BOUNDARY VALUE PROBLEMS FOR SECOND ORDER 
DIFFERENTIAL EQUATIONS OF SOBOLEV TYPE 

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Abstract—In this paper we provide necessary and sufficient conditions for the existence and uniqueness 
of solutions of second order differential equations of Sobolev type satisfying some boundary conditions. 
We also give sufficient conditions so that the Picard and approximate Picard method converge to the 
unique solution of these boundary value problems. 

1. INTRODUCTION 
In the year 1954, Sobolev [1] proposed an imbedding method for solving linear Fredholm integral 
équations which leads to a new class of differential equations. This imbedding technique to nonlinear 
Fredholm integral equations was extended by Kagiwada and Kalaba [2, 3] which gives 
rise to a similar class of integro-differential equations. In a series of papers Lakshmikantham and 
his coworkers [4–6] named these type of equations as differential equations of Sobolev type. For 
such equations together with initial functions they have studied Picard and Peano type of existence 
results, extension of solutions over the entire square, comparison theorems, variation of constants 
formula and Bellman–Gronwall type of inequality. These results naturally include as a special case 
the results for ordinary differential equations [7]. 

In this paper we shall consider the following second order system of differential equations of 
Sobolev type: 

\[ u''(t, x) = f[t, x, u(t, x), u(x, t)] \] 

(1) 

\[ u(a, x) = \alpha(x), u(b, x) = \beta(x) \] 

(2) 

where \( \alpha, \beta \in C[J, R^*], J = [a, b], f \in C[J \times J \times R^* \times R^*, R^*] \). Obviously, the boundary value 
problem (1) and (2) includes in particular prototype boundary value problems considered 
extensively in Refs [8–11] and references therein. Boundary value problems for the delay–
differential equations and for the discrete systems which are not included in (1) and (2) are 
examined recently in Refs [12–14] and references therein. 

The plan of this paper is as follows: in Section 2, we shall provide necessary and sufficient 
conditions for the existence and uniqueness of solutions of the boundary value problem (1) and 
(2). In Section 3, we obtain a priori conditions so that the Picard iterative scheme converges to 
the unique solution \( u^*(t, x) \) of the problem (1) and (2). In practical evaluation of Picard’s iterative 
sequence \( \{u_m(t, x)\} \) only an appropriate sequence \( \{v_m(t, x)\} \) is constructed and this depends on 
approximating \( f \) by some simpler function. In Section 4, the approximate Picard’s method 
developed in Refs [15–17] for the usual boundary value problems is used for (1) and (2) to construct 
the sequence \( \{v_m(t, x)\} \). We also provide necessary and sufficient conditions so that the sequence 
\( \{v_m(t, x)\} \) converges to \( u^*(t, x) \). Finally, in Section 5 two examples are illustrated. 

2. EXISTENCE AND UNIQUENESS 

Theorem 2.1 

Assume that the following conditions hold: 

(i) \( K > 0 \) is a given positive number and let \( Q \) be the maximum of \( \|f(t, x, u, v)\| \) on the compact 
set \( J \times J \times D_0 \), where 

\[ D_0 = \{(u, v) : \|u\| \leq 2K, \|v\| \leq 2K\}; \]
(ii) for every $x_1, x_2 \in J$,
\[
\lim_{x_1 \to x_2} \left( \sup_u \left\{ \int_a^b \| f[s, x_1, u(s, x_1), u(x_1, s)] - f[s, x_2, u(s, x_2), u(x_2, s)] \| \, ds, \right. \right)
\]
\[
\left\{ u \in C[J \times J, R^n], \sup_{a \leq t, s \leq b} \| u(t, x) \| = \sup_{a \leq x, t \leq b} \| u(x, t) \| \leq 2K \right\} = 0;
\]

(iii) \[
\max \left\{ \sup_{a \leq t, x \leq b} \| \alpha(x) \|, \sup_{a \leq x \leq b} \| \beta(x) \| \right\} = I \leq K;
\]

(iv) \[
(b - a) \leq \left( \frac{8k}{Q} \right)^{1/2}.
\]

Then, there exists a solution $u(t, x)$ of the boundary value problem (1) and (2) which is such that
\[
\sup_{a \leq t, x \leq b} \| u(t, x) \| \leq 2K.
\]

**Proof.** As in ordinary differential equations [10] the problem (1) and (2) is equivalent to the following Fredholm integral equation:
\[
u(t, x) = l(t, x) + \int_a^b G(t, s)f[s, x, u(s, x), u(x, s)] \, ds,
\]
where $G(t, s)$ is the Green's function defined by
\[
G(t, s) = \begin{cases} 
\frac{(b - t)(s - a)}{(b - a)}, & a \leq s \leq t \leq b \\
\frac{(b - s)(t - a)}{(b - a)}, & a \leq t \leq s \leq b 
\end{cases}
\]
and the function $l(t, x)$ is
\[
l(t, x) = \alpha(x) + \frac{\beta(x) - \alpha(x)}{(b - a)} (t - a).
\]

The set
\[
B[J] = \left\{ (u(t, x))_{a \leq t, x \leq b} \in C[J \times J, R^n]: \sup_{a \leq t, x \leq b} \| u(t, x) \| \leq 2K \right\}
\]
is closed, bounded and convex subset of the Banach space $C[J \times J, R^n]$. Obviously, if $u \in B[J]$, then it also satisfies
\[
\sup_{a \leq t, x \leq b} \| u(x, t) \| \leq 2K,
\]
and hence $B[J]$ can also be defined as
\[
B[J] = \left\{ (u(t, x))_{a \leq t, x \leq b} \in C[J \times J, R^n]: \sup_{a \leq t, x \leq b} \| u(t, x) \| \leq 2K, \sup_{a \leq t, x \leq b} \| u(x, t) \| \leq 2K \right\}.
\]

Consider an operator $T: C[J \times J, R^n] \to C[J \times J, R^n]$ as
\[
(Tu)(t, x) = l(t, x) + \int_a^b G(t, s)f[s, x, u(s, x), u(x, s)] \, ds.
\]

It is clear that any fixed point of $T$ is a solution of (1) and (2). For any $u \in B[J]$, from (4) and the hypotheses, we find that
\[
\| (Tu)(t, x) \| \leq \sup_{a \leq t, x \leq b} \| l(t, x) \| + \int_a^b \| G(t, s) \| \| f[s, x, u(s, x), u(x, s)] \| \, ds
\]
\[
\leq \max \left\{ \sup_{a \leq t \leq b} \| \alpha(x) \|, \sup_{a \leq x \leq b} \| \beta(x) \| \right\} + Q \frac{(t - a)(b - t)}{2}
\]
Thus, $T$ maps $B[J]$ into itself, and $TB[J]$ is uniformly bounded.

Next, we shall show that $TB[J]$ is equicontinuous. Let $\epsilon > 0$ be given, and let $t_1, x_1, t_2, x_2 \in J$.

Then, from (4) we have

$$\| (Tu)(t_1, x_1) - (Tu)(t_2, x_2) \| \leq \| l(t_1, x_1) - l(t_2, x_2) \|$$

$$+ \int_a^b |G(t_1, s)| \| f[s, x_1, u(s, x_1), u(x_1, s)] - f[s, x_2, u(s, x_2), u(x_2, s)] \| \, ds$$

$$+ \int_a^b |G(t_1, s) - G(t_2, s)| \| f[s, x_2, u(s, x_2), u(x_2, s)] \| \, ds$$

$$= I_1 + I_2 + I_3,$$

say.

Since $\alpha(x)$ and $\beta(x)$ are continuous on $J$, the function $l(t, x)$ is uniformly continuous, and hence we can choose $\delta_1$ so that $|t_1 - t_2| < \delta_1$ and $|x_1 - x_2| < \delta_1$ implies that $I_1 < \epsilon/3$.

Since

$$\sup_{a < t, s < b} |G(t, s)| \leq \frac{1}{4}(b - a),$$

from condition (ii) we can choose $\delta_2$ so that $|x_1 - x_2| < \delta_2$ implies that $I_2 < \epsilon/3$.

Finally, since $G(t, s)$ is continuous on $J \times J$, it is also uniformly continuous, and hence we can choose $\delta_3$ so that $|t_1 - t_2| < \delta_3$ implies that

$$|G(t_1, s) - G(t_2, s)| \leq \frac{\epsilon}{3(b - a)\Omega}.$$

The above inequality easily gives that $I_3 < \epsilon/3$.

Thus, if

$$\max\{|t_1 - t_2|, |x_1 - x_2|\} \leq \min\{\delta_1, \delta_2, \delta_3\}, \quad \text{then} \quad \| (Tu)(t_1, x_1) - (Tu)(t_2, x_2) \| < \epsilon.$$

This implies that the set $\{TB[J]\}$ is an equicontinuous family and therefore its closure is compact.

Next, let $\{u^n(t, x)\} \subseteq B[J]$ be a sequence converging to $u(t, x)$. It is obvious that the sequence $\{u^n(t, x)\}$ also converges to $u(x, t)$. Since $f$ is continuous, we have

$$f[t, x, u^n(t, x), u^n(x, t)] \rightarrow f[t, x, u(t, x), u(x, t)].$$

Thus, from the bounded convergence theorem, it follows that

$$\int_a^b G(t, s)f[s, x, u^n(s, x), u^n(x, s)] \, ds \rightarrow \int_a^b G(t, s)f[s, x, u(s, x), u(x, s)] \, ds.$$

Hence $Tu^n \rightarrow Tu$, which shows that $T$ is continuous.

Combining the above considerations, we see that the Schauder fixed point theorem is applicable. Thus, $T$ has a fixed point in $B[J]$.

**Corollary 2.1**

Assume that the following conditions hold: (i) condition (ii) of Theorem 2.1; and (ii) for all $(t, x, u, v) \in J \times J \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\| f(t, x, u, v) \| \leq L + L_1 \| u \|^{\alpha} + L_2 \| v \|^{\beta}$$

(5)

where $0 \leq \alpha, \beta < 1$ and $L, L_1, L_2$ are non-negative constants. Then, there exists a solution $u(t, x)$ of the boundary value problem (1) and (2).

**Proof.** Inequality (5) implies that on $J \times J \times D_0$,

$$\| f(t, x, u, v) \| \leq L + L_1(2K)^\alpha + L_2(2K)^\beta = Q_1,$$

say.
Now, we can choose $K$ so large that condition (iii) of Theorem 2.1 is satisfied, and
\[ \frac{1}{3}Q_1(b-a)^2 \leq K. \]

**Theorem 2.2**

Assume that (i) for all $(t, x, u, v) \in J \times J \times D_1$, the inequality (5) with $\alpha = \beta = 1$ is satisfied, where $D_1 = \{(u, v) : \|u\| \leq r, \|v\| \leq r\}$, and
\[ r = \left(1 + \frac{1}{3}(b-a)^2L\right)(1-\theta)^{-1}; \quad (6) \]
\[ \theta = \frac{1}{3}(b-a)^2(L_1 + L_2) < 1; \quad (7) \]
and (ii) condition (ii) of Theorem 2.1 holds with $2K$ replaced by $r$. Then, there exists a solution $u(t, x)$ of the boundary value problem (1) and (2) which is such that
\[ \|u(t, x)\| \leq r. \]

**Proof.** From the proof of Theorem 2.1 it suffices to show that the operator $T$ defined in (4) maps the set
\[ B_1[J] = \{u(t, x) \in C[J \times J, \mathbb{R}^n] : \|u(t, x)\| \leq r\} \]
into itself. For this, let $u \in B_1[J]$, then it also satisfies
\[ \sup_{a \leq t, x \leq b} \|u(t, x)\| \leq r, \]
and hence using (5) in (4) we find
\[ \|(Tu)(t, x)\| \leq l + \int_a^b \|G(t, s)\| \left[L + L_1 \|u(s, x)\| + L_2 \|u(x, s)\| \right] ds \]
\[ \leq l + \frac{1}{3}(b-a)^2(L + L_1 r + L_2 r) = r. \]

**Theorem 2.3**

Assume that (i) for all $(t, x, u, v) \in J \times J \times D_2$,
\[ \|f(t, x, u, v)\| \leq L \sin \frac{\pi(x-a)}{(b-a)} + L_1 \|u - l(t, x)\| + L_2 \|v - l(x, t)\| \]
where
\[ D_2 = \{(u, v) : \|u - l(t, x)\| \leq r_1(t, x), \|v - l(x, t)\| \leq r_1(x, t)\}, \]
and
\[ r_1(t, x) = r_1(x, t) = \frac{1}{2\pi} (b-a)^2 L(1-\theta_1)^{-1} \sin \frac{\pi(t-a)}{(b-a)} \sin \frac{\pi(x-a)}{(b-a)}, \quad (9) \]
\[ \theta_1 = \frac{1}{\pi^2}(b-a)^2(L_1 + L_2) < 1; \quad (10) \]
and (ii) condition (ii) of Theorem 2.1 holds with $2K$ replaced by $r_1(t, x)$. Then, there exists a solution $u(t, x)$ of the boundary value problem (1) and (2) which is such that
\[ \|u(t, x) - l(t, x)\| \leq r_1(t, x). \]

**Proof.** As in Theorem 2.2 it suffices to show that the operator $T$ defined in (4) maps the set
\[ B_2[J] = \{u(t, x) \in C[J \times J, \mathbb{R}^n] : \|u(t, x) - l(t, x)\| \leq r_1(t, x)\} \]
into itself. For this, let $u \in B_1[J]$ then from $r_1(t, x) = r_1(x, t)$ it is obvious that $\|u(x, t) - l(x, t)\| \leq r_1(t, x)$, and hence using (8) in (4) we find
\[ \|(Tu)(t, x) - l(t, x)\| \leq l + \int_a^b \|G(t, s)\| \left[L \sin \frac{\pi(x-a)}{(b-a)} \right. \]
\[ + L_1 \|u(s, x) - l(s, x)\| + L_2 \|u(x, s) - l(x, s)\| \right] ds \]
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\[ \begin{aligned}
\int_a^b |G(t, s)||L \sin \frac{\pi(x-a)}{(b-a)} + (L_1 + L_2) r_t(s, x)| \, ds \\
\leq \left[ \frac{1}{2\pi} \frac{(b-a)^2 L + \frac{1}{\pi^2} (b-a)^2 (L_1 + L_2)}{(b-a)^2} \right] \int_a^b \frac{\pi(t-a)}{(b-a)} \sin \frac{\pi(x-a)}{(b-a)} = r_t(t, x).
\end{aligned} \]

**Theorem 2.4**

Assume that the boundary value problem (1) and (2) has a solution \( u'(t, x) \) which is different from \( l(t, x) \) and the condition (8) is satisfied with \( L = 0 \) for all \( (t, x, u, 0) \in J \times J \times D_3 \), where

\[ D_3 = \{(u, v): \| u - l(t, x) \| \leq \sup_{a \leq t, x \leq b} \| u'(t, x) - l(t, x) \| \}. \]

Then, it is necessary that \( \theta \geq 1 \).

**Proof.** Since \( u'(t, x) \) is a solution of (1) and (2) different from \( l(t, x) \), it is necessary that

\[ \sup_{a \leq t, x \leq b} \| u'(t, x) - l(t, x) \| = \sup_{a \leq t, x \leq b} \| u'(x, t) - l(x, t) \| \neq 0. \]

Obviously \( (u'(t, x), u'(x, t)) \in D_3 \), and hence using (8) in (3) we get

\[ \| u'(t, x) - l(t, x) \| \leq \int_a^b \sup_{a \leq t, x \leq b} \| u'(t, x) - l(t, x) \| \, ds \]

\[ \leq \frac{1}{2}(b-a)^2(L_1 + L_2) \sup_{a \leq t, x \leq b} \| u'(t, x) - l(t, x) \|. \]

Thus, it is necessary that \( \theta \geq 1 \).

**Remark 2.1**

If \( u'(t, x) \) is a solution of (1) and (2) different from \( l(t, x) \), then it is clear that

\[ \sup_{a \leq t, x \leq b} \| u'(t, x) - l(t, x) \| = \sup_{a \leq t, x \leq b} \| u'(x, t) - l(x, t) \| \neq 0. \]

is finite and different from zero. Since

\[ \sup_{a \leq t, x \leq b} \| u'(t, x) - l(t, x) \| \leq \| u'(t, x) - l(t, x) \| \ast, \]

it is obvious that

\[ \sup_{a \leq t, x \leq b} \| u'(x, t) - l(x, t) \| \leq \| u'(t, x) - l(t, x) \| \ast. \]

Using these facts in (11), we find that

\[ \| u'(t, x) - l(t, x) \| \ast \leq \left( \sup_{a \leq t, x \leq b} \frac{1}{\sin \frac{\pi(x-a)}{(b-a)}} \int_a^b |G(t, s)||L \sin \frac{\pi(s-a)}{(b-a)} + (L_1 + L_2) s ds \right) \times \| u'(t, x) - l(t, x) \| \ast \leq \theta_2 \| u'(t, x) - l(t, x) \| \ast, \]

where

\[ \theta_2 = \left( \frac{1}{\pi_2} + \frac{1}{2\pi} L_2 \right)(b-a)^2. \]

Thus, in Theorem 2.4 the necessary condition \( \theta \geq 1 \) can be replaced by \( \theta_2 \geq 1 \).

**Remark 2.2**

If \( \theta < 1 \) or \( \theta_2 < 1 \) and the condition (8) is satisfied on \( J \times J \times R^+ \times R^+ \) with \( L = 0 \), then \( u(t, x) = l(t, x) \) is the only solution of (1) and (2).
To prove the convergence of Picard's iterates for the boundary value problem (1) and (2) we shall need the following:

**Lemma 3.1** [10]

Let $B$ be a Banach space and let for $\eta > 0$, $\eta \in \mathbb{R}$, $S(u_0, \eta) = \{u \in B: \|u - u_0\| \leq \eta \}$. Let $T$ map $S(u_0, \eta)$ into $B$ and (i) for all $u, v \in S(u_0, \eta)$, $\|Tu - Tv\| \leq \alpha \|u - v\|$ where $0 \leq \alpha < 1$ and (ii) $\eta_0 = (1 - \alpha)^{-1} \|Tu_0 - u_0\| \leq \eta$.

Then, the following hold: (1) $T$ has a fixed point $u^* \in S(u_0, \eta_0)$; (2) $u^*$ is the unique fixed point of $T$ in $S(u_0, \eta)$; (3) the sequence $\{u^n\}$ defined by $u^{n+1} = Tu^n$, $n = 0, 1, \ldots$ converges to $u^*$ with $\|u^n - u^*\| \leq \alpha^m \eta_0$; (4) with any $u \in S(u_0, \eta_0)$, $u^* = \lim_{n \to \infty} Tu^n$; and (5) any sequence $\{u^n\}$ such that $u^n \in S(u^n, \alpha^m \eta_0)$, $m = 0, 1, \ldots$ converges to $u^*$.

**Definition 3.1**

A function $\bar{u}(t, x)$ which is such that $\bar{u}^n(t, x) \in C[J \times J, \mathbb{R}^n]$ is called an approximate solution of (1) and (2) if there exist $\delta$ and $\epsilon$, non-negative constants, such that

$$\sup_{a \leq t, x \leq b} \|\bar{u}^n(t, x) - f[t, x, \bar{u}(t, x), \bar{u}(x, t)]\| \leq \delta$$

and

$$\sup_{a \leq t, x \leq b} \|l(t, x) - \bar{l}(t, x)\| \leq \frac{1}{4} \epsilon (b - a)^2,$$

where the function $\bar{l}(t, x)$ is

$$\bar{l}(t, x) = \bar{u}(a, x) + \frac{\bar{u}(b, x) - \bar{u}(a, x)}{(b - a)} (t - a).$$

(13)

The approximate solution $\bar{u}(t, x)$ can be expressed as

$$\bar{u}(t, x) = \bar{l}(t, x) + \int_a^b G(t, s) \{ f[s, x, \bar{u}(s, x), \bar{u}(x, s)] + p(s, x) \} \, ds,$$

where $p(t, x) = \bar{u}^n(t, x) - f[t, x, \bar{u}(t, x), \bar{u}(x, t)]$ and

$$\sup_{a \leq t, x \leq b} \|p(t, x)\| \leq \delta.$$

**Definition 3.2**

The function $f(t, x, u, v)$ is said to be of Lipschitz class if for all $(t, x, u, v)$, $(t, x, u', v') \in J \times J \times D$, $D \subset \mathbb{R}^n \times \mathbb{R}^n$, the following is satisfied:

$$\|f(t, x, u, v) - f(t, x, u', v')\| \leq L_1 \|u - u'\| + L_2 \|v - v'\|.$$

In what follows we shall consider the Banach space $B = C[J \times J, \mathbb{R}^n]$ and for all $u(t, x) \in B$,

$$\|u\| = \sup_{a \leq t, x \leq b} \|u(t, x)\|.$$

**Theorem 3.1**

Assume that the following conditions hold: (i) there exists an approximate solution $\bar{u}(t, x)$ of (1) and (2) and the function $f(t, x, u, v)$ is of Lipschitz class on $J \times J \times D_4$, where $D_4 = \{(u, v): \|u - \bar{u}(t, x)\| \leq N, \|v - \bar{u}(x, t)\| \leq N\}$; and (ii) $\theta < 1$ and

$$N_0 = (1 - \theta)^{-1} \epsilon + \delta \frac{1}{4} (b - a)^2 \leq N.$$

Then, the following hold: (1) there exists a solution $u^*(t, x)$ of (1) and (2) in $S(\bar{u}, N_0)$; (2) $u^*(t, x)$
is the unique solution of (1) and (2) in \( \mathcal{S}(\bar{\alpha}, N) \); (3) the Picard sequence \( \{u^m(t,x)\} \) defined by

\[
u^{m+1}(t,x) = I(t,x) + \int_a^b G(t,s)f[s,x,u^m(s,x),u^m(x,s)] \, ds; \quad m = 0, 1, \ldots;
\]

\[
u^0(t,x) = \tilde{u}(t,x)
\]

converges to \( u^*(t,x) \) with \( \|u^* - u^m\| \leq \theta^m N_0 \); (4) with \( u^0(t,x) = u(t,x) \in \mathcal{S}(\bar{\alpha}, N_0) \) the iterative process (16) converges to \( u^*(t,x) \); and (5) any sequence \( \{\tilde{u}^m(t,x)\} \) such that \( \tilde{u}^m(t,x) \in \mathcal{S}(u^m, \theta^m N_0) \), \( m = 0, 1, \ldots \), converges to \( u^*(t,x) \).

**Proof.** We shall show that the operator \( T \) defined in (4) on \( \mathcal{S}(\bar{\alpha}, N) \) to \( B \) satisfies the conditions of Lemma 3.1. For this, if \( u \in \mathcal{S}(\bar{\alpha}, N) \), i.e.,

\[
sup_{a \leq t, x \leq b} \|u(t,x) - \tilde{u}(t,x)\| \leq N,
\]

then it is obvious that

\[
sup_{a \leq x, t \leq b} \|u(t,x) - \tilde{u}(x,t)\| \leq N,
\]

and hence \( [u(t,x), u(x,t)] \in D_4 \). Let \( u, v \in \mathcal{S}(\bar{\alpha}, N) \), then from (4) it follows that

\[
\| (Tu)(t,x) - (Tv)(t,x) \| \leq \int_a^b |G(t,s)| |L_1\| u(s,x) - v(s,x)\| + L_2\| u(x,s) - v(x,s)\| \| \leq \frac{1}{b-a}(L_1 + L_2)\| u - v \|
\]

and hence

\[
\| (Tu) - (Tv) \| \leq \theta \| u - v \|.
\]

Further, from (4) and (14), we have

\[
(T\tilde{u})(t,x) - \tilde{u}(t,x) = I(t,x) - \tilde{I}(t,x) - \int_a^b G(t,s)p(s,x) \, ds
\]

and hence

\[
\| (T\tilde{u})(t,x) - \tilde{u}(t,x) \| \leq \| I(t,x) - \tilde{I}(t,x) \| + \int_a^b |G(t,s)| \| p(s,x) \| \, ds
\]

\[
\leq \frac{1}{b-a}(b-a)^2 + \frac{1}{2}\delta(b-a)^2 = (\epsilon + \delta)\frac{1}{2}(b-a)^2,
\]

which implies that

\[
\| Tu^0 - u^0 \| \leq (\epsilon + \delta)\frac{1}{2}(b-a)^2.
\]

Using condition (15) in the above inequality we conclude that

\[
(1 - \theta)^{-1}\| Tu^0 - u^0 \| \leq N.
\]

**Remark 3.1**

If \( N = \infty \), then obviously (15) is satisfied, and hence Theorem 3.1 ensures the existence of \( u^*(t,x) \) in \( \mathcal{S}(\bar{\alpha}, N_0) \) whereas its uniqueness in \( \mathcal{S}(\bar{\alpha}, \infty) \).

**Remark 3.2**

From conclusion (3) of Theorem 3.1 and equation (15), we have

\[
\| u^* - \tilde{u} \| \leq (1 - \theta)^{-1} (\epsilon + \delta)\frac{1}{2}(b-a)^2.
\]

## 4. APPROXIMATE PICARD'S METHOD

In Theorem 3.1 conclusion (3) ensures that the sequence \( \{u^m(t,x)\} \) obtained from (16) converges to the solution \( u^*(t,x) \) of the boundary value problem (1) and (2). However, in practical evaluation this sequence is approximated by the computed sequence, say, \( \{v^m(t,x)\} \). To obtain \( v^{m+1}(t,x) \) the function \( f \) is approximated by \( f^m \). Therefore, the computed sequence \( \{v^m(t,x)\} \) satisfies the
recurrence relation

\[ v^{m+1}(t, x) = l(t, x) + \int_a^b G(t, s)f^m[s, x, v^m(s, x), v^m(x, s)] \, ds; \quad m = 0, 1, \ldots; \]

\[ v^0(t, x) = u^0(t, x) = \bar{u}(t, x). \]  

(17)

With respect to \( f^m \), we shall assume the following condition:

**Condition C**

For \( v^m(t, x) \) obtained from (17), the following inequality is satisfied:

\[
\sup_{a \leq t, s \leq b} \| f[t, x, v^m(t, x), v^m(x, t)] - f^m[t, x, v^m(t, x), v^m(x, t)] \| \leq \Delta \sup_{a \leq t, s \leq b} \| f[t, x, v^m(t, x), v^m(x, t)] \|; \quad m = 0, 1, \ldots, \]  

(18)

where \( \Delta \) is a non-negative constant.

Inequality (18) corresponds to the relative error in approximating the function \( f \) by \( f^m \) for the \((m + 1)\)th iteration.

**Theorem 4.1**

Assume that the following conditions hold: (i) condition (i) of Theorem 3.1; (ii) condition C; and (iii) \( 0 < (1 + \theta) < 1 \), and

\[ N_i = (1 - \theta_i)^{-1}(\epsilon + \delta + \Delta F) \frac{1}{2}(b - a)^2 \leq N, \]

where

\[ F = \sup_{a \leq t, x \leq b} \| f[t, x, \bar{u}(t, x), \bar{u}(x, t)] \|. \]

Then, the following hold: (1) all the conclusions (1)–(5) of Theorem 3.1 hold; (2) the sequence \( \{v^m(t, x)\} \) obtained from (17) remains in \( S(\bar{u}, N) \); (3) the sequence \( \{v^m(t, x)\} \) converges to \( u^*(t, x) \) the solution of (1) and (2) if and only if

\[ \lim_{m \to \infty} w^m = 0, \]

where

\[ w^m = \sup_{a \leq t, x \leq b} \| v^{m+1}(t) - l(t, x) - \int_a^b G(t, s)f[s, x, v^m(s, x), v^m(x, s)] \| \]

and

\[ \| u^* - v^{m+1} \| \leq (1 - \theta)^{-1} \{ \theta \| v^m - u^* \| + \Delta \frac{1}{2}(b - a)^2 \sup_{a \leq t, x \leq b} \| f[t, x, v^m(t, x), v^m(x, t)] \| \}. \]  

(20)

**Proof.** Since \( \theta_i < 1 \) implies \( \theta < 1 \) and obviously \( N_0 \approx N_i \), the conditions of Theorem 3.1 are satisfied and conclusion (1) follows.

To prove (2) we note that \( \bar{u} \in S(\bar{u}, N) \) and from (14) and (17), we find

\[ v^1(t, x) - \bar{u}(t, x) = l(t, x) - \bar{l}(t, x) \]

\[ + \int_a^b G(t, s)f_0[s, x, \bar{u}(s, x), \bar{u}(x, s)] - f[s, x, \bar{u}(s, x), \bar{u}(x, s)] - p(s, x) \, ds, \]

which implies that

\[ \| v^1(t, x) - \bar{u}(t, x) \| \leq \| \epsilon(b - a)^2 + \delta(b - a)^2 + \frac{1}{2}(b - a)^2 \Delta F = (\epsilon + \delta + \Delta F) \frac{1}{2}(b - a)^2 \leq N_i \]

and hence \( v^1 \in S(\bar{u}, N_i) \).
Next, we shall assume that \( v^m \in \mathcal{S}(\bar{u}, N_1) \) and show that \( v^{m+1} \in \mathcal{S}(\bar{u}, N_1) \). For this, from (14) and (17), we have

\[
v^{m+1}(t, x) - \bar{u}(t, x) = I(t, x) - \bar{I}(t, x) + \int_a^b G(t, s) \left\{ f^m\left[s, x, v^m(s, x), v^m(x, s)\right] - f\left[s, x, v^m(s, x), v^m(x, s)\right] - f\left[s, x, \bar{u}(s, x), \bar{u}(x, s)\right] - p(s, x) \right\} ds
\]

and hence

\[
\|v^{m+1}(t, x) - \bar{u}(t, x)\| \leq (\epsilon + \delta)\frac{1}{2}(b - a)^2 + \frac{1}{8}(b - a)^2 \sum_{s < t, x < b} \|f[t, x, v^m(t, x), v^m(x, t)]\| + (L_1 + L_2) \|v^m - \bar{u}\|
\]

\[
\leq (\epsilon + \delta)(b - a)^2 + \frac{1}{2}(b - a)^2 \left[(\Delta F + (1 + \Delta)(L_1 + L_2)) \|v^m - \bar{u}\|\right]
\]

\[
\leq (\epsilon + \delta + \Delta F)\frac{1}{2}(b - a)^2 + \theta N_1 = (1 - \theta) N_1 + \theta N_1 = N_1.
\]

This completes the proof of (2).

From the definition of \( u^{m+1}(t, x) \) and \( v^{m+1}(t, x) \) we have

\[
u^{m+1}(t, x) - v^{m+1}(t, x) = I(t, x) + \int_a^b G(t, s) f\left[s, x, v^m(s, x), v^m(x, s)\right] ds - v^{m+1}(t, x)
\]

and hence

\[
\|u^{m+1}(t, x) - v^{m+1}(t, x)\| \leq w^m + \theta \|u^m - v^m\|.
\]

Since \( u^0(t, x) = v^0(t, x) \) the above inequality provides that

\[
\|u^{m+1} - v^{m+1}\| \leq \sum_{i=0}^m \theta^{m-i} w^i.
\]

(21)

Using (21) in the triangle inequality, we get

\[
\|u^* - u^{m+1}\| \leq \sum_{i=0}^m \theta^{m-i} w^i + \|u^{m+1} - u^*\|.
\]

(22)

In (22), Theorem 3.1 ensures that

\[
\lim_{m \to \infty} \|u_{m+1} - u^*\| = 0.
\]

Thus, the condition

\[
\lim_{m \to \infty} w^m = 0
\]

is necessary and sufficient for the convergence of the sequence \( \{v^m(t, x)\} \) to \( u^*(t, x) \) follows from Toeplitz's lemma "for any \( 0 \leq \alpha < 1 \), let

\[
s_m = \sum_{i=0}^m \alpha^{m-i} d_i; \quad m = 0, 1, \ldots
\]

then

\[
\lim_{m \to \infty} s_m = 0
\]

if and only if

\[
\lim_{m \to \infty} d_m = 0^*.
\]
Finally, to prove (20) we note that

\[ u^*(t, x) - \nu^{m+1}(t, x) = \int_a^b G(t, s) \left\{ f[s, x, u^*(s, x), u^*(x, s)] - f[s, x, v^m(s, x), v^m(x, s)] \right\} \]

\[ + \int_a^b \left\{ f[s, x, v^m(s, x), v^m(x, s)] - f^m[s, x, v^m(s, x), v^m(x, s)] \right\} ds \]

and as earlier, we find

\[ \|u^*(t, x) - \nu^{m+1}(t, x)\| \leq \theta \|u^* - \nu^m\| + \Delta \frac{1}{2}(b - a)^2 \sup_{a \leq t, x \leq b} \|f[t, x, v^m(t, x), v^m(x, t)]\|. \]  \hspace{1cm} (23)

From (23) inequality (20) is obvious.

**Remark 4.1**

If in (18), \( \Delta < 1 \), then it is easy to obtain

\[ \sup_{a \leq t, x \leq b} \|f[t, x, v^m(t, x), v^m(x, t)]\| \leq (1 - \Delta)^{-1} \sup_{a \leq t, x \leq b} \|v^m[t, x, v^m(t, x), v^m(x, t)]\|. \]

Thus, in this case (20) can be replaced by a more practical error estimate:

\[ \|u^* - \nu^{m+1}\| \leq (1 - \theta)^{-1} \int_a^b \left\{ \theta \|v^{m+1} - \nu^m\| + \Delta(1 - \Delta)^{-1}\frac{1}{2}(b - a)^2 \sup_{a \leq t, x \leq b} \|f^m[t, x, v^m(t, x), v^m(x, t)]\| \right\}. \]

In our next result, we shall assume the following condition:

**Condition C**

For \( \nu^m(t, x) \) obtained from (17), the following inequality is satisfied:

\[ \sup_{a \leq t, x \leq b} \|f[t, x, v^m(t, x), v^m(x, t)] - f^m[t, x, v^m(t, x), v^m(x, t)]\| \leq \Delta_i \] \hspace{1cm} (24)

where \( \Delta_i \) is a non-negative constant.

Inequality (24) corresponds to an absolute error in approximating the function \( f \) by \( f^m \) for the \((m + 1)\)th iteration.

**Theorem 4.2**

Assume that the following conditions hold: (i) condition (i) of Theorem 3.1; (ii) condition C; (iii) \( \theta < 1 \), and

\[ N_2 = (1 - \theta)^{-1}(\epsilon + \delta + \Delta_i)^{\frac{1}{2}}(b - a)^2 \leq N. \]

Then, the following hold: (1) all the conclusions (1)-(5) of Theorem 3.1 hold; (2) the sequence \( \{v^m(t, x)\} \) obtained from (17) remains in \( \mathcal{S}(\mathcal{A}, N_2) \); (3) conclusion (3) of Theorem 4.1 holds with (20) replaced by

\[ \|u^* - \nu^{m+1}\| \leq (1 - \theta)^{-1}[\theta \|v^{m+1} - \nu^m\| + \Delta \frac{1}{2}(b - a)^2]. \] \hspace{1cm} (25)

*Proof.* The proof is similar to that of Theorem 4.1.

**5. SOME EXAMPLES**

**Example 5.1**

Consider the scalar boundary value problem:

\[ u^*(t, x) = -\frac{2}{(1 + t^2)} u(t, x) + \frac{8t^2}{(1 + x^2)^2} u(x, t); \]

\[ u(0, x) = \frac{1}{(1 + x^2)^3}; \quad u(b, x) = \frac{1}{(1 + b^2)(1 + x^2)^3}. \] \hspace{1cm} (26)
for which
\[ u(t, x) = \frac{1}{(1 + t^2)(1 + x^2)} \]
is a solution.

Obviously, in (26) the function \( f \) is of Lipschitz class on \([0, b] \times [0, b] \times R \times R\) with \( L_1 = 2, L_2 = 8b^2 \). Thus,
\[ \theta = \frac{1}{8}b^2(2 + 8b^2) < 1 \]
is satisfied provided \( b < 0.9395649 \ldots \)

For problem (26) we assume that \( \tilde{u}(t, x) \equiv 0 \), so that \( \delta = 0 \) and \( \tilde{I}(t, x) \equiv 0 \). Thus,
\[ \sup_{0 \leq t, x \leq b} \| I(t, x) - \tilde{I}(t, x) \| = \sup_{0 \leq t, x \leq b} \| I(t, x) \| = \max \left\{ \frac{1}{0 \leq x \leq b} \left( \frac{1}{1 + x^2} \right)^3, \frac{1}{0 \leq x \leq b} \left( \frac{1}{1 + b^2(1 + x^2)^3} \right) \right\} = 1, \]
and hence \( \varepsilon = \frac{1}{8}b^2 \).

Now, from the definition of \( N_0 \), we find that \( N_0 = (1 - \theta)^{-1} \). Thus, if \( b < 0.9395649 \ldots \), Theorem 3.1 and Remark 3.1 ensures that (1) the boundary value problem (26) has a solution \( u^*(t, x) \) such that
\[ \sup_{0 \leq t, x \leq b} \| u^*(t, x) - e^{-\varepsilon} \| \leq (1 - \theta)^{-1} \]
(2) \( u^*(t, x) \) is the unique solution of (1) and (2) in \( R^2 \), (3) the Picard iterative method (16) for the boundary value problem (26), converges to \( u^*(t, x) \) with
\[ \sup_{0 \leq t, x \leq b} \| u^*(t, x) - u^m(t, x) \| \leq \theta^m(1 - \theta)^{-1}. \]

**Example 5.2**

For the scalar boundary value problem
\[ u''(t, x) = e^{-t^2} \sin u'(t, x) + x^2u(x, t) + \frac{\cos tx^2}{1 + t^2 + x^2}; \]
\[ u(0, x) = 1, u(1, x) = e^{-x}, \] (27)
we assume that \( \tilde{u}(t, x) = e^{-\varepsilon}x \), so that \( \delta = 2, \varepsilon = 0 \). With this choice of \( \tilde{u}(t, x) \) the function \( f \) is of Lipschitz class on \([0, 1] \times [0, 1] \times D_4 \) with \( L_1 = 2(N + 1), L_2 = 1 \). Thus, \( \theta = 1/(2N + 3) < 1 \) is satisfied provided \( N < 2.5 \). Next, condition (15) reduces to
\[ N_0 = (1 - \frac{1}{4}N - \frac{3}{8})^{-1}(2 + 0)\frac{1}{8} \leq N, \]
which is satisfied if \( \frac{1}{4} \leq N \leq 2 \). Hence, the conditions of Theorem 3.1 are satisfied provided \( \frac{1}{4} \leq N \leq 2 \). Further, if \( N = \frac{1}{2} \), then \( N_0 = \frac{1}{2} \) and hence we have the following conclusions: (1) the boundary value problem (27) has a solution \( u^*(t, x) \) such that
\[ \sup_{0 \leq t, x \leq 1} \| u^*(t, x) - e^{-t^2} \| \leq \frac{1}{2}; \]
(2) \( u^*(t, x) \) is the unique solution of (27) in \( S(e^{-\varepsilon}, 2) \); and (3) the Picard iterative method (16) for the boundary value problem (27) converges to \( u^*(t, x) \) with
\[ \sup_{0 \leq t, x \leq 1} \| u^*(t, x) - u^m(t, x) \| \leq (\frac{1}{2})^{m+1}. \]

**REFERENCES**


