SUMMARIES

Among the geometers who contributed to the rise of physical science, I mention Desargues and Poncelet, who created projective space by adding ideal elements to Euclidean space; also Cayley and Klein, who used a polarity to equip this projective space with a non-Euclidean metric; Riemann and Schlafli, who first understood that an n-dimensional continuum can be of finite extent without having a boundary (not only when \( n = 1 \) or 2 but also for greater values); and Study, who boldly stepped outside Klein's absolute quadric to discover the exterior-hyperbolic space which Du Val (seventeen years later) identified with De Sitter's world, thus providing a convincing explanation for the observed departure of the most remote objects in the universe. Most particularly I mention Minkowski, because he enriched affine space by inserting a real isotropic cone at every point, and because he invented the world line (which was so fruitfully developed by Robb and Synge).

Unter den Vertretern der Geometrie, die zum Aufschwung der Naturwissenschaften beigetragen haben, möchte ich insbesonders Desargues und Poncelet erwähnen, die den projektiven Raum als einen durch ideale Elemente erweiterten Euklidischen Raum einführten; weiter Cayley und Klein, die eine Polarität hinzufügten und so den projektiven Raum mit einer nicht-Euklidischen Metrik versahen; dann Riemann und Schlaffi, die als erste begriffen, dass ein \( n \)-dimensionales Kontinuum endliche Ausdehnung haben kann, ohne begrenzt zu sein (nicht nur für \( n = 1 \) oder 2, sondern auch für größere Werte); und schließlich Study, der aus dem Feld Kleinscher Geometrie des Inneren einer Quadrik ausbrach und so den wüssten hyperbolischen Raum entdeckte, welcher siebzehn Jahre später von Du Val als De Sitter's Welt erkannt wurde, wodurch sich eine einleuchtende Erklärung für die Flucht-Bewegung der Objekte an der Grenze des Universums ergab. Vor allem aber möchte ich Minkowski erwähnen, weil er den affinen Raum zu bereichern wusste, indem er an jedem Punkt einen rell-isotropen Kegel ansetzte, und weil er das Konzept einer Weltlinie erfand, das so erfolgreicht von Robb und Synge weiterentwickelt werden konnte.
1. Early Modifications of Euclidean Geometry

The independent discovery of hyperbolic geometry by Gauss, Bolyai and Lobachevsky is so famous that we are apt to forget some other examples of ideas appearing simultaneously in different countries when the time is ripe. As Bolyai's father said: "Many things ... are found at the same time in several places, just as violets appear on every side in spring." The possibility of extending Euclidean space to projective space (by adding the "points" and "lines" of an "ideal" plane) was hinted at by a much earlier pair of independent geniuses: Johann Kepler and Girard Desargues. Kepler [1604] considered what happens when the eccentricity of an ellipse is gradually increased towards 1, while one focus and the corresponding vertex are kept fixed. He suggested that the focus of a parabola is still one of two, the other being infinitely distant in both of two opposite directions, so that any point on the curve can be joined to this "blind focus" (which we now call a point at infinity) by a line parallel to the axis of the parabola. A few years later, Desargues [1639] declared that parallel lines have a common end at an infinite distance, and that when no point of a line is accessible the whole line is at an infinite distance. He thus introduced what we now call a line at infinity: the ideal intersection of two parallel planes. The deeply rooted notion that Euclidean space is the only possible kind of space was thus beginning to be upset, though the peculiarities were infinitely far away.

Two hundred years later, Poncelet [1822] showed that the lines at infinity on various planes in space behave like the lines in a plane: the plane at infinity. In 1827, K.W. Feuerbach and A.F. Möbius independently represented a point in the plane by three (instead of two) coordinates [Struik 1953, 61]. Two years later, Julius Plücker noticed that only the ratios of these three coordinates are needed. He thus discovered homogeneous coordinates, which provide an algebraic explanation for the "ideal" elements.

2. Spherical Space

The interaction of physics and mathematics is nicely illustrated by a remark of Schlafli [1858, 227; Cayley's rather poor translation]:

The first traces that I know of a theory of the general equation of the second degree appear in Laplace's Mécanique Céleste on the occasion of the secular perturbations of the solar system.... We proceed to one of its most particular cases, the general spheric equation

\[ x_1^2 + x_2^2 + \ldots + x_n^2 = a^2. \]

... For the curved continuum itself, represented by this
equation, I propose the term polysphere.... We are assuming for shortness $a = 1$ .... The total polyspheric continuum [has] for its measure $2\pi^{n/2}/\Gamma(n/2)$.

(For a plausible reconstruction of the way Schlafli obtained this expression, see Coxeter 1973, 125.) Using spherical coordinates, analogous to latitude and longitude, Schlafli explored this finite but unbounded space and described its trigonometry in great detail. He seemed especially pleased with his discovery that the whole three-dimensional spherical continuum (that is, the case when $n = 4$) can be dissected, by means of sixty planes (or "great spheres") into 14400 tetrahedra, each having, as its dihedral angles, three right angles, two of 60° and one of 36°. (In Figure 1, each edge has been marked with the dihedral angle between the planes that meet there.) Since the volume of the whole spherical 3-space is $2\pi^2$, this means that the volume of one such tetrahedron is $\pi^2/7200$.

As Schlafli achieved all this (and much more) before 1854, why is the discovery of spherical space nearly always ascribed to Riemann alone? The answer seems to be that Riemann stepped outside the ivory tower of pure mathematics and philosophized about applications to the real world. People were excited by his famous words about Unbegrenztheit and Unendlichkeit [Riemann 1866]:

The unboundedness of space possesses a greater empirical certainty than any other experience. But its infinite extent by no means follows from this.
It thus seems reasonable to assert that, although Schläfli stands alone as the discoverer of the regular spherical honeycombs, and Riemann as the discoverer of his manifolds of variable curvature, the Swiss and the German should be honoured equally for the common part of these two subjects: the manifolds of constant positive curvature $K = 1/a^2$. In other words, just as Gauss, Bolyai and Lobachevsky were codiscoverers of the first kind of non-Euclidean geometry (hyperbolic geometry, which E. Beltrami recognized as the geometry of manifolds of constant negative curvature), so Schläfli and Riemann were codiscoverers of the second kind (spherical geometry).

3. Hyperbolic Space

Schläfli did not confine himself to the case of positive curvature. In two brief paragraphs [1858, 162] he allowed $K$ to be negative. For instance, he found that a tetrahedron of the kind described above, but with 30° instead of 36° (see Figure 2) and $K = -1$, has volume $(1/4)(L(\pi/3) - 2L(\pi/6))$, where

$L(\alpha) = \int_0^\alpha \log \sec \theta \, d\theta$. Incidentally, the expression $L(\pi/3)$

$-2L(\pi/6)$, in the form

$$\frac{1}{4\sqrt{3}} \left(1 + \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{72} + \frac{1}{8^2} - \frac{1}{10^2} - \frac{1}{11^2} + \ldots\right)$$

$$= -\frac{1}{2\sqrt{3}} \left(\frac{\pi^2}{27} + \int_0^1 \frac{\log x}{1 + x^3} \, dx\right)$$

appeared recently in an "advanced problem" proposed by F. Haring and G.T. Nelson [Bach 1974; see also Coxeter 1968a, 20 and 174].

![Figure 2](image-url)
Obviously Schlafli was unaware that Lobachevsky had already solved many such problems of hyperbolic mensuration.

4. Projective Space

By accepting the possibility of an $n$-dimensional continuum of positive or negative curvature, Schlafli and Riemann prepared the ground for relativistic cosmology. Arthur Cayley [1843; 1859] and Felix Klein [1871] made another contribution by embedding both kinds of non-Euclidean space in a projective space. In particular, Klein regarded hyperbolic space as the part of projective space that lies inside a non-ruled quadric, the points on the quadric itself being the "points at infinity" determined by the various bundles of parallel lines. Then Eduard Study [1907] asked himself: "What about the rest of the projective space, the part outside the absolute quadric?"

This exterior-hyperbolic space, which Study investigated as pure geometry, is strikingly different. Although its points are just the poles of the planes in hyperbolic space, its lines and planes are of various types. For instance, from any such exterior point we can draw an enveloping cone to the quadric.

The generators of the cone, being tangents to the quadric, are isotropic (that is, self-perpendicular) lines, along which all distances are zero!

5. Minkowski's World

By having a cone of isotropic lines at each point, Study's exterior-hyperbolic space resembles the flat space-time which Minkowski [1909] proposed as the geometry for Einstein's special theory of relativity: an affine 4-space with a special cone $x^2 + y^2 + z^2 - t^2 = 0$. Each point $(x,y,z,t)$ is the centre (or vertex) of such an isotropic cone whose generators are parallel to those of the first cone. In the physical application, each point is an event in space-time, and the generators of the isotropic cone are world lines of photons [Synge 1960, 109].

The representation of the whole "life" of any particle by a "world line" (a continuous curve in a four-dimensional continuum, length along it representing proper time) was praised by J.L. Synge [1958] as being an idea of crucial importance, putting relativity theory on a firm geometric foundation. For an unaccelerated particle the world line is straight. If the particle is not a photon, the line is timelike; that is, interior to the isotropic cone at each of its points, and measurements along it are measurements of the particle's proper time. The world line of an accelerated particle is a curve having timelike tangents.

The Minkowskian 4-space thus contains straight lines of three kinds: timelike, isotropic and spacelike, the last being exterior to the isotropic cone at each of its points. Any two points...
on a spacelike line represent events each of which occurs neither before nor after the other, illustrating the principle that simultaneity at different places cannot be consistently defined [Robb 1936, 14-21].

6. De Sitter's World

All these geometric ideas, and their physical interpretation, remain valid when Minkowski's affine 4-space with a special cone is replaced by Study's projective 4-space with an "absolute" quadric hypersurface whose tangents are the isotropic lines. In both cases, timelike lines are infinitely long; eternity is infinite. The essential difference is that, whereas Minkowski's spacelike lines are likewise infinite, Study's are finite, like the great circles on a sphere.

Du Val [1924] recognized this 4-dimensional exterior-hyperbolic space as being practically identical with the world of de Sitter [1917] concerning which Synge wrote:

*The de Sitter universe is interesting in itself. It opens up new vistas, introducing us to the idea that space (a slice of space-time) may be finite, and this seems to satisfy some mental need in us, for infinity is one of those things which we find difficulty in comprehending.* [Synge 1960]

Timelike lines remain infinite, but eternity is easier to accept than a cyclic view of time.

If we represent the world line of our Earth as a directed secant MN of the absolute quadric, the points M and N, where this line penetrates the hypersurface, are the infinitely distant beginning and end of our proper time. Let M'N' (Figure 3) be the analogous world line of a distant star. The section of the absolute quadric hypersurface by the hyperplane MNM'N' is a non-ruled quadric surface. Since projective 3-space admits only one kind of non-ruled quadric, we lose no generality by visualizing this section as a hyperboloid of two sheets.

7. The Expanding Universe

It is easier to interpret another nice geometric concept: the tangent hyperplane to the quadric at N, say v. This hyperplane, which appears in Figure 3 as the tangent at N to the hyperbola, is our astronomical horizon, the set of events that have just ceased to be observable. This concept may be explained as follows. Every point on the secant M'N' represents an event in the history of the "distant star." One such point is B', on the tangent at N. Any earlier event in the history of the star appears as a point A' between M' and B'. One of the photons emitted by the star at that earlier event has a world line A'T which touches the quadric and intersects our own world line at A. This point A is the event of our seeing
Figure 3

the star. In a similar manner we continue to see the star so long as the star's events precede \( B' \). But the event \( B' \) itself is like the familiar sight of a ship disappearing over the horizon: a photon emitted then will never reach us, and it will seem that the star has been receding from us with an ever-increasing velocity that has ultimately reached the velocity of light so that we cannot observe it any more.

This simple geometric picture removes most of the mystery that used to obscure the concept of the expanding universe.

8. The Twin (or Clock) Paradox

Another principle that is easily explained is Einstein's Twin Paradox. ("Travel keeps you young.") One of two twin brothers travels to a star and returns home. Figure 4 shows his world line during the trip as \( AB + BC \). His brother stays at home, so that his world line is \( AC \). By the "non-triangle inequality," \( AB + BC < AC \) for time like lines [Coxeter 1943,
the total proper time for the traveller is less than for his brother: at the event C when they meet again, the traveller is younger.

The same projective diagram provides an interpretation for velocity. Let $b = AC$ and $c = AB$ be the world lines of the Earth and of the traveller's space-ship, respectively; also let $b'$ and $c'$ be lines conjugate to $b$ and $c$ through $A$, that is, joining $A$ to the poles of $b$ and $c$ (see Figure 4). Then the velocity of the space-ship (relative to the Earth) on its outward journey, is the velocity of light multiplied by the square root of the cross ratio $\{bb', cc'\}$ [Coxeter 1968a, 258] However, this approximate description ignores the acceleration of the space-ship when it leaves the Earth and when it lands on a star. As acceleration bends the world line, a more realistic picture would be Figure 5.
Some people try to argue that each twin is travelling the same way relative to the other. But they are forgetting that the principle of relativity applies only to unaccelerated relative motion: although velocity is relative, acceleration is absolute.

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