t)\rangle,$$

and let H be given by the pro-3 presentation

$$H = \langle x, y \mid x^3, t^{-1}\sigma(t)\rangle.$$

Lemma 2.2. *G_n is a central cyclic extension of H_n; and all H_n's are natural quotients of H.*

Proof. The first relation of G_n is $x^6 = y^{2\cdot 3^n}$, so x^6 is central in G_n. Now the relator x^6 is equivalent to the relator x^3 in a 3-group; and the same argument holds for the relator y^{3^n}. It follows that the kernel of the natural map $G_n \to H_n$ is generated by x^3. The second assertion of the lemma is obvious. □

The next lemma exhibits an explicit, matrix representation of H. Let $\alpha \in \mathbb{Z}_3$ satisfy $\alpha^2 = -2$.

Lemma 2.3. *The map $\rho : H \to \mathrm{SL}_2(\mathbb{Z}_3)$, given by*

$$x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad y \mapsto \alpha \begin{pmatrix} 0 & 1/2 \\ 1 & -1 \end{pmatrix},$$

is an isomorphism between H and a pro-3 Sylow subgroup of $\mathrm{SL}_2(\mathbb{Z}_3)$.

We recall the recursive definition of the lower p-central series of a pro-p group G: a series of closed subgroups of G

$$G = P_1(G) \geqslant P_2(G) \geqslant P_3(G) \geqslant \cdots$$

defined by $P_k(G) = P_{k-1}(G)^p[G, P_{k-1}(G)]$ for each $k \geqslant 1$. Here the group on the right-hand side is the closed subgroup generated by all pth powers of elements in $P_{k-1}(G)$, and commutators of elements from G and $P_{k-1}(G)$.

Proof. We first claim that ρ is a homomorphism. Let $\sigma' : \mathrm{SL}_2 \to \mathrm{SL}_2$ be conjugation by $\left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. It is then easy to check $\sigma'\rho = \rho\sigma$ and $\rho(x)^3 = \rho(t^{-1}\sigma(t)) = 1$.

We will now show simultaneously that ρ is injective, and that its image P is a pro-3 Sylow subgroup of $\mathrm{SL}_2(\mathbb{Z}_3)$.

Consider the subgroup K of index 3 in H that is the normal closure of yx^{-1}. It is generated by the $z_i = x^{-i}yx^{i-1}$ for all $i \in \{0, 1, 2\}$. Its presentation, obtained by rewriting the presentation of H with respect to the Schreier transversal $\{1, x, x^2\}$, is given by

$$K = \langle z_0, z_1, z_2 \mid z_i z_{i+1}^2 z_i^2 z_{i+1} \text{ for } i = 0, 1, 2 \rangle.$$

The relators of K may be written as $[z_i, z_{i+1}^{-1}]z_{i+1}^3 z_i^3$; therefore, inductively, every element of $[K, K]$ may be written as a cube (K is said to be "powerful," see [6,12,13]). It follows that the lower central series $(\gamma_k(K))$ coincides with $(P_k(K))$ for $p = 3$. Then $\gamma_k(K)$ is generated, modulo $\gamma_{k+1}(K)$, by $\{z_i^{3^{k-1}}\}_{0 \leqslant i \leqslant 2}$. We conclude that $\gamma_k(K)/\gamma_{k+1}(K)$ has rank at most 3.

Recall that $\mathrm{SL}_2(\mathbb{Z}_3)$ has congruence kernels $N_k = 1 + 3^k M_2(\mathbb{Z}_3)$. The lower central series of N_1 is given by $\gamma_k(N_1) = N_k$, and the rank of N_k/N_{k+1} is 3.

All the claims will follow if we show that $\{\rho(z_i^{3^{k-1}})\}$ spans N_k/N_{k+1} for all $k \geqslant 1$; indeed then $\rho(K) = N_1$, and since the ranks along the lower central series of K are bounded by 3, they must equal 3 and ρ is then injective. We compute:

$$\rho\left(z_0^{3^{k-1}}\right) = \begin{pmatrix} \alpha^{-3^{k-1}} & 0 \\ 0 & \alpha^{3^{k-1}} \end{pmatrix};$$

and $\alpha^{3^{k-1}} \in 1+3^k\mathbb{Z}_3 \setminus 1+3^{k+1}\mathbb{Z}_3$; or, in other words, α is a topological generator of the torsion-free part of $\mathbb{Z}_3^\times$. Similarly,

$$\rho\big(z_1^{3^{k-1}}\big) = \begin{pmatrix} \alpha^{3^{k-1}} & \alpha^{-3^{k-1}} - \alpha^{3^{k-1}} \\ 0 & \alpha^{-3^{k-1}} \end{pmatrix},$$
$$\rho\big(z_2^{3^{k-1}}\big) = \begin{pmatrix} \alpha^{3^{k-1}} & 0 \\ \alpha^{-3^{k-1}} - \alpha^{3^{k-1}} & \alpha^{-3^{k-1}} \end{pmatrix},$$

and the off-diagonal entries are in $3^k\mathbb{Z}_3 \setminus 3^{k+1}\mathbb{Z}_3$.

We conclude by considering $P = N_1\langle\rho(x)\rangle$, which is a pro-3 Sylow subgroup of $\mathrm{SL}_2(\mathbb{Z}_3)$, and noting that $\rho(H) = P$ since they have isomorphic index-3 subgroups K and N_1 respectively. $\square$

Remark 2.4. (i) The proof is similar to a construction of presentations of congruence kernels in [1].

(ii) The following simple and general argument was generously communicated to us by Nigel Boston and Jordan Ellenberg, see [3]. Suppose $f : T \to U$ is a surjective homomorphism of pro-p groups such that $H^1 f : H^1(U, \mathbb{F}_p) \to H^1(T, \mathbb{F}_p)$ is an isomorphism, and $H^2 f : H^2(U, \mathbb{F}_p) \to H^2(T, \mathbb{F}_p)$ is injective. Then f is an isomorphism.

We may apply it to $T = K$ and $U = N_1$. It is not difficult to show that f is surjective and that $H^1 f$ is an isomorphism. Now the cup product map $\bigwedge^2 H^1(U, \mathbb{F}_p) \to H^2(U, \mathbb{F}_p)$ is an isomorphism, because U is uniform; so to prove injectivity of $H^2 f$ it suffices to show that $\bigwedge^2 H^1(T, \mathbb{F}_p) \to H^2(T, \mathbb{F}_p)$ is injective; this holds because $T/\Phi(\Phi(T))$ is abelian.

We may now identify H_n with an appropriate quotient of P:

Lemma 2.5. *H_n is the quotient of P by the subgroup of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfying the congruences*

$$\begin{aligned} a, d &\equiv 1 \quad \big(\mathrm{mod}\ 3^n\big), \\ b, c &\equiv 0 \quad \big(\mathrm{mod}\ 3^n\big), \\ a+d &\equiv 2 \quad \big(\mathrm{mod}\ 3^{n+2}\big), \\ a+b &\equiv 1 \quad \big(\mathrm{mod}\ 3^{n+1}\big), \\ a+b-c &\equiv 1 \quad \big(\mathrm{mod}\ 3^{n+2}\big). \end{aligned}$$

Proof. This amounts to computing the normal closure R of $\rho(y^{3^n})$ in P. We note that y^3 is conjugate to $z_2z_1z_0$, which implies $\rho(y^{3^n}) \equiv \rho(z_0^{3^{n-1}} z_1^{3^{n-1}} z_2^{3^{n-1}})$ in N_n/N_{n+1}, and so the intersection of R with N_n/N_{n+1} is one-dimensional.

Then, taking commutators with z_i, we have $[y^3, z_i] \equiv z_{i-1}^{3^n} z_i^{3^n} / z_i^{3^n} z_{i+1}^{3^n} \equiv z_{i-1}^{3^n} / z_i^{3^n}$ in N_{n+1}/N_{n+2}; so the intersection of R with N_{n+1}/N_{n+2} is two-dimensional.

Finally, taking a commutator again, we have

$$\big[z_{i-1}^{3^n}/z_i^{3^n}, z_{i+1}\big] \equiv z_{i+1}^{-3^{n+1}}$$

in N_{n+2}/N_{n+3}, so the intersection of R with N_{n+2}/N_{n+3} is three-dimensional, and the same holds for N_k/N_{k+1} for all $k \geqslant n+2$.

Writing out the matrices $\rho(z_i^{3^n})$ then proves the lemma. $\square$

Finally, we identify better the relation between G_n and H_n:

Lemma 2.6. *The kernel of the natural map $G_n \to H_n$ is cyclic of order* 3.

Proof. The kernel is cyclic by Lemma 2.2. The order of y^{3^n} in G_n is at most 3, since $y^{3^n} = z_0^{3^{n-1}} z_1^{3^{n-1}} z_2^{3^{n-1}}$ and the relations in H_n force cubes of $z_i^{3^{n-1}}$ to be commutators, and therefore to vanish in any central extension.

On the other hand, construct the $\mathbb{Z}/3\mathbb{Z}$-extension E of H_n by the central element z, with relation $y^{3^n} = z$. Then it is easy to see that the relations of G_n are satisfied in E; so there exist images of G_n in which y has order 3^{n+1}, which is therefore the order of y in G_n. We conclude $|G_n| = 3|H_n|$. □

We are finally ready to prove the main theorem of this section:

Proof of Theorem 2.1. (i) We have $|G_n| = 3|H_n|$ by Lemma 2.6, and $|H_n| = 3^{3n+1}$ because the normal closure R_n of y^{3^n} in H has index 3^{3n} in N_1, and therefore has index 3^{3n+1} in P.

(ii) We first compute the lower central series of H_n. It is obtained from that of P, as follows: $\gamma_1(P) = P$; and for $k \geqslant 1$, $\gamma_{2k}(P) = N_{k+1}\langle (z_0/z_1)^{3^k}, (z_1/z_2)^{3^k}\rangle$ and $\gamma_{2k+1}(P) = N_{k+1}\langle (z_0 z_1 z_2)^{3^k}\rangle$. The last index k such that R_n is not contained in N_k is $n+1$, so the nilpotency class of H_n is $2n+1$. Finally, the action by conjugation of x on $y^{3^n} \equiv (z_0 z_1 z_2)^{3^{n-1}}$ is trivial, so the nilpotency class of G_n is the same as that of H_n, namely $2n+1$; the last quotient $\gamma_{2n+1}(G_n)/\gamma_{2n+2}(G_n) = \langle x, y^{3^n}\rangle$.

(iii) The derived length of G_n can also be obtained from the derived series of P: one has $P^{(2k)} = \gamma_{(2^{2k+2}-1)/3}(P)$ and $P^{(2k+1)} = \gamma_{(2^{2k+3}-2)/3}(P)$ using $[N_k, N_\ell] = N_{k+\ell}$, which comes from the identity

$$\left[1 + 3^m A, 1 + 3^n B\right] \equiv 1 + 3^{m+n}(AB - BA)$$

and the fact that the Lie algebra sl_2 is simple.

By (ii), we have $\gamma_{2n+1}(P) > R_n > \gamma_{2n+2}(P)$, so $P^{(k)} > R_n > P^{(k+1)}$ for $k = \lfloor \log_2(3n+3) \rfloor$. The same argument as above shows that the derived length of G_n is equals that of H_n. □

Remark 2.7. The groups G_n are finite 3-groups with the same number of relations as generators in their pro-3 presentations. It is an open question as to whether or not this implies that such groups must have an *abstract* presentation with equal numbers of generators and relations. Finite groups with this latter property are said to have *deficiency zero*. It is also open whether or not there exist abstract groups of deficiency zero with arbitrarily large derived length. To date the maximum length achieved is 6 (see [10]). If G_n has deficiency zero then this question will be resolved.

We note that examples similar to ours have appeared in the literature before. In [1] a family of finite 3-generated p-groups (for odd prime p) with increasing nilpotency class and derived length is constructed. However our family of groups are the first 2-generated candidates to appear in the literature, as far as we know.

3. Explicit computations of Gal($k^{nr,3}/k$)

In [4] and [5] the p-group generation algorithm is used to compute the Galois groups of several p-extensions with restricted ramification. Here we use it to verify that $\mathrm{Gal}(k^{nr,3}/k) \cong G_1$

for several different imaginary quadratic fields k. For the reader's convenience we recall some definitions and give a brief description of the method.

Recall that $(P_k(G))$ denotes the lower p-central series of G. If G is a finite p-group then this series is finite and the smallest c such that $P_{c+1}(G) = \{1\}$ will be called the *p-class* of G. A p-group H is called a *descendant* of G if $H/P_c(H) \cong G$ where c is the p-class of G. It is an *immediate descendant* if it has p-class $c+1$. The p-group generation algorithm [14] finds representatives (up to isomorphism) of all the immediate descendants of a given finite p-group G. Starting with the elementary abelian p-group on d generators (for some fixed d) the algorithm allows one to compute a tree containing all finite d-generated p-groups. The p-class of a group determines the level of the tree in which it occurs.

For the Galois groups we are interested in we have additional information about the maximal abelian quotients of various subgroups of small index. This information is obtained by computing class groups of various extensions and applying the Artin reciprocity isomorphism from class field theory. This information is sometimes sufficient to eliminate all but finitely many groups from the tree of descendants described above, in which case we are left with a finite number candidates for the Galois group. A more precise formulation of the method and several examples in the case $p = 2$ can be found in [5].

In the case $p = 3$ we have obtained the following result using the symbolic computation package MAGMA [2]. (Note: we describe abelian groups by listing the orders of their cyclic components. So for instance $[3, 3]$ is the direct product of a cyclic group of order 3 with itself.)

Proposition 3.1. *Let G be a pro-3 group and suppose $G/[G, G] \cong [3, 3]$. Then G has four closed subgroups of index 3. If these four subgroups have maximal abelian quotients $[3, 9]$, $[3, 9]$, $[3, 9]$ and $[3, 3, 3]$, then G is a finite 3-group.*

In fact, after starting the p-group generation algorithm on the 2-generated elementary abelian 3-group $[3, 3]$ with the restrictions described in the proposition, the computation terminates having found two candidates for G. These will be denoted by Q_1 and Q_2 and are generated by $\{x_i\}_{i=1}^5$ subject to the following power-commutator presentations.

$$
\begin{array}{lll}
(Q_1) & x_1^3 = x_4, & [x_2, x_1] = x_3, \\
 & x_2^3 = x_4, & [x_3, x_1] = x_4, \\
 & & [x_3, x_2] = x_5, \\
(Q_2) & x_1^3 = x_4^2, & [x_2, x_1] = x_3, \\
 & & [x_3, x_1] = x_4, \\
 & & [x_3, x_2] = x_5.
\end{array}
$$

Q_1 and Q_2 are the groups $(243, 5)$ and $(243, 6)$ respectively in MAGMA's or GAP's `SmallGroups` database [2,7].

Remark 3.2. Note that in these power-commutator presentations if a power x_r^3 or commutator $[x_r, x_s]$ does not occur on the left-hand side of the given relations then it is assumed to be trivial.

Corollary 3.3. *For discriminants d satisfying $-50\,000 \leqslant d \leqslant 0$, the field $k = \mathbb{Q}(\sqrt{d})$ has* $\mathrm{Gal}(k^{nr,3}/k) \cong G_1$ *if and only if*

$$d \in \{-4027, -8751, -19\,651, -21\,224, -22\,711, -24\,904, -26\,139, -28\,031, -28\,759, -34\,088, -36\,807, -40\,299, -40\,692, -41\,015, -42\,423, -43\,192, -44\,004, -45\,835, -46\,587, -48\,052, -49\,128, -49\,812\}.$$

Proof. For each of these fields $Cl_3(k) \cong [3,3]$. MAGMA's class field theory package can be used to construct and verify that the four unramified extensions $\{k_i\}_{i=1}^4$ of degree 3 over k have $Cl_3(k_i) \cong [3,9]$ for three choices of i, and $Cl_3(k_i) \cong [3,3,3]$ for the remaining choice. By Proposition 3.1, $\mathrm{Gal}(k^{nr,3}/k)$ is isomorphic to Q_1 or Q_2. The group Q_2 has non-trivial Schur multiplier and hence can be eliminated (see [11]) leaving Q_1 as the only possibility. One can verify (by hand or by machine computation) that G_1 satisfies the conditions in Proposition 3.1. Hence we must also have $Q_1 \cong G_1$. □

Remark 3.4. Since G_1 has derived length 2 the fields described in the corollary all have 3-class towers of length 2. In [15] the field $\mathbb{Q}(\sqrt{-4027})$ is shown to have 3-class tower of length 2 by different means.

The following question remains to be answered:

Question 3.5. Is it possible, for all $n \geqslant 1$, to find an imaginary quadratic field k such that $\mathrm{Gal}(k^{nr,3}/k) \cong G_n$?

If the answer is yes then this would imply that the lengths of finite p-class towers are unbounded. As a first step towards answering this question one might compute the abelian quotient invariants of the index 3 subgroups in G_n for various $n \geqslant 2$, and then search for fields k which have unramified extensions with matching 3-class groups. Using standard methods [8,9] one can compute the abelian quotient invariants and it turns out that the result is independent of n. More precisely one obtains the following proposition.

Proposition 3.6. *For $n \geqslant 2$ the group G_n has four subgroups of index* 3 *with abelian quotient invariants* $[3,9]$, $[3,3,3]$, $[3,3,3]$, *and* $[3,3,3]$.

Moreover when one looks for examples of imaginary quadratic fields having unramified extensions with matching 3-class groups they seem relatively easy to find. The discriminants d with $|d| < 50\,000$ for which there is a match are $d = -3896, -6583, -23\,428, -25\,447, -27\,355, -27\,991, -36\,276, -37\,219, -37\,540, -39\,819, -41\,063, -43\,827, -46\,551$.

At this point it becomes difficult to make further progress. Clearly one cannot use the abelian quotient invariants of the index 3 subgroups to separate out any of the groups G_n for $n \geqslant 2$ as we did with G_1. If one restricts attention to the smallest groups G_2 and G_3 then differences in the abelian quotient invariants only show up when one looks at subgroups of index at least 27. The corresponding class group computations that one would need to carry out do not seem feasible currently.

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