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An adaptive fuzzy observer-based approach for chaotic synchronization

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Abstract

This paper presents an adaptive fuzzy observer design to synchronize chaotic systems. The chaotic system is expressed in the form of Takagi–Sugeno fuzzy model (T–S fuzzy system), which considers the effect of model mismatches. Based on this model, an adaptive fuzzy observer is developed to deal with the synchronization of nonidentical chaotic systems. In contrast to the framework of parallel distributed compensation for T–S fuzzy system, the proposed method does not rely on the existence of common matrix *P* which is imposed in stability conditions. The computer simulation examines the performance of two well-known chaotic systems, Lorenz system and Chua circuit. The results show that the proposed approach can not only attain synchronization but also is robust to parameter perturbations in the drive system.

Keywords: T–S fuzzy model; Adaptive fuzzy observer

1. Introduction

Since the pioneering works of [5,18], the secure communication via chaotic signals has been extensively discussed in the past decade. The chaotic systems are essentially

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nonlinear systems with the particular properties of broadband, noise-like, and difficult to predict. With these properties the secure communication can perform transmitting messages, masked by chaotic signals or modulated in chaotic systems. The idea of secure communication is to encrypt a plain text at the transmitter and decrypt the cipher text at the receiver. The transmission channels are public in general. That means anyone is able to detect or obtain the transmitted data. Therefore, it is advisable to mask or modulate the information within a chaotic signal and retrieve it from the received signal. For these purposes, the principle of synchronization of chaotic system is utilized.

Synchronization of chaotic system has attracted much interest in study for its potential applications in many areas such as secure communication, chemical reactions, biological systems, and information processing [3]. The primary scheme of chaos synchronization is the drive-driven systems addressed by Pecora and Carroll [13], where the output of drive system is to control the response of driven system such that coherent response of both systems can be attained. From the control point of view, synchronization can be viewed as designing proper observer [11]. The driving system usually transmits some of its states to the driven system at the receiving end. Then, based on the limited signals developing control of the observer is to recover all states of the drive system. Accordingly, the state variables of the receiver (driven) system have synchronous responses with the original (drive) system.

There are various methodologies for chaos synchronization studied in the past decade [3]. Many research works are developed on the basis of exact mathematic model of the system and noise-free condition. However, for practical application such perfection seems to be unrealistic. It is very common that the perturbations of parameters are inevitable due to device aging or varying of the operating environment. Besides, noise or disturbance does exist physically. The robust control strategy usually deals with these problems and provides an effective solution to synchronization of chaotic system.

Several adaptive synchronization methods have been reported recently. For examples, Liao and Tsai [9] addressed an adaptive observer to estimate the unknown parameter and disturbance of a chaotic system with output feedback term. Chen and Lu [4] derived an adaptive algorithm to a specific uncertain unified chaotic model. Feki [7] designed complete adaptive observer-based response system to synchronize chaos with parameter uncertainties. Having been employed in the continuous-time chaotic systems, adaptive control is also applicable to synchronization of discrete-time chaos. The principle of parameter identification was cited in [6]. Based on this principle, the slave system parameters were tuned by both synchronization and optimization algorithms. Synchronization of model-unknown discrete-time generalized Henon map was discussed in [19].

The above research works are essentially based on the explicit mathematic model to analyze and design the synchronization of continuous-time or discrete-time chaotic system. In this work, the synchronization by fuzzy control methodology is addressed. The fuzzy model of chaotic system can be obtained by model construction [15] or linearization method [14] which is adopted in this paper. Generally, the nonlinear system is linearized around some operating points to produce certain local linear dynamics. These local models are then smoothly aggregated by the fuzzy inferences, which give the T–S fuzzy model. In fact a wide class of chaotic systems can be represented by T–S fuzzy models with one premise variable in fuzzy rules [8].

As to the controller and observer design of T–S fuzzy system, much research works are based on the concept of parallel distributed compensation [16,17] to derive the sufficient conditions of stability and observability and yield fruitful results. According to the approach, a common matrix P is required for each fuzzy local system, which is to guarantee the stabilization of the global fuzzy system (in the sense of Lyapunov). Generally, the exploration of the common matrix for synthesis of controller or observer can be recast as convex problem that can be solved efficiently by linear matrix inequality (LMI) optimization techniques. In spite of LMI's advantages, the existence of the solution, which satisfies the sufficient conditions, is still not guaranteed. Specifically as the number of fuzzy rule becomes large or too many constraints are imposed, the solution could be infeasible [10].

As opposed to LMI's method, we explore an adaptive fuzzy observer scheme to deal with the synchronization problem. In each fuzzy subspace, the local dynamics is regarded as the nominal model and the effect of other rules stand for the uncertainty. Accordingly, the T–S fuzzy model can be treated as the linear uncertain system. Further, by appropriately selecting the observer gain and deriving an updating law for estimation of the uncertainty, the synchronization and stability of the global system can be guaranteed by the Lyapunov approach. Finally, the computer simulations demonstrate the effectiveness and robustness of proposed scheme.

2. T–S fuzzy model of chaos systems

Consider a general chaotic system described by

$$\begin{aligned} \dot{x}(t) &= f(x(t)), \\ y(t) &= h(x(t)), \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector, $y(t) \in R^m$ is the output vector, $f(\cdot)$ and $h(\cdot)$ are nonlinear functions with appropriate dimensions. Further, it is assumed that $f(\cdot)$ is known or partly known and has continuous differentiation to $x(t)$, and the output equation is of the linear form $y(t) = Cx(t)$. The dynamics of the chaos system can be approximated around some operating points x_i's. Then, the linearized state equation [14] can be written as

$$\dot{x}(t) = A_i x(t) + d_i(t), \quad i = 1, 2, \ldots, q, \tag{2}$$

where $A_i = (\partial f/\partial x)_{x=x_i}$, $d_i(t) = (f(x_i) - A_i x_i + \text{higher order terms})$ stands for the approximation error, and q is the number of operating point. Note that in contrast to general linearization method x_i needs not be an equilibrium point. Since the linearized models depict system dynamics in local region around the operating points, it is advisable to apply T–S fuzzy inference to aggregate these models smoothly [20]. With proper selection and definition of input variables and the corresponding membership functions, the T–S fuzzy representation for chaotic system is written as

$$\begin{aligned} R^l:\ & \text{If } z_1(t) \text{ is } M_1^l \text{ and } \dots \text{ and } z_j(t) \text{ is } M_j^l \\ & \text{then } \dot{x}(t) = A_l x(t) + d_l(t),\ y(t) = Cx(t) \\ & l = 1, 2, \dots, q, \end{aligned} \tag{3}$$

where M_k^l denotes the fuzzy set $(k = 1, 2, \dots, j)$ and $z(t) = [z_1(t), z_2(t), \dots, z_j(t)]^{\mathrm{T}}$ is the premise variable vector associated with the system states.

By center of gravity defuzzification, the inferred result of state equation is

$$\dot{x}(t) = \frac{\sum_{l=1}^{q} w_l(z)[A_l x(t) + d_l(t)]}{\sum_{l=1}^{q} w_l(z)}, \tag{4}$$

where $w_l(z) = \prod_{i=1}^{j} M_i^l(z_i)$ and $M_i^l(z_i)$ is the grade of membership function M_i^l corresponding $z_i(t)$. Since the output equation in each fuzzy rule has the same form, therefore it can be viewed as the inferred result directly. Let $\mu_l(z)$ be defined as

$$\mu_l(z) = \frac{w_l(z)}{\sum_{l=1}^{q} w_l(z)}. \tag{5}$$

Then (4) becomes

$$\dot{x}(t) = \sum_{l=1}^{q} \mu_l(x)[A_l x(t) + d_l(t)]. \tag{6}$$

It is obviously $\sum_{l=1}^{q} \mu_l(z) = 1$ and $\mu_l(z) \geqslant 0$ for $l = 1, 2, \dots, q$.

The principle of parallel distributed compensation [17] provides an effective methodology to systematic analysis and design of T–S fuzzy system. Many research works follow this concept to derive the fuzzy controller and fuzzy observer [12]. According to their results there should exist a common matrix P for each fuzzy local dynamics, which is to guarantee stabilization of the global system in the sense of Lyapunov. However, the common matrix P was not easy to obtain until Tanaka et al. [16] introduced linear matrix inequality (LMI) method to solve this problem. In fact, the LMI is a very powerful tool to the design of T–S fuzzy systems. It is shown that LMI method can lead to an effective way for deducing sufficient conditions of stabilization [12,16]. In spite of its success the existence of the solution is not guaranteed. When the number of fuzzy rules is large or constraints increase, the solutions may be infeasible [10].

To avoid the problem of searching matrix P, the concept of piecewise smooth quadratic Lyapunov functions is addressed in [2]. Based on their results the state space of fuzzy system is participated into q fuzzy subspaces, S_l, where $S_l = \{z | \mu_l(z) \geqslant \mu_i(z), i = 1, 2, \dots, q, i \neq l\}$. Then, assume that the partition is well behaved, i.e., the system dynamics is completely described by the union of each local dynamics in every fuzzy subspace. In each subspace S_l, the state equation is given by

$$\begin{aligned}\dot{x}(t) &= \mu_l(z)A_lx(t) + \sum_{\substack{i=1\\i\neq l}}^{q}\mu_i(z)A_ix(t) + \sum_{i=1}^{q}\mu_i(z)d_i(t)\\ &= A_lx(t) + \Delta A_lx(t) + \sum_{i=1}^{q}\mu_i(z)d_i(t),\end{aligned} \tag{7}$$

where

$$\Delta A_l = \sum_{\substack{i=1\\i\neq l}}^{q}\mu_i(z)(A_i - A_l). \tag{8}$$

From (7), it reveals that in subspace S_l the system dynamics is dominated by local model A_l and other fuzzy inferences are regarded as uncertainties. The design of fuzzy system is not only to focus on the local dynamics A_l but also deal with the effect of interactions coming from the other fuzzy rules. When the stabilization in each fuzzy subspace is achieved, the global system stability is achieved in the sequel.

This concept prompts us to modify (6) as

$$\dot{x}(t) = (A_0 + \Delta A(t,x))x(t) + D(t,x), \tag{9}$$

where

$$\begin{aligned}A_0 &= \frac{1}{q}\sum_{i=1}^{q}A_i,\\ \Delta A(t,x) &= \sum_{i=1}^{q}\mu_i(z)(A_i - A_0),\\ D(t,x) &= \sum_{i=1}^{q}\mu_i(z)d_i.\end{aligned} \tag{10}$$

The T–S fuzzy system is regarded as a perturbed linear system consisting of the nominal dynamics A_0x, the perturbation $\Delta A(t,x)$, and the modeling error or disturbance $D(t,x)$.

3. Adaptive observer for T–S fuzzy systems

Before proceeding derivation, we impose the matched conditions on the system (9) over the uncertainties $\Delta A(t,x)$ and $D(t,x)$. It is assumed that there are some matrices $B_0 : R^{n\times p}$, $E(\cdot)$: $R\times R^n \to R^p$, and $F(\cdot)$: $R\times R^n \to R^p$ such that

$$\begin{aligned}\Delta A(t,x) &= B_0E(t,x),\\ D(t,x) &= B_0F(t,x).\end{aligned} \tag{11}$$

Accordingly, one may rearrange the fuzzy system (9) as

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + B_0(E(t,x) + F(t,x)) = A_0x(t) + B_0\xi(t,x), \\ y(t) &= Cx(t), \end{aligned} \tag{12}$$

where $\xi(t,x) \in R^p$ denotes the lumped uncertainty. Herein we assume (A_0, B_0) to be controllable and (A_0, C) to be observable. An adaptive observer is designed to achieve asymptotic state estimation in the presence of uncertainty. Let the observer for T–S fuzzy model be defined as

$$\begin{aligned} \dot{\hat{x}}(t) &= A_0\hat{x}(t) + B_0(u(t) + \hat{\xi}(\hat{x}|\theta)) + L(\hat{y}(t) - y(t)), \\ \hat{y}(t) &= C\hat{x}(t), \end{aligned} \tag{13}$$

where $u(t) \in R^p$ is the control vector, $\hat{\xi}(\hat{x}|\theta)$ is the estimation of $\xi(t,x)$, and $L \in R^{n\times m}$ is the constant observer gain. Since $\xi(t,x)$ is unknown and estimation of the state can only depend on the output injection, a unifying adaptive fuzzy observer scheme is proposed as follows.

As been employed in published literatures, the fuzzy systems can approximate any real continuous function on a compact set to an arbitrary accuracy. Accordingly we use the following Mamdani type fuzzy inference to estimate the ith element of ξ, ξ_i, as

$$\begin{aligned} R^j:\ &\text{If } x_1(t) \text{ is } \widetilde{M}_1^j \text{ and } \ldots \text{ and } x_n(t) \text{ is } \widetilde{M}_n^j \\ &\text{then } \hat{\xi}_i \text{ is } \widetilde{D}_{ij},\ j = 1, 2, \ldots, r. \end{aligned}$$

The output of fuzzy inference is given by

$$\hat{\xi}_i(x|\theta_i) = \frac{\sum_{j=1}^{r} \theta_{ij}\left(\prod_{h=1}^{n} \mu_{\widetilde{M}_h^j}(x_h)\right)}{\sum_{j=1}^{r}\prod_{h=1}^{n} \mu_{\widetilde{M}_h^j}(x_h)} = \theta_i^{\mathrm{T}}\omega(x), \tag{14}$$

where $\theta_i = (\theta_{i1}, \theta_{i2}, \ldots, \theta_{ir})^{\mathrm{T}}$ is an adjustable parameter vector, θ_{ij} is the center of $\widetilde{D}_{ij}$ for $i = 1, 2, \ldots, p$, and $j = 1, 2, \ldots, r$, $\omega(x)$ is called the fuzzy basis function. Then, we have $\hat{\xi}(x|\theta) = \theta^{\mathrm{T}}\omega(x)$, $\theta \in R^{r\times p}$. Assume that there exists an optimal parameter matrix

$$\theta^* = \arg\min_{\theta\in\Omega_\theta}\left\{\sup_x \left\|\hat{\xi}(x|\theta) - \xi(t,x)\right\|\right\} \tag{15}$$

such that

$$\left\|\hat{\xi}(x|\theta^*) - \xi(t,x)\right\| \leqslant \varepsilon, \tag{16}$$

where $\Omega_\theta = \{\theta|\mathrm{tr}(\theta^{\mathrm{T}}\theta) < M_\theta\}$, M_θ is a given constant, and ε is an unknown upper-bound and can be determined via adaptive mechanism. Since the system state is unavailable for measurement, we replace x by $\hat{x}$ in the premise of fuzzy rules and yield the estimation $\hat{\xi}(\hat{x}|\theta)$.

Let $\tilde{x} = \hat{x} - x$ be the estimation error. Then, from (12) and (13) the error dynamics is described by

$$\begin{aligned}\dot{\tilde{x}} &= A_0\tilde{x} + B_0(u + \hat{\xi}(\hat{x}|\theta) - \xi(t,x)) + LC\tilde{x} \\ &= (A_0 + LC)\tilde{x} + B_0(\hat{\xi}(\hat{x}|\theta) - \xi(t,x) + \hat{\xi}(\hat{x}|\theta^*) - \hat{\xi}(\hat{x}|\theta^*) + \hat{\xi}(x|\theta^*) - \hat{\xi}(x|\theta^*) + u) \\ &= \bar{A}\tilde{x} + B_0(\tilde{\theta}^{\mathrm{T}}\omega(\hat{x}) + \hat{\xi}(\hat{x}|\theta^*) - \hat{\xi}(x|\theta^*) + \hat{\xi}(x|\theta^*) - \xi(t,x) + u),\end{aligned} \tag{17}$$

where $\tilde{\theta} = \theta - \theta^*$. The design work is to explore adaptive law for adjusting parameters θ and determine appropriate control $u(t)$ such that the state estimation error converges to zero while maintaining all signals bounded.

To ensure the stability of adaptive observer, we give the following hypotheses:

(A1) The fuzzy basis function $\omega(x)$ is Lipschitz in x with constant γ, i.e.,

$$\|\omega(\hat{x}) - \omega(x)\| \leqslant \gamma\|\widehat{x} - x\|. \tag{18}$$

(A2) There exist two symmetric positive definite matrices P and Q satisfying

$$\begin{aligned}&\overline{A}^{\mathrm{T}}P + P\overline{A} = -Q, \\ &PB_0 = C^{\mathrm{T}}.\end{aligned} \tag{19}$$

(A3) The matrices P and Q have the following property

$$2\gamma M_\theta\|B_0\| \leqslant \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}. \tag{20}$$

The Lipschitz condition of $\omega(x)$ can be verified by investigating the norm value of its Jacobian matrix. The second assumption is exactly the celebrated Kalman–Yakubovich lemma [14]. The assumption (A3) refers to the sufficient condition to design of stable adaptive observer [1].

Choose the control input and the adaptive laws for updating parameters as

$$u = \begin{cases} -\dfrac{\hat{\varepsilon}B_0^{\mathrm{T}}P\tilde{x}}{\|\tilde{x}^{\mathrm{T}}PB_0\|} & \text{if } \|\tilde{x}^{\mathrm{T}}PB_0\| \neq 0, \\ 0 & \text{if } \|\tilde{x}^{\mathrm{T}}PB_0\| = 0, \end{cases} \tag{21}$$

$$\dot{\theta} = \begin{cases} -2\eta_1\omega(\widehat{x})\tilde{x}^{\mathrm{T}}PB_0 \\ \quad \text{if } (\|\theta\| < M_\theta) \text{ or } \left(\|\theta\| = M_\theta \text{ and } \mathrm{tr}\left(\theta^{\mathrm{T}}\omega(\hat{x})\tilde{x}^{\mathrm{T}}PB_0\right) \geqslant 0\right), \\ -2\eta_1\omega(\widehat{x})\tilde{x}^{\mathrm{T}}PB_0 + 2\eta_1\mathrm{tr}\left(\theta^{\mathrm{T}}\omega(\hat{x})\tilde{x}^{\mathrm{T}}PB_0\right)\dfrac{1 + \|\theta\|^2}{M_\theta^2}\theta \\ \quad \text{if } \|\theta\| = M_\theta \text{ and } \mathrm{tr}(\theta^{\mathrm{T}}\omega(\hat{x})\tilde{x}^{\mathrm{T}}PB_0) < 0, \end{cases} \tag{22}$$

$$\dot{\hat{\varepsilon}} = 2\eta_2\|B_0^{\mathrm{T}}P\tilde{x}\|, \tag{23}$$

where $\hat{\varepsilon}$ is the estimation of ε, and η_1 and η_2 are adaptation gains. Then, we have the following theorem.

Theorem 1. *Consider the chaotic system (12) satisfying the assumptions* (A1)–(A3). *If the adaptive laws (22) and (23) and control input (21) are applied, then the adaptive fuzzy observer (13) is convergent to the states x asymptotically, i.e., $\tilde{x} \to 0$ as $t \to \infty$. Furthermore, the boundedness of all signals in the closed-loop is guaranteed.*

Proof. We first prove the bounded property of θ using adaptation (22). Define a positive function $V_\theta = \frac{1}{2}\mathrm{tr}(\theta^{\mathrm{T}}\theta) = \frac{1}{2}\|\theta\|^2$. It is true that $\dot{V}_\theta = \frac{\mathrm{d}V_\theta}{\mathrm{d}t} = \mathrm{tr}(\theta^{\mathrm{T}}\dot{\theta})$. Applying the adaptive law (22) one yields

$$\mathrm{tr}(\theta^{\mathrm{T}}\dot{\theta}) = \begin{cases} -2\eta_1 \mathrm{tr}(\theta^{\mathrm{T}}\omega(\hat{x})\tilde{x}^{\mathrm{T}}PB_0) \\ \quad \text{for } (\|\theta\| < M_\theta) \text{ or } (\|\theta\| = M_\theta \text{ and } \mathrm{tr}(\theta^{\mathrm{T}}\omega(\hat{x})\tilde{x}^{\mathrm{T}}PB_0) \geqslant 0), \\ -2\eta_1 \mathrm{tr}(\theta^{\mathrm{T}}\omega(\hat{x})\tilde{x}^{\mathrm{T}}PB_0) + 2\eta_1 \mathrm{tr}(\theta^{\mathrm{T}}\omega(\hat{x})\tilde{x}^{\mathrm{T}}PB_0)\dfrac{1+\|\theta\|^2}{M_\theta^2}\mathrm{tr}(\theta^{\mathrm{T}}\theta) \\ \quad \text{for } \|\theta\| = M_\theta \text{ and } \mathrm{tr}(\theta^{\mathrm{T}}\omega(\hat{x})\tilde{x}^{\mathrm{T}}PB_0) < 0. \end{cases} \tag{24}$$

As to the first case, it is obviously $\dot{V}_\theta \leqslant 0$. In the second case, since $\|\theta\| = M_\theta$, we have $\mathrm{tr}(\theta^{\mathrm{T}}\dot{\theta}) = 2\eta_1 \mathrm{tr}(\theta^{\mathrm{T}}\omega(\hat{x})\tilde{x}^{\mathrm{T}}PB_0)\|\theta\|^2 < 0$. Therefore, the boundedness of θ, $\|\theta\| \leqslant M_\theta$, is assured.

Consider the Lyapunov function candidate

$$V = \tilde{x}^{\mathrm{T}}P\tilde{x} + \frac{1}{2\eta_1}\mathrm{tr}(\tilde{\theta}^{\mathrm{T}}\tilde{\theta}) + \frac{1}{2\eta_2}\tilde{\varepsilon}^2, \tag{25}$$

where $\tilde{\varepsilon} = \hat{\varepsilon} - \varepsilon$. The time derivative of V is

$$\begin{aligned} \dot{V} &= \tilde{x}^{\mathrm{T}}(\overline{A}^{\mathrm{T}}P + P\overline{A})\tilde{x} + 2\tilde{x}^{\mathrm{T}}PB_0\theta^{*^{\mathrm{T}}}(\omega(\hat{x}) - \omega(x)) + 2\tilde{x}^{\mathrm{T}}PB_0\tilde{\theta}^{\mathrm{T}}\omega(\hat{x}) \\ &\quad + 2\tilde{x}^{\mathrm{T}}PB_0\left(\hat{\xi}(x|\theta^*) - \xi(t,x) - \frac{\varepsilon B_0^{\mathrm{T}}P\tilde{x}}{\|\tilde{x}^{\mathrm{T}}PB_0\|} - \frac{\tilde{\varepsilon}B_0^{\mathrm{T}}P\tilde{x}}{\|\tilde{x}^{\mathrm{T}}PB_0\|}\right) + \frac{1}{\eta_1}\mathrm{tr}(\tilde{\theta}^{\mathrm{T}}\dot{\theta}) + \frac{1}{\eta_2}\tilde{\varepsilon}\dot{\hat{\varepsilon}} \\ &\leqslant -\tilde{x}^{\mathrm{T}}Q\tilde{x} + 2\|\tilde{x}^{\mathrm{T}}PB_0\|\|\theta^*\|\|\omega(\hat{x}) - \omega(x)\| + \frac{1}{\eta_1}\mathrm{tr}\left[\tilde{\theta}^{\mathrm{T}}\left(\dot{\theta} + 2\eta_1\omega(\hat{x})\tilde{x}^{\mathrm{T}}PB_0\right)\right] \\ &\quad + 2\|\tilde{x}^{\mathrm{T}}PB_0\|\left(\left\|\hat{\xi}(x|\theta^*) - \xi(t,x)\right\| - \varepsilon\right) + \frac{1}{\eta_2}\tilde{\varepsilon}\left(\dot{\hat{\varepsilon}} - 2\eta_2\|\tilde{x}^{\mathrm{T}}PB_0\|\right). \end{aligned} \tag{26}$$

Substituting (22) and (23) into (26) and using the fact $\|\theta^*\| \leqslant M_\theta$ result in

$$\begin{aligned}\dot{V} &\leqslant -\tilde{x}^{\mathrm{T}} Q\tilde{x} + 2\gamma M_\theta \|\tilde{x}^{\mathrm{T}} PB_0\| \|\tilde{x}\| + 2\eta_1 I_1 \mathrm{tr}\left(\theta^{\mathrm{T}}\omega(\hat{x})\|\tilde{x}^{\mathrm{T}} PB_0\|\right) \frac{1+\|\theta\|^2}{M_\theta^2} \mathrm{tr}(\tilde{\theta}^{\mathrm{T}}\theta) \\ &\leqslant -\tilde{x}^{\mathrm{T}}(\lambda_{\min}(Q) - 2\gamma M_\theta \|B_0\| \lambda_{\max}(P))\tilde{x} \\ &\quad + 2\eta_1 I_1 \mathrm{tr}\left(\theta^{\mathrm{T}}\omega(\hat{x})\|\tilde{x}^{\mathrm{T}} PB_0\|\right) \frac{1+\|\theta\|^2}{M_\theta^2} \mathrm{tr}(\tilde{\theta}^{\mathrm{T}}\theta),\end{aligned} \tag{27}$$

where $I_1 = 0$ for the first case in (22), and $I_1 = 1$ for the second case of (22). If $I_1 = 0$ and with (A3), we have $\dot{V} \leqslant 0$. When $I_1 = 1$, using the following equality

$$\mathrm{tr}(\tilde{\theta}^{\mathrm{T}}\theta) = \frac{1}{2}\mathrm{tr}(\theta^{\mathrm{T}}\theta) - \frac{1}{2}\mathrm{tr}(\theta^{*\mathrm{T}}\theta^*) + \frac{1}{2}\mathrm{tr}(\tilde{\theta}^{\mathrm{T}}\tilde{\theta}) \tag{28}$$

with the fact $\mathrm{tr}(\theta^{*\mathrm{T}}\theta^*) \leqslant M_\theta^2 = \mathrm{tr}(\theta^{\mathrm{T}}\theta)$, we have $\mathrm{tr}(\tilde{\theta}^{\mathrm{T}}\theta) \geqslant 0$. Thus

$$I_1 \mathrm{tr}(\theta^{\mathrm{T}}\omega(\hat{x})\|\tilde{x}^{\mathrm{T}} PB_0\|) \frac{1+\|\theta\|^2}{M_\theta^2} \mathrm{tr}(\tilde{\theta}^{\mathrm{T}}\theta) \leqslant 0. \tag{29}$$

That implies

$$\dot{V} \leqslant -\beta\tilde{x}^{\mathrm{T}}\tilde{x}, \quad \beta > 0. \tag{30}$$

Therefore, $V \in L_\infty$, which indicates $\tilde{x}$, $\tilde{\theta}$, $\tilde{\varepsilon}$, $u \in L_\infty$. Integrating (30) from 0 to ∞ results in

$$\beta \int_0^\infty \tilde{x}^{\mathrm{T}}\tilde{x}\,\mathrm{d}t \leqslant V(0) - V(\infty) < \infty. \tag{31}$$

That is, $\tilde{x} \in L_2$. From (17) and Lipschitz condition on the fuzzy estimation, it can be seen $\dot{\tilde{x}} \in L_\infty$. By Babalat lemma [14], we have $\tilde{x} \to 0$ as $t \to \infty$. □

Remark 1. In Theorem 1 no conclusion is drawn about the convergence of parameters θ and ε to their optimal values. However, if the uncertainty $\xi(t,x)$ is completely described by the fuzzy estimation $\hat{\xi}(x|\theta^*)$, then the convergence of parameter can be attained. Since $\tilde{x} \to 0$ and $\dot{\tilde{x}} \to 0$, from (20) and using the Lipschitz continuity of $\omega(x)$ we may have $\tilde{\theta} \to 0$. Compared to convergence properties in [7], where the chaos system is given by an explicit mathematic model, our approach is more flexible.

Remark 2. The assumptions (19) and (20) actually impose constraints on selecting appropriate observer gain vector L. In general, the problem of determining L to satisfy (19) and (20) can be solved by using LMI technique.

If the uncertainty in (12) is merely relevant to the output, i.e., $\xi(t,x) = \xi(y)$, then the fuzzy estimation $\hat{\xi}(\hat{x}|\theta)$ is replaced by $\hat{\xi}(y|\theta) = \theta^{\mathrm{T}}\omega(y)$. As to this condition, we introduce the following theorem.

Theorem 2. *Consider the chaos system described by*

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + B_0(E(t,x) + F(t,x)) = A_0x(t) + B_0\xi(y), \\ y(t) &= Cx(t). \end{aligned} \tag{32}$$

If the assumption (A2) *is satisfied, by using the following adaptive laws and control input*

$$u = \begin{cases} -\dfrac{\hat{\varepsilon}B_0^{\mathrm{T}}P\tilde{x}}{\left\|\tilde{x}^{\mathrm{T}}PB_0\right\|} & \text{if } \left\|\tilde{x}^{\mathrm{T}}PB_0\right\| \neq 0, \\ 0 & \text{if } \left\|\tilde{x}^{\mathrm{T}}PB_0\right\| = 0, \end{cases} \tag{33}$$

$$\dot{\theta} = \begin{cases} -2\eta_1\omega(y)\tilde{x}^{\mathrm{T}}PB_0 \\ \quad \text{if } (\|\theta\| < M_\theta) \text{ or } (\|\theta\| = M_\theta \text{ and } \mathrm{tr}(\theta^{\mathrm{T}}\omega(y)\tilde{x}^{\mathrm{T}}PB_0) \geqslant 0), \\ -2\eta_1\omega(y)\tilde{x}^{\mathrm{T}}PB_0 + 2\eta_1\mathrm{tr}(\theta^{\mathrm{T}}\omega(y)\tilde{x}^{\mathrm{T}}PB_0)\dfrac{1+\|\theta\|^2}{M_\theta^2}\theta \\ \quad \text{if } \|\theta\| = M_\theta \text{ and } \mathrm{tr}(\theta^{\mathrm{T}}\omega(y)\tilde{x}^{\mathrm{T}}PB_0) < 0, \end{cases} \tag{34}$$

$$\dot{\hat{\varepsilon}} = 2\eta_2\left\|B_0^{\mathrm{T}}P\tilde{x}\right\|, \tag{35}$$

the adaptive fuzzy observer (13) *is convergent*, *i.e.*, $\tilde{x} \to 0$ *as* $t \to \infty$. *The boundedness of all signals in the closed-loop is also guaranteed.*

Proof. The proof is in the same spirit as that of Theorem 1. Eliminating (A1) and (A3) from Theorem 1 and selecting Lyapunov function as (29), one can easily obtain the result via the same procedure. □

Remark 3. Notice that if the uncertainty $\xi(y)$ is exactly estimated by $\hat{\xi}(y|\theta^*)$, then the convergence of parameter is attained. The same conclusions are drawn in [7,9].

4. Simulation examples

In this section we use two well-known chaotic systems to depict the design procedure and verify the effectiveness of the proposed algorithm.

Example 1. Consider the Lorenz system with the dynamics

$$\begin{aligned} \dot{x}_1 &= -10x_1 + 10x_2, \\ \dot{x}_2 &= \alpha x_1 - x_2 - x_1x_3, \\ \dot{x}_3 &= x_1x_2 - \beta x_3, \\ y &= Cx. \end{aligned}$$

It is assumed $x_1 \in [-\varsigma\ \varsigma]$ with $\varsigma = 30$, $\alpha = 28$, $\beta = 8/3$. Then, the T–S fuzzy representation for Lorenz system is given by [8]:

$$\begin{aligned} R^l &: \text{If } x_1(t) \text{ is } M_l \\ &\quad \text{then } \dot{x}(t) = A_l x(t) + d_l(t), \quad y(t) = Cx(t) \\ &\quad l = 1, 2, \end{aligned}$$

where the fuzzy sets are defined as $M_1(x_1) = (1 + (x_1/\varsigma))/2$, $M_2(x_1) = (1-(x_1/\varsigma))/2$, and the state matrices are defined as

$$A_1 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & -\varsigma \\ 0 & \varsigma & -8/3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & \varsigma \\ 0 & -\varsigma & -8/3 \end{bmatrix}.$$

Accordingly, the T–S fuzzy model can be rearranged in the form of

$$\dot{x} = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix} x + \begin{bmatrix} 0 \\ \xi_1(x) \\ \xi_2(x) \end{bmatrix} = A_0 x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1(x) \\ \xi_2(x) \end{bmatrix} = A_0 x + B_0 \xi(x).$$

Since there are two uncertainties contained in the system, it is assumed that output matrix C is of the form

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B_0^{\mathrm{T}}.$$

Moreover, to satisfy (19), the following matrices are determined by LMI's method:

$$L = \begin{bmatrix} -10.5118 & 0 \\ -420.7458 & 0 \\ 0 & -418.7964 \end{bmatrix}, \quad P = \begin{bmatrix} 54.8874 & -0.003 & 0 \\ -0.003 & 0.9993 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$Q = \begin{bmatrix} 548.8811 & 0 & 0 \\ 0 & 421.4671 & 0 \\ 0 & 0 & 421.4671 \end{bmatrix}.$$

The fuzzy inferences for estimation of uncertainty are given by

$$\begin{aligned} R^j &: \text{If } x_1(t) \text{ is } \widetilde{M}_1^j \text{ and } x_2(t) \text{ is } \widetilde{M}_2^j \text{ and } x_3(t) \text{ is } \widetilde{M}_3^j \\ &\quad \text{then } \hat{\xi}_i \text{ is } \widetilde{D}_{ij}, \quad j = 1, 2, \ldots, r. \end{aligned}$$

The normalized membership functions for each premise variables are shown in Fig. 1, where the Gaussian function $\mu(x) = \exp(-\frac{(x-m)^2}{2\sigma^2})$ is used with $\sigma = 0.5$ and $m = -1$, 0, and 1 respectively. In addition, the Lipschitz constant γ is equal to 1.2473, which is obtained by calculating the norm value of $\omega(x)$ over the operating space. Let $M_\theta = 3$ and from (20) we have

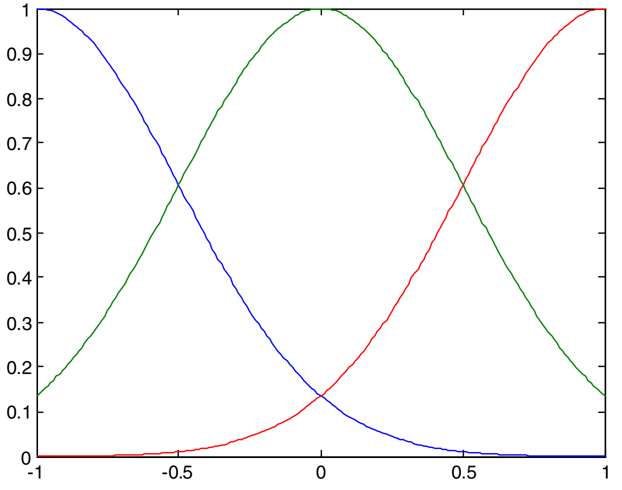

Fig. 1. The membership functions.

$$2 \times 1.2473 \times 3 = 2\gamma M_{\theta}\|B_0\| \leqslant \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} = 7.6788.$$

The initial conditions of states are chosen as $x(0) = (5, 10, 10)$ and $\hat{x}(0) = (-5, -10, -10)$. $\theta(0)$ is a zero matrix and $\hat{\varepsilon}$ is set to 0. Applying Theorem 1, the adaptive fuzzy observer has good tracking performance as shown in Fig. 2. In order to

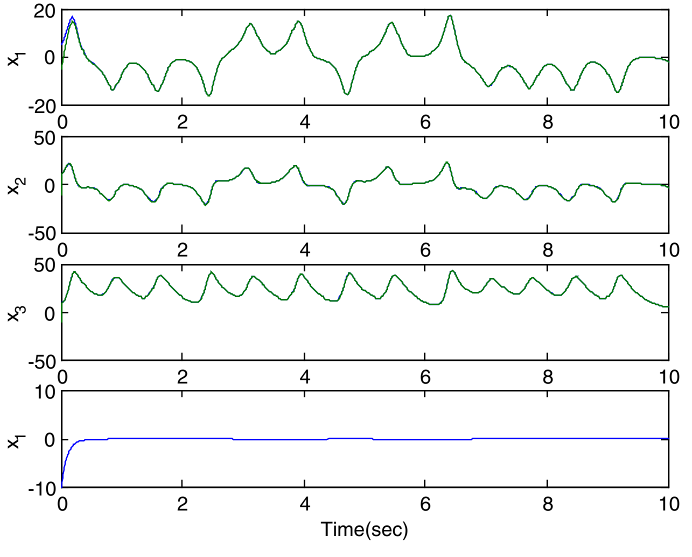

Fig. 2. Responses of synchronization of Lorenz system.

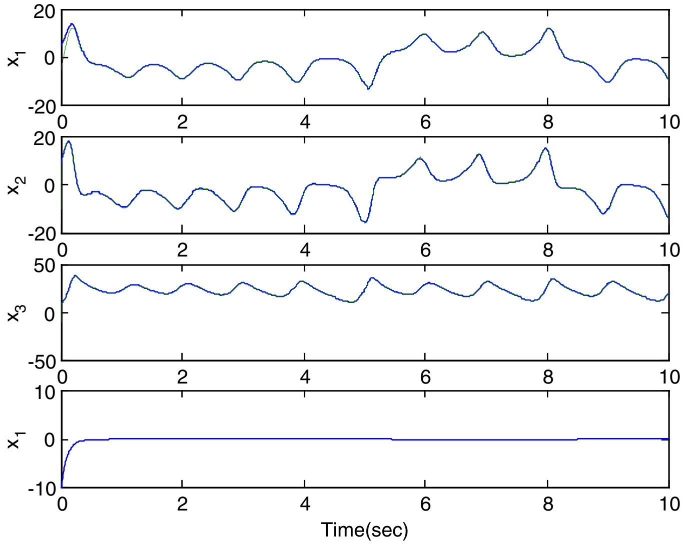

Fig. 3. Responses of synchronization of Lorenz system under perturbed conditions.

verify the robust behavior, the parameters α and β are changed to 25 and 1.5, respectively. From Fig. 3, one can see that synchronization of chaotic system is achieved asymptotically as well. In the figures, the solid lines depict the states of system and the dashed lines reflect the states of observer.

Example 2. This example is used to illustrate a chaotic synchronization based on Theorem 2. A generic Chua circuit is described by

$$\begin{aligned}
\dot{x}_1 &= \sigma_1(-x_1 + x_2 - f(x_1)),\\
\dot{x}_2 &= x_1 - x_2 + x_3,\\
\dot{x}_3 &= -\sigma_2 x_2,\\
y &= x_1
\end{aligned}$$

with $f(x_1) = bx_1 + 0.5(a - b)(|x_1 + 1| - |x_1 - 1|)$, where $\sigma_1 = 10$, $\sigma_2 = 15$, $a = -1.28$ and $b = -0.69$. Define a function $g(x_1)$ as

$$g(x_1) = \begin{cases} f(x_1)/x_1, & x_1 \neq 0,\\ b, & x_1 = 0. \end{cases}$$

Then, the T–S fuzzy model of Chua circuit is described by [8]:

$$\begin{aligned}
R^l :\ & \text{If } x_1(t) \text{ is } M_l\\
& \text{then } \dot{x}(t) = A_l x(t) + d_l(t), \quad y(t) = x_1(t)\\
& l = 1, 2,
\end{aligned}$$

where

$$A_1 = \begin{bmatrix} (\alpha-1)\sigma_1 & \sigma_1 & 0 \\ 1 & -1 & 1 \\ 0 & -\sigma_2 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -(\alpha+1)\sigma_1 & \sigma_1 & 0 \\ 1 & -1 & 1 \\ 0 & -\sigma_2 & 0 \end{bmatrix}$$

with $M_1(x_1) = (1 - (g(x_1)/\alpha))/2$, $M_2(x_1) = 1 - M_1(x_1)$ and $\alpha = \sup_{x \in \Omega} | g(x_1) | = 13$. Accordingly we have

$$\dot{x} = \begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -15 & 0 \end{bmatrix} x + \begin{bmatrix} \xi(x) \\ 0 \\ 0 \end{bmatrix} = A_0 x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xi(x) = A_0 x + B_0 \xi(x).$$

Since the uncertainty is relevant to x_1 only, i.e., $\xi(x) = \xi(x_1)$, the observer is built on the basis of Theorem 2. To satisfy (19), the following matrices are determined by LMIs

$$L = \begin{bmatrix} -209.2721 \\ -1.0149 \\ -0.0107 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 698.1972 & -35.5533 \\ 0 & -35.5533 & 49.1654 \end{bmatrix},$$

$$Q = \begin{bmatrix} 219.2702 & 0 & 0 \\ 0 & 164.8976 & 1.865 \\ 0 & 1.865 & 35.5533 \end{bmatrix}.$$

The estimation of $\xi(x_1)$ is $\theta^{\mathrm{T}}\omega(x_1)$, where five fuzzy sets spaced equally over the operating space of x_1 are implemented. The initial values of states are chosen $x(0) = (0.1, 0.1, 0.1), \hat{x}(0) = (1, 1, 1)$, and $\theta(0) = [0\ 0\ 0\ 0\ 0]^{\mathrm{T}}$ and $\hat{\varepsilon} = 0$. Figs. 4 and 5 illustrate the system behaviors, the states of system (solid lines) and observer (dashed lines), under the nominal and perturbed conditions respectively, where the perturbed values of a and b are -3 and -0.4.

Example 3. One of the most interesting applications of chaotic synchronization is the secure communication. Herein we simulate a constant value message defined interval by interval is to be transmitted by chaotic signals. For this purpose, a chaotic system of T–S fuzzy model form is modulated as

$$\begin{aligned} R^l : \ & \text{If } z_1(t) \text{ is } M_1^l \text{ and } \ldots \text{ and } z_j(t) \text{ is } M_j^l \\ & \text{then } \dot{x}(t) = A_l x(t) + d_l(t) + B_0 g(y) s(t), \quad y(t) = Cx(t) \\ & l = 1, 2, \ldots, q, \end{aligned} \tag{36}$$

where $g(y) \in R^{p \times s}$ is an output feedback function and $s(t) \in R^{s \times 1}$ is the transmitted signal. Note the hypotheses (A1)–(A3) are still held to the design of adaptive obser-

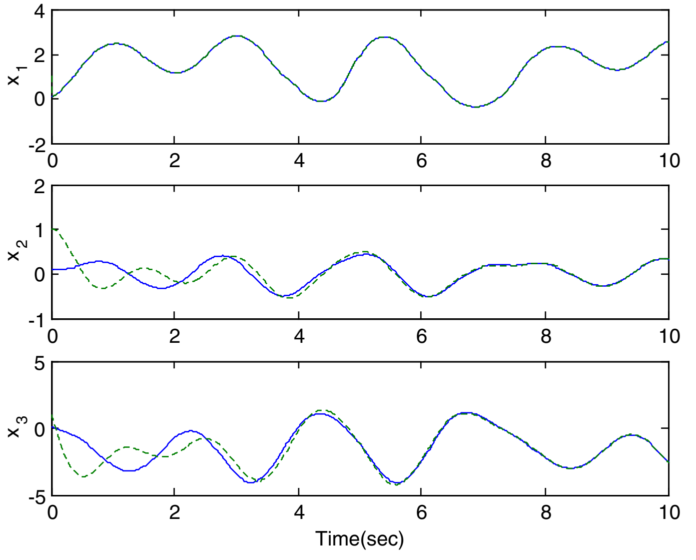

Fig. 4. Responses of synchronization of Chua circuit.

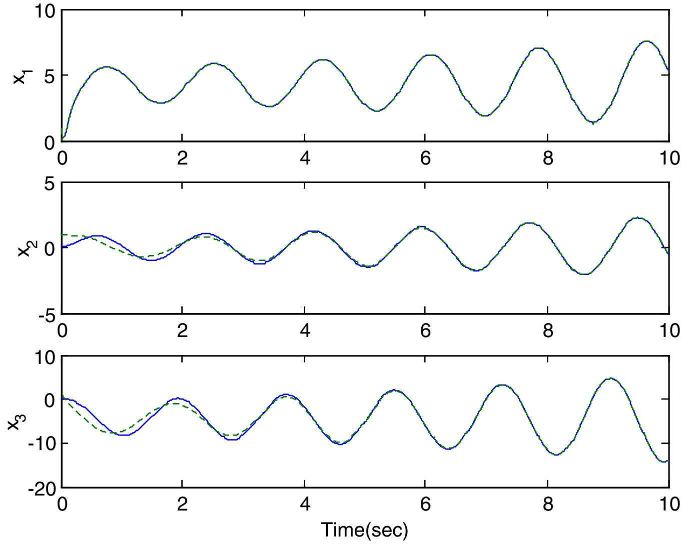

Fig. 5. Responses of synchronization of Chua circuit under perturbed conditions.

ver. It is also assumed that the time interval of each level of $s(t)$ is much longer than the convergent time of synchronization.

The output of inferences (36) is written as

$$\begin{aligned}\dot{x}(t) &= \sum_{l=1}^{q} \mu_l(x)[A_l x(t) + d_l(t)] + B_0 g(y)s(t)\\ &= A_0 x(t) + B_0(\xi(t,x) + g(y)s(t)),\\ y &= Cx(t),\end{aligned} \tag{37}$$

and the corresponding observer is in the following form

$$\begin{aligned}\dot{\hat{x}}(t) &= A_0\hat{x}(t) + B_0(u(t) + \hat{\xi}(\hat{x}|\theta)) + L(\hat{y}(t) - y(t)) + B_0 g(y)\hat{s}(t),\\ \hat{y}(t) &= C\hat{x}(t).\end{aligned} \tag{38}$$

Subtracting (37) from (38) leads to the error dynamics

$$\begin{aligned}\dot{\tilde{x}} &= A_0\tilde{x} + B_0(u + \hat{\xi}(\hat{x}|\theta) - \xi(t,x)) + LC\tilde{x} + B_0 g(y)\tilde{s}\\ &= (A_0 + LC)\tilde{x} + B_0(\hat{\xi}(\hat{x}|\theta) - \xi(t,x) + \hat{\xi}(\hat{x}|\theta^*) - \hat{\xi}(\hat{x}|\theta^*) + \hat{\xi}(x|\theta^*)\\ &\quad - \hat{\xi}(x|\theta^*) + u) + B_0 g(y)\tilde{s}\\ &= \overline{A}\tilde{x} + B_0(\tilde{\theta}^{\mathrm{T}}\omega(\hat{x}) + \hat{\xi}(\hat{x}|\theta^*) - \hat{\xi}(x|\theta^*) + \hat{\xi}(x|\theta^*) - \xi(t,x) + u) + B_0 g(y)\tilde{s}\end{aligned} \tag{39}$$

with $\tilde{s} = \hat{s} - s$. Comparing (39) with (17) one can see that these equations are almost the same except additional term $B_0 g(y)\tilde{s}$ in (39).

Choose the Lyapunov function candidate as

$$V = \tilde{x}^{\mathrm{T}} P\tilde{x} + \frac{1}{2\eta_1}\mathrm{tr}(\tilde{\theta}^{\mathrm{T}}\tilde{\theta}) + \frac{1}{2\eta_2}\tilde{\varepsilon}^2 + \frac{1}{2\eta_3}\tilde{s}^2 \tag{40}$$

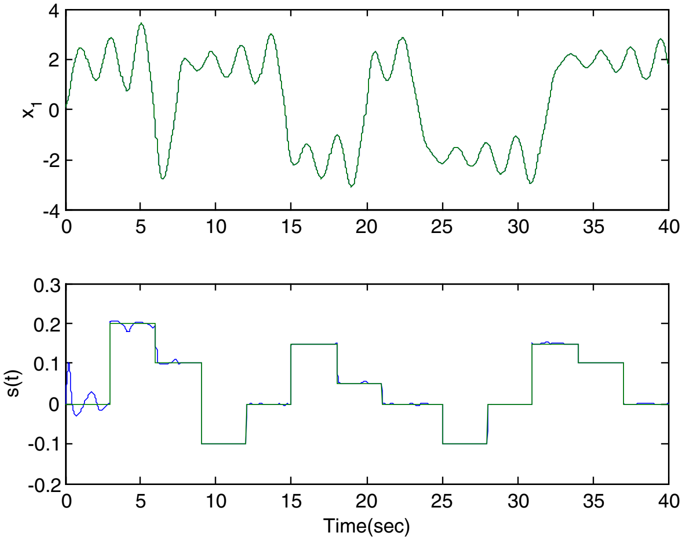

Fig. 6. Responses of the transmitted signal, message and retrieved message by proposed method.

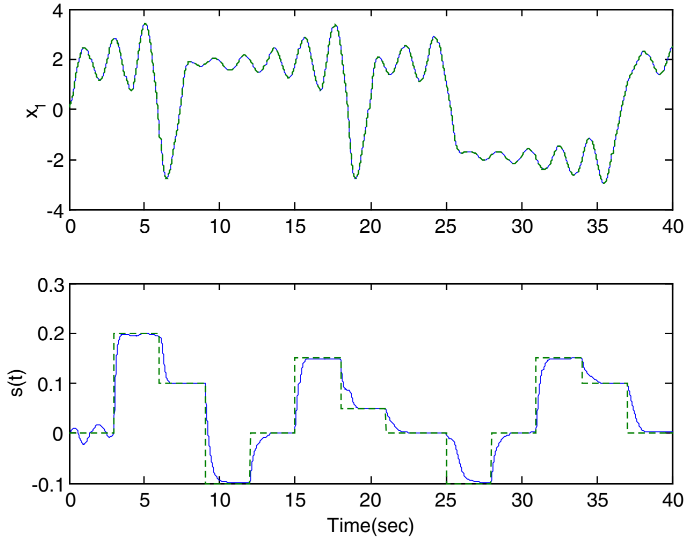

Fig. 7. Responses of the transmitted signal, message and retrieved message by Feki's method.

and perform the same derivation given in Theorem 1. Finally, it can be proved that the adaptive observer is convergent by applying (21)–(23) with additional adaptation law for estimation of $s(t)$ as

$$\dot{\hat{s}} = -2\eta_3 g(y)\tilde{x}^{\mathrm{T}} P B_0. \tag{41}$$

In simulation we employ Chua's circuit in the synchronization-based secure communication scheme with $g(y) = x_1$ and compare our result with the Feki's method [7]. According to the work of [7], the system model is explicitly defined with parameter uncertainties and the Lyapunov stability theory is used to derive adaptation law for estimation of the unknown parameters. The initial states of the system are $x(0) = (0.2, 0.2, 0.2), \hat{x}(0) = (0, 0, 0)$. The system responses are given in Figs. 6 and 7, where the dashed line depicts the message and the solid line stands for retrieved message. The adaptive observers perform adaptation to synchronize the chaotic system in the first few seconds, and finally information can be retrieved in both cases.

5. Conclusion

In this paper we presents the adaptive fuzzy observer for synchronization of chaotic systems. In contrast to the earlier studies [7,9], the chaotic system is specifically expressed in T–S fuzzy model. Based on the concept of system linearization, the adaptive algorithms are developed to the design of observer. It is demonstrated that if the chaotic system satisfies the hypotheses then synchronization can be guaranteed by the proposed approach. Computer simulations have verified that the adaptive fuzzy observer has well performance both in synchronization of chaotic system and control robustness.

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