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The Bellman Equations of Dynamic Programming Concerning a New Class of Boundary Value Problems with "Total Derivatives"

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Submitted by Richard Bellman

1. In some of his previous papers [1-5] the author has developed from different points of view the boundary value problems concerning equations of nonelliptic type.[1]

He has established relations between the characteristic values of the boundary value problems

$$[A(x)\,u' + \lambda B(x)\,u]' + \lambda[B(x)\,u' + C(x)\,u] = 0 \tag{1}$$

on R, $u = 0$ on the boundary of R, and the minimum values of the functional

$$D(f) = \int_R A(x)\,f'^2(x)\,dx, \tag{2}$$

subject to the conditions that

$$H(f) = \int_R (2B(x)\,f(x)\,f'(x) + C(x)\,f^2(x))\,dx = \pm 1, \tag{3}$$

and $f = 0$ on the boundary of R.

The novel aspects of the theorems lies in the interpretation of R as a $2n$-dimensional rectangular domain and the symbol $'$ as designating total differentiation in the sense that

$$u' = \frac{\partial^{2n} u}{\partial x_1\,\partial x_2 \cdots \partial x_{2n}} \tag{4}$$

(Picone's sense [13, 14]).

[1] This problems received due mention in a large field of subsequent research papers belonging to well known scientists [6-12].

In order to simplify we begin with the boundary value problem

$$\frac{\partial^4 u}{\partial x^2\, \partial y^2} - \lambda A(x, y)\, u = 0, \tag{5}$$

$$u \Big|_{FR} = 0, \qquad R \equiv \begin{pmatrix} 0 \leqslant x \leqslant 1 \\ 0 \leqslant y \leqslant 1 \end{pmatrix}, \tag{6}$$

that, under light conditions on $A(x, y)$, is equivalent to the problem of determining the relative minima of

$$\int_0^1 \int_0^1 \left(\frac{\partial^2 u}{\partial x\; \partial y}\right)^2 dx\, dy, \tag{7}$$

subject to the constraints

$$u(0, y) = u(x, 0) = u(1, y) = u(x, 1) = 0, \tag{8}$$

$$\int_0^1 \int_0^1 A(x, y)\, u^2\, dx\, dy = 1, \tag{9}$$

or, conversely, to that of determining the relative maxima of

$$\int_0^1 \int_0^1 A(x, y)\, u^2\, dx\, dy, \tag{10}$$

subject to the constraints (8) and

$$\int_0^1 \int_0^1 \left(\frac{\partial^2 u}{\partial x\; \partial y}\right)^2 dx\, dy = 1. \tag{11}$$

2. In the first approach, following step by step the well known work by Bellman [15], closely related in Chapter IX with a new formalism in the Calculus of Variations, we consider the minimization of

$$J(u) = \int_{a_1}^1 \int_{a_2}^1 \left(\frac{\partial^2 u}{\partial x\; \partial y}\right)^2 dx\, dy, \tag{12}$$

over all u satisfying the conditions

$$u(a_1, y) = k_1(y), \qquad u(x, a_2) = k_2(x), \qquad k_1(a_2) = k_2(a_1) = k,$$
$$u(x, 1) = u(1, y) = 0, \tag{13}$$

$$\int_{a_1}^1 \int_{a_2}^1 A(x, y)\, u^2\, dx\, dy = 1. \tag{14}$$

Here the new state variables a_1, a_2 satisfy the conditions

$$0 \leqslant a_1 < 1, \qquad 0 \leqslant a_2 < 1. \tag{15}$$

We assume that the function $A(x, y)$ satisfies the constrain

$$0 < b_1 \leqslant A(x, y) \leqslant b_2$$

for all $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ and is continuous over

$$R = [0, 1] \times [0, 1].$$

Let us set a *functional*

$$f(a_1, a_2, k_1(y), k_2(x)) = \text{Min} \int_{a_1}^{1} \int_{a_2}^{1} \left(\frac{\partial^2 u}{\partial x\, \partial y}\right)^2 dx\, dy, \tag{16}$$

subject to the constraints (13), (14).

We write, along an extremal,

$$\int_{a_1+s_1}^{1} \int_{a_2+s_2}^{1} A(x, y)\, u^2\, dx\, dy$$

$$= \left[\int_{a_1}^{1} \int_{a_2}^{1} - \int_{a_1}^{a_1+s_1} \int_{a_2}^{a_2+s_2} - \int_{a_1}^{a_1+s_1} \int_{a_2+s_2}^{1} - \int_{a_1+s_1}^{1} \int_{a_2}^{a_2+s_2}\right] A(x, y)\, u^2\, dx\, dy$$

$$= 1 - s_1 \int_{a_2+s_2}^{1} A(a_1, y)\, k_1^2(y)\, dy - s_2 \int_{a_1+s_1}^{1} A(x, a_2)\, k_2^2(x)\, dx,$$

$$(a_1 + s_1 \leqslant x \leqslant 1; \qquad a_2 + s_2 \leqslant y \leqslant 1), \tag{17}$$

$$u(a_1 + s_1, a_2 + s_2) = k + s_1 v_1 + s_2 v_2,$$

$$v_1 = \left.\frac{\partial u}{\partial x}\right|_{x=a_1}, \qquad v_2 = \left.\frac{\partial u}{\partial y}\right|_{y=a_2}, \tag{18}$$

$$f(a_1, a_2, k_1(y), k_2(x)) = \int_{a_1+s_1}^{1} \int_{a_2+s_2}^{1} \left(\frac{\partial^2 u}{\partial x\, \partial y}\right)^2 dx\, dy$$

$$+ s_1 \int_{a_2+s_2}^{1} v_{1y}^2\, dy + s_2 \int_{a_1+s_1}^{1} v_{2x}^2\, dx,$$

$$v_{1y} = \left.\frac{\partial v_1}{\partial y}\right|_{x=a_1}, \qquad v_{2x} = \left.\frac{\partial v_2}{\partial x}\right|_{y=a_2}, \tag{19}$$

to terms in $o(s_1, s_2)$.

We now make the change of variable

$$u(x, y) = \left[1 - \frac{s_1}{2} \int_{a_2+s_2}^{1} A(a_1, y)\, k_1^2(y)\, dy - \frac{s_2}{2} \int_{a_1+s_1}^{1} A(x, a_2)\, k_2^2(x)\, dx\right] w(x, y) \qquad (20)$$

in order to maintain the condition (14). We then have

$$w(a_1 + s_1, a_2 + s_2) = k + s_1 v_1 + s_2 v_2 + \frac{ks_1}{2} \int_{a_2+s_2}^{1} A(a_1, y)\, k_1^2(y)\, dy + \frac{ks_2}{2} \int_{a_1+s_1}^{1} A(x, a_2)\, k_2^2(x)\, dx, \qquad (21)$$

$$f(a_1, a_2, k_1(y), k_2(x)) = \int_{a_1+s_1}^{1} \int_{a_2+s_2}^{1} \left(\frac{\partial^2 w}{\partial x\, \partial y}\right)^2 dx\, dy$$

$$\times \left(1 - \frac{s_1}{2}\right) \int_{a_2+s_2}^{1} A(a_1, y)\, k_1^2(y)\, dy - \frac{s_2}{2} \int_{a_1+s_1}^{1} A(x, a_2)\, k_2^2(x)\, dx + s_1 \int_{a_2+s_2}^{1} v_{1y}^2\, dy + s_2 \int_{a_1+s_1}^{1} v_{2x}^2\, dx, \qquad (22)$$

to terms in $o(s_1, s_2)$.

Combining the above results, we obtain the approximate functional equation, reproduced in ref. 16,

$$f(a_1, a_2, k_1(y), k_2(x)) = \underset{v_1, v_2}{\operatorname{Min}} \left[s_1 \int_{a_2+s_2}^{1} v_{1y}^2\, dy + s_2 \int_{a_1+s_1}^{1} v_{2x}^2\, dx\right.$$

$$+ \left(1 - \frac{s_1}{2} \int_{a_2+s_2}^{1} A(a_1, y)\, k_1^2(y)\, dy - \frac{s_2}{2} \int_{a_1+s_1}^{1} A(x, a_2)\, k_2^2(x)\, dx\right)$$

$$\times f\left(a_1 + s_1, a_2 + s_2, k + s_1 v_1 + \frac{s_1 k}{2} \int_{a_2+s_2}^{1} A(a_1, y)\, k_1^2(y)\, dy, k + s_2 v_2\right.$$

$$\left.\left. + \frac{s_2 k}{2} \int_{a_1+s_1}^{1} A(x, a_2)\, k_2^2(x)\, dx\right)\right] + o(s_1, s_2). \qquad (23)$$

Letting $s_1 \to 0$ and $s_2 \to 0$ one obtains the *generalized Bellman equations* related to his new technique concerning the classical variational solution of the eigenvalue problem

$$\frac{d^2u}{dt^2} + \lambda^2 g(t)\, u = 0, \qquad u(0) = u(1) = 0 \tag{*}$$

in the form of integro-differential equations with partial derivatives. Its approximate solution combined with some very recent results of Picone and L. S. Pontrjagin and his collaborators, related respectively to a very important sufficient criterion concerning the absolute minimum of a polydimensional integral [17], and to the mathematical theory of optimal processes [18], as well as various developments of Bellman's new variational technique [19] for both boundary value problems for ordinary differential equations of higher order and for partial or integro-differential equations with partial or "total derivatives," was presented [20] or published [21, 22, 23], or will be given in later papers.

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