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The existence and the non-existence of joint t-universality for Lerch zeta functions

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Abstract

In this paper, we show the following theorems. Suppose $0 < a_l < 1$ are algebraically independent numbers and $0 < \lambda_l \leqslant 1$ for $1 \leqslant l \leqslant m$. Then we have the joint t-universality for Lerch zeta functions $L(\lambda_l, a_l, s)$ for $1 \leqslant l \leqslant m$. Next we generalize Lerch zeta functions, and obtain the joint t-universality for them. In addition, we show examples of the non-existence of the joint t-universality for Lerch zeta functions and generalized Lerch zeta functions.

MSC: primary 11M35

Keywords: Lerch zeta functions; t-Universality; Joint t-universality

1. Introduction

Definition 1.1. The Lerch zeta function $L(\lambda, a, s)$, for $0 < \lambda \leqslant 1$, $0 < a \leqslant 1$ and $\Re(s) > 1$, is defined by

$$L(\lambda, a, s) := \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda n}}{(n+a)^s}. \tag{1.1}$$

When $\lambda = 1$, the Lerch zeta function $L(\lambda, a, s)$ reduces to the Hurwitz zeta function $\zeta(a, s)$. If $\lambda \neq 1$, the function $L(\lambda, a, s)$ is analytically continuable to an entire function. But the func-

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tion $\zeta(a,s)$ is analytically continuable to a meromorphic function, which has a simple pole at $s=1$.

We prepare some notation for t-universality. By meas$\{A\}$ we denote the Lebesgue measure of the set A, and, for $T>0$, we use the notation

$$\nu_T^\tau\{\dots\} := \frac{1}{T}\operatorname{meas}\bigl\{\tau\in[0,T];\dots\bigr\}$$

where in place of dots some condition satisfied by τ is to be written. Let $D := \{s\in\mathbb{C};\ 1/2<\Re(s)<1\}$ and $K_1,\dots,K_m$ ($m\geqslant 2$) be compact subsets of the strip D with connected complements. The next theorem is proved by A. Laurinčikas and K. Matsumoto in [6, Theorem 2] (see also [4, p. 122, Theorem 3.1] and [5, Theorem 1]).

Theorem 1.2 *(Joint t-universality). (See [6, Theorem 2].) For $1\leqslant l\leqslant m$, let a_l be algebraically independent numbers, $b_l,q_l\in\mathbb{N}$, q_l which are distinct, $\lambda_l=b_l/q_l$, $(b_l,q_l)=1$ and $b_l<q_l$. Let $f_l(s)$ be functions analytic in the interior of K_l and continuous on K_l. Then for every $\varepsilon>0$ it holds that*

$$\liminf_{T\to\infty}\nu_T^\tau\Bigl\{\sup_{1\leqslant l\leqslant m}\sup_{s\in K_l}\bigl|L(\lambda_l,a_l,s+i\tau)-f_l(s)\bigr|<\varepsilon\Bigr\}>0.$$

When all a_l's are the same, this property is called the λ-joint t-universality in [8, Definition 7]. Firstly we show the next theorem, which gives the universality under the assumption weaker than that in Theorem 1.2.

Theorem 1.3. *Suppose $0<a_l<1$ are algebraically independent numbers and $0<\lambda_l\leqslant 1$ for $1\leqslant l\leqslant m$. Let $f_l(s)$ be functions analytic in the interior of K_l and continuous on K_l. Then for every $\varepsilon>0$ it holds that*

$$\liminf_{T\to\infty}\nu_T^\tau\Bigl\{\sup_{1\leqslant l\leqslant m}\sup_{s\in K_l}\bigl|L(\lambda_l,a_l,s+i\tau)-f_l(s)\bigr|<\varepsilon\Bigr\}>0.$$

Next we introduce a generalization of Lerch zeta functions and consider their joint universality.

Definition 1.4. The generalized Lerch zeta functions $\mathfrak{L}(\lambda,a,b,c;s)$, for $0<\lambda\leqslant 1$, $0<a\leqslant 1$, $0<b\leqslant 1$, $\Re(s)>1$ and $c\in\mathbb{C}$, are defined by

$$\mathfrak{L}(\lambda,a,b,c;s) := \sum_{n=0}^{\infty}\frac{e^{2\pi i\lambda n}}{(n+a)^{s-c}(n+b)^{c}}. \tag{1.2}$$

The cases of $a=b$ or $c=0$, these functions coincide with Lerch zeta functions. We remark that for $0<\lambda\leqslant 1$, $\mathfrak{L}(\lambda,a,b,c;s)$ is meromorphic in the half-plane $\sigma>0$, since

$$\mathfrak{L}(\lambda,a,b,c;s) = L(\lambda,a,s)+\sum_{n=0}^{\infty}\frac{e^{2\pi i\lambda n}}{(n+a)^{s-c}(n+b)^{c}}\left(1-\frac{(n+b)^c}{(n+a)^c}\right),$$

and the series on the right-hand side converges in the half-plane $\sigma > 0$. For $0 < \lambda < 1$, this series converges uniformly on any compact subset in the half-plane $\sigma > \sigma_0$ for any $\sigma_0 > 0$. The case of $b = 1, c = -1$ is

$$\sum_{n=0}^{\infty} \frac{e^{2\pi i\lambda n}(n+1)}{(n+a)^{s+1}} = \sum_{n_1,n_2=0}^{\infty} \frac{e^{2\pi i\lambda(n_1+n_2)}}{(n_1+n_2+a)^{s+1}}.$$

(See for example [9, p. 85, (10)].) Hence $\mathfrak{L}(\lambda, a, b, c; s)$ contain a special case of Barnes double zeta functions. The following theorem gives a joint universality property of $\mathfrak{L}(\lambda, a, b, c; s)$. This is a partial solution of the problem of (joint) universality of multiple zeta functions presented in [7, Section 2].

Theorem 1.5. *Suppose $0 < a_l < 1$ are algebraically independent numbers and $0 < \lambda_l \leqslant 1$, $0 < b_l \leqslant 1$ for $1 \leqslant l \leqslant m$. Let $f_l(s)$ be functions analytic in the interior of K_l and continuous on K_l. Then for every $\varepsilon > 0$ it holds that*

$$\liminf_{T\to\infty} \nu_T^{\tau}\Big\{ \sup_{1\leqslant l\leqslant m} \sup_{s\in K_l} \big|\mathfrak{L}(\lambda_l, a_l, b_l, c; s+i\tau) - f_l(s)\big| < \varepsilon \Big\} > 0.$$

This paper is divided into six sections. Section 2 is a preparation for the proof of these theorems. In Sections 3 and 4, we prove Theorems 1.3 and 1.5, respectively. We consider the proofs of Theorems 1.3 and 1.5 in Section 5. We show examples of the non-existence of the joint t-universality for Lerch zeta functions in Section 6.

2. Preliminaries

In this section, we quote definitions and theorems from [2] and [5], and we omit the proofs of those theorems. Denote by $H(D)$ the space of analytic on D functions equipped with the topology of uniform convergence on compacta. Let $\mathfrak{B}(S)$ stands for the class of Borel sets of the space S. Define on $(H^m(D), \mathfrak{B}(H^m(D)))$ the probability measure

$$P_{\underline{L}}^T(A) := \nu_T^{\tau}\big(L(\lambda_1, a_1, s+i\tau), \ldots, L(\lambda_m, a_m, s+i\tau) \in A\big), \quad A \in \mathfrak{B}\big(H^m(D)\big).$$

What we need is a limit theorem in the sense of weak convergence of probability measures for $P_{\underline{L}}^T$ as $T \to \infty$, with an explicit form of the limit measure. Denote by γ the unit circle on $\mathbb{C}$, and let

$$\Omega := \prod_{n=0}^{\infty} \gamma_n,$$

where $\gamma_n = \gamma$ for all $n \in \mathbb{N} \cup \{0\}$. With the product topology and pointwise multiplication the infinite dimensional torus Ω is a compact topological Abelian group. Denoting by m_{H_m} the probability Haar measure on $(\Omega^m, \mathfrak{B}(\Omega^m))$, where $\Omega^m := \Omega \times \cdots \times \Omega$, we obtain a probability space $(\Omega^m, \mathfrak{B}(\Omega^m), m_{H_m})$. Let $\omega_l(n)$ be the projection of $\omega_l \in \Omega$ to the coordinate space γ_n, and define on the probability space $(\Omega^m, \mathfrak{B}(\Omega^m), m_{H_m})$ the $H^m(D)$-valued random element $\underline{L}(s, \underline{\omega})$ by

$$\underline{L}(s, \underline{\omega}) := \big(L(\lambda_1, a_1, s, \omega_1), \ldots, L(\lambda_m, a_m, s, \omega_m)\big),$$

where

$$L(\lambda_l, a_l, s, \omega_l) := \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda_l n}\omega_l(n)}{(n+a_l)^s}, \quad s \in D,\ \omega_l \in \Omega,\ 1 \leqslant l \leqslant m.$$

The function $L(\lambda_l, a_l, s, \omega_l)$ is an $H(D)$-valued random element. Let $P_{\underline{L}}$ stand for the distribution of the random element $\underline{L}(s, \underline{\omega})$, i.e.

$$P_{\underline{L}}(A) := m_{H_m}\big(\underline{\omega} \in \Omega^m;\ \underline{L}(s, \underline{\omega}) \in A\big), \quad A \in \mathfrak{B}\big(H^m(D)\big).$$

In [6, Theorem 1], the following lemma is proved in the case of $0 < \lambda_l < 1$. But we can prove the case of $0 < \lambda_l \leqslant 1$ similarly.

Lemma 2.1. *(See [6, Theorem 1].) Suppose* $0 < a_l < 1$ *are algebraically independent numbers and* $0 < \lambda_l \leqslant 1$ *for* $1 \leqslant l \leqslant m$. *The probability measure* $P_{\underline{L}}^T$ *converges weakly to* $P_{\underline{L}}$ *as* $T \to \infty$.

Next we consider the support of the measure P. We recall that the minimal closed set $S_P \subseteq H^m(D)$ such that $P(S_P) = 1$ is called the support of P. The set S_P consists of all $\underline{f} \in H^m(D)$ such that for every neighborhood V of $\underline{f}$ the inequality $P(V) > 0$ is satisfied. The support of the distribution of the random element X is called the support of X and is denoted by S_X.

Lemma 2.2. *(See [5, Lemma 2].) Let* $\{X_n\}$ *be a sequence of independent* $H^m(D)$*-valued random elements, and suppose that the series*

$$\sum_{n=1}^{\infty} X_n$$

converges almost surely. Then the support of the sum of this series is the closure of the set of all $\underline{f} \in H^m(D)$ *which may be written as a convergent series*

$$\underline{f} := \sum_{n=1}^{\infty} \underline{f}_n, \quad \underline{f}_n \in S_{X_n}.$$

We quote some results on Hilbert spaces from [2, Chapter 6]. The subset $L \subset X$ is called a linear manifold if for all $x, y \in L$ and for all $\alpha, \beta \in \mathbb{C}$ the linear combination $\alpha x + \beta y \in L$. Let L be a linear manifold of X. The set of elements $x \in X$ such that $x \perp L$ is called the orthogonal complement of L and is denoted by $L^{\perp}$.

Lemma 2.3. *(See [2, Theorem 6.1.8].) Let* L *be a linear manifold of* X. *Then* L *is dense in* X *if and only if* $L^{\perp} = \{0\}$.

Let X be a Hilbert space with an inner product $\langle x, y\rangle$ and a norm $\|x\| := \sqrt{\langle x, x\rangle}$.

Lemma 2.4. *(See [2, Theorem 6.1.11].) Let* f *be a continuous linear functional on a Hilbert space* X. *Then there exists a unique element* $y \in X$ *such that* $f(x) = \langle x, y\rangle$ *for all* $x \in X$.

Lemma 2.5. *(See [2, Theorem 6.1.16].) Let $\{x_n\}$ be a sequence in a Hilbert space X satisfying the following conditions*:

(a) $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$;
(b) $\sum_{n=1}^{\infty} |\langle x_n, x\rangle| = \infty$ *for all* $0 \neq x \in X$.

Then the set of all convergent series $\sum_{n=1}^{\infty} a_n x_n$, $|a_n| = 1$, $n \in \mathbb{N}$, *is dense in* X.

Finally, we quote some results on Hardy spaces. Let D_0 be an arbitrary simply connected domain with at least two boundary points. A set of functions $\underline{f}$ analytic in D_0 is said to belong to the Hardy space $H_2^m(D_0)$, if the subharmonic function $\sum_{l=1}^{m} |f_l(s)|^2$ has a harmonic majorant in D_0. We remark that $H_2^m(D_0)$ is a Hilbert space. A proof of the following lemmas in the case of $m = 1$ is given in [2, Theorems 6.3.6 and 6.3.7]. The proof of the general case is obtained in a similar way.

Lemma 2.6. *(See [2, Theorem 6.3.6].) Let $\{\underline{f}_n\}$ be a sequence in $H_2^m(D_0)$ such that*

$$\lim_{n\to\infty} \underline{f}_n(s) = \underline{f}(s)$$

in the topology of $H_2^m(D_0)$. Then this relation is true uniformly on every compact subset of D_0.

Lemma 2.7. *(See [2, Theorem 6.3.7].) Let $\underline{g} \in H_2^m(D_0)$. There exist complex Borel measures μ_{g_l} $(1 \leqslant l \leqslant m)$ with their support contained in the boundary ∂D_0 of D_0 such that if $\underline{f} \in H_2^m(D_0)$ has a continuous extension to $\overline{D_0}$, then the inner product can be expressed by the formula*

$$\langle \underline{f}, \underline{g}\rangle = \sum_{l=1}^{m} \int_{\partial D_0} f_l \, d\mu_{g_l}.$$

We define the norm of $\underline{f} \in H_2^m(D_0)$ by

$$\|\underline{f}\| := \sqrt{\langle \underline{f}, \underline{f}\rangle}.$$

Lemma 2.8. *(See [2, Theorem 6.3.9].) Let the boundary of D_0 be an analytic simple closed curve. The set of polynomials is dense in the space $H_2^m(D_0)$.*

3. Joint universality I

In this section, we prove Theorems 1.3. We define the Hilbert space X^m by $X^m = X \times \cdots \times X$. For convenience, we define the next symbols:

$$\underline{a} \cdot \underline{x} := \{a_1 x_1, \ldots, a_m x_m\}, \quad \underline{a} \in \mathbb{C}^m, \ \underline{x} \in X^m,$$
$$\Pi^m := \left\{\underline{c} = (c_1, \ldots, c_m) \in \mathbb{C}^n;\ |c_l| = 1,\ 1 \leqslant l \leqslant m\right\}.$$

The following theorem is a generalization of Lemma 2.5.

Theorem 3.1. *Let* $\{\underline{x}_n\} := \{(x_{1,n}, \ldots, x_{m,n});\ n \in \mathbb{N}\}$ *be a sequence in the Hilbert space* X^m *satisfying the following conditions*:

(a) $\sum_{n=1}^{\infty} \|\underline{x}_n\|^2 < \infty$;
(b) *There exist* $\underline{c}_n \in \Pi^m$, $n \in \mathbb{N}$, *such that* $\sum_{n=1}^{\infty} |\langle \underline{c}_n \cdot \underline{x}_n, \underline{x}\rangle| = \infty$ *for all* $\underline{0} \neq \underline{x} \in X^m$.

Then the set of all convergent series $\sum_{n=1}^{\infty} \underline{a}_n \cdot \underline{x}_n$, $\underline{a}_n \in \Pi^m$, $n \in \mathbb{N}$, *is dense in* X^m.

Proof. Put $\underline{y}_n := \underline{c}_n \cdot \underline{x}_n$. By using Lemma 2.5 as $X = X^m$, the set of all convergent series

$$\sum_{n=1}^{\infty} b_n \underline{y}_n, \quad |b_n| = 1,\ n \in \mathbb{N},$$

is dense in X^m. Hence by taking $\underline{d}_n := b_n \underline{c}_n$, the set of all convergent series

$$\sum_{n=1}^{\infty} \underline{d}_n \cdot \underline{x}_n, \quad \underline{d}_n \in \Pi^m,$$

is dense in X^m. Since this set is contained in the set of all convergent series $\sum_{n=1}^{\infty} \underline{a}_n \cdot \underline{x}_n$, $\underline{a}_n \in \Pi^m$, we obtain this theorem. □

The following theorem is a generalization of [2, Theorem 6.3.10] and [5, Lemma 3].

Theorem 3.2. *Let* D_1 *be a simply connected domain in* $\mathbb{C}$. *Let* $\{\underline{f}_n\}$ *be a sequence in* $H^m(D_1)$ *which satisfies*:

(a) *If* μ_l, $1 \leqslant l \leqslant m$, *are complex measures on* $(\mathbb{C}, \mathfrak{B}(\mathbb{C}))$ *with compact supports contained in* D_1 *such that there exist* $\underline{c}_n \in \Pi^m$, $n \in \mathbb{N}$, *which satisfy* $\sum_{n=1}^{\infty} |\sum_{l=1}^{m} \int_{\mathbb{C}} c_{l,n} f_{l,n}\, d\mu_l| < \infty$, *then* $\int_{\mathbb{C}} s^r\, d\mu_l = 0$ *for all* $r \in \mathbb{N} \cup \{0\}$, $1 \leqslant l \leqslant m$;
(b) *There exist* $\underline{d}_n \in \Pi^m$, $n \in \mathbb{N}$, *for which the series* $\sum_{n=1}^{\infty} \underline{d}_n \cdot \underline{f}_n$ *converges in* $H^m(D_1)$;
(c) *For any compact set* $K \subseteq D_1$, *we have* $\sum_{n=1}^{\infty} \sup_{1 \leqslant l \leqslant m} \sup_{s \in K} |f_{l,n}(s)|^2 < \infty$.

Then the set of all convergent series $\sum_{n=1}^{\infty} \underline{a}_n \cdot \underline{x}_n$, $a_n \in \Pi^m$, $n \in \mathbb{N}$, *is dense in* $H^m(D_1)$.

Proof. We modify the proof of [2, Theorem 6.3.10]. Let K be a compact subset of D_1. We choose a simply connected domain G such that $K \subseteq G$, $\overline{G}$ is a compact subset of D_1 and the boundary of G is an analytic simple closed curve. We will consider the space $H_2^m(D_1)$. In view of Lemma 2.7 (see [2, proof of Theorem 6.3.10]), we have

$$\|\underline{f}_n\|^2 = \int_{\partial G} \sum_{l=1}^{m} f_{l,n}\, d\mu_{f_{l,n}} \leqslant \sup_{1 \leqslant l \leqslant m} \sup_{s \in \overline{G}} \bigl|f_{l,n}(s)\bigr| \int_{\partial G} |d\mu_{f_{l,n}}|$$
$$\leqslant c \sup_{1 \leqslant l \leqslant m} \sup_{s \in \overline{G}} \bigl|f_{l,n}(s)\bigr|^2.$$

Hence by assumption (c), we have

$$\sum_{n=1}^{\infty} \|\underline{f}_n\|^2 < \infty. \tag{3.1}$$

Suppose $\underline{c}_n \in \Pi^m$, $n \in \mathbb{N}$. Let $\underline{g} \in H^m(G)$ be such that

$$\sum_{n=1}^{\infty} \left|\langle \underline{c}_n \cdot \underline{f}_n, \underline{g}\rangle\right| < \infty. \tag{3.2}$$

By Lemma 2.7 again, we have

$$\langle \underline{c}_n \cdot \underline{f}_n, \underline{g}\rangle = \sum_{l=1}^{m} \int_{\mathbb{C}} c_{l,n} f_{l,n}\, d\mu_{g_l}, \tag{3.3}$$

where μ_{g_l}, $1 \leqslant l \leqslant m$, are complex Borel measures with support contained in the boundary of G. Thus in view of (3.2), we have

$$\sum_{n=1}^{\infty} \left| \sum_{l=1}^{m} \int_{\mathbb{C}} c_{l,n} f_{l,n}\, d\mu_{g_l} \right| < \infty.$$

This and assumption (a) give that

$$\int_{\mathbb{C}} s^r\, d\mu_{g_l} = 0, \quad \text{for all } r \in \mathbb{N} \cup \{0\},\ 1 \leqslant l \leqslant m.$$

Hence in view of (3.3), we deduce that $\underline{g}$ is orthogonal to all polynomials. Therefore it follows from Lemmas 2.3 and 2.8 that $\underline{g}$ is the zero element of $H_2^m(G)$. Consequently,

$$\sum_{n=1}^{\infty} \left|\langle \underline{c}_n \cdot \underline{f}_n, \underline{g}\rangle\right| = \infty, \quad \text{for all } 0 \neq \underline{g} \in H_2^m(G).$$

Whence and from (3.1), using Theorem 3.1, we obtain that the set of all convergent series

$$\sum_{n=1}^{\infty} \underline{a}_n \cdot \underline{f}_n, \quad \underline{a}_n \in \Pi^m,$$

is dense in $H_2^m(G)$.

Let $f \in H^m(D_1)$ and $\varepsilon > 0$. Then by Theorem 3.1 and Lemma 2.6, there exists a series

$$\sum_{n=1}^{\infty} \underline{\alpha}_n \cdot \underline{f}_n, \quad \underline{\alpha}_n \in \Pi^m,$$

convergent uniformly on K and

$$\sup_{1\leqslant l\leqslant m}\sup_{s\in K}\left|\sum_{n=1}^{\infty}\alpha_{l,n}f_{l,n}(s)-f_l(s)\right|<\frac{\varepsilon}{4}.$$

Thus we can choose a positive integer M such that

$$\sup_{1\leqslant l\leqslant m}\sup_{s\in K}\left|\sum_{n=1}^{M}\alpha_{l,n}f_{l,n}(s)-f_l(s)\right|<\frac{\varepsilon}{2}, \tag{3.4}$$

and in view of (b)

$$\sup_{1\leqslant l\leqslant m}\sup_{s\in K}\left|\sum_{n=M+1}^{\infty}d_{l,n}f_{l,n}(s)\right|<\frac{\varepsilon}{2}. \tag{3.5}$$

Now let

$$\underline{a}_n:=\begin{cases}\underline{\alpha}_n & \text{if } 1\leqslant m\leqslant M,\\ \underline{d}_n & \text{if } m>M.\end{cases}$$

Then we have the convergent series $\sum_{n=1}^{\infty}\underline{a}_n\cdot\underline{x}_n$ in $H^m(D)$ and the inequalities (3.4) and (3.5) yield the inequality

$$\sup_{1\leqslant l\leqslant m}\sup_{s\in K}\left|\sum_{n=1}^{\infty}a_{l,n}f_{l,n}(s)-f_l(s)\right|<\varepsilon$$

which completes the proof of this theorem. □

Proof of Theorem 1.3. We modify the proof of [6, Theorem 10]. It follows from the definition of Ω^m that $\{\underline{\omega}(n)\}$ is a sequence of independent random variables with respect to the measure m_{H_m}. Hence $\{\underline{f}_n(s,\underline{\omega}(n)),\ n\in\mathbb{N}\cup\{0\}\}$ is a sequence of independent $H^m(D)$-random elements, where

$$\underline{f}_n\big(s,\underline{\omega}(n)\big):=\left(\frac{e^{2\pi i\lambda_1 n}\omega_1(n)}{(n+a_1)^s},\dots,\frac{e^{2\pi i\lambda_m n}\omega_m(n)}{(n+a_m)^s}\right).$$

The support of each $\omega_l(n)$ $(n\in\mathbb{N}\cup\{0\}, 1\leqslant l\leqslant m)$ is the unit circle γ. Therefore the set $\{\underline{f}_n(s,\underline{\alpha});\ \underline{\alpha}\in\Pi^m\}$ is the support of the random elements $\underline{f}_n(s,\underline{\omega}(n))$. Consequently, by Lemma 2.2 the closure of the set of all convergent series

$$\sum_{n=0}^{\infty}\left(\frac{e^{2\pi i\lambda_1 n}\alpha_{1,n}}{(n+a_1)^s},\dots,\frac{e^{2\pi i\lambda_m n}\alpha_{m,n}}{(n+a_m)^s}\right),\quad \alpha_{l,n}\in\gamma,\ n\in\mathbb{N}\cup\{0\},\ 1\leqslant l\leqslant m,$$

is the support of the random element $\underline{L}(s,\underline{\omega}):=(L(\lambda_1,a_1,s,\omega_1),\dots,L(\lambda_m,a_m,s,\omega_m))$. It remains to check that the latter set is dense in $H^m(D)$. First, we check assumption (c) of Theorem 3.2. By the definition of D, for every compact subset K of D,

$$\sup_{1\leqslant l\leqslant m}\sup_{s\in K}\sum_{n=0}^{\infty}\left|\frac{e^{2\pi i\lambda_l n}}{(n+a_l)^{2\sigma}}\right|<\infty.$$

Next, we verify assumptions (a) and (b) of Theorem 3.2. We put $\eta_l := \eta + l/m$, $\eta \in \mathbb{R}\setminus\mathbb{Q}$, $1 \leqslant l \leqslant m$. We define $\underline{c}_n \in \Pi^m$ by

$$e^{2\pi i\lambda_l n}c_{l,n} := e^{2\pi i\eta_l n}, \quad n \in \mathbb{N}\cup\{0\},\ 1\leqslant l\leqslant m.$$

By the definition of $\underline{c}_n$ and Abel's partial summation, we can check assumption (b). Therefore it remains to confirm only assumption (a) of Theorem 3.2. Let μ_l, $1 \leqslant l \leqslant m$, be complex measures on $(\mathbb{C}, \mathfrak{B}(\mathbb{C}))$ with compact supports contained in D such that

$$\sum_{n=0}^{\infty}\left|\int_{\mathbb{C}}\sum_{l=1}^{m}\frac{e^{2\pi i\eta_l n}}{(n+a_l)^s}\,d\mu_l\right|<\infty. \tag{3.6}$$

By the same argument as in the proof of [8, Theorem 4.1] (using the "positive density method"), we deduce that

$$\int_{\mathbb{C}} s^r\,d\mu_l = 0 \quad \text{for all } r\in\mathbb{N}\cup\{0\},\ 1\leqslant l\leqslant m.$$

Therefore we obtain that all assumptions of Theorem 3.2 are satisfied. Hence we obtain Theorem 1.3 by the same argument as in the proof of [5, Theorem 1]. □

Remark 3.3. In the proof of [6, Theorem 10], the authors showed $S_{P_{\underline{L}_0}} = H^m(D)$ where

$$\underline{L}_0(s,\omega) := \bigl(L(\lambda_1,a_1,s,\omega),\dots,L(\lambda_m,a_m,s,\omega)\bigr), \quad \omega\in\Omega.$$

Clearly $S_{P_{\underline{L}_0}} \subseteq S_{P_{\underline{L}}}$. The fact $S_{P_{\underline{L}}} = H^m(D)$ is therefore an immediate consequence of $S_{P_{\underline{L}_0}} = H^m(D)$ in the situation of [6]. However, if at least two of λ_l's are equal, which is a special case of the present weaker assumptions, we will show $S_{P_{\underline{L}_0}} \neq H^m(D)$ in Proposition 5.3, hence we have shown $S_{P_{\underline{L}}} = H^m(D)$ directly in the proof of Theorem 1.3.

By Theorem 1.3, we obtain the following theorem. The proof of this theorem is completely the same as in the proof of [5, Theorem 2].

Theorem 3.4. *Let λ_l and a_l $(1\leqslant l\leqslant m)$ be as in Theorem 1.3. Suppose F_k, $0\leqslant k\leqslant n$, are continuous functions on $\mathbb{C}^{Nm}$. Suppose*

$$\sum_{k=0}^{n}s^kF_k\bigl(L(\lambda_1,a_1,s),\dots,L(\lambda_m,a_m,s),L'(\lambda_1,a_1,s),\dots,$$
$$L'(\lambda_m,a_m,s),\dots,L^{(N-1)}(\lambda_1,a_1,s),\dots,L^{(N-1)}(\lambda_m,a_m,s)\bigr)=0$$

identically for all $s\in\mathbb{C}$. Then $F_k\equiv 0$, $0\leqslant k\leqslant n$.

4. Joint universality II

In this section, we will prove Theorem 1.5. Firstly we show the limit theorem for $\mathfrak{L}(\lambda, a, b, c; s)$. Define on $(H^m(D), \mathfrak{B}(H^m(D)))$ the probability measure

$$P_{\underline{\mathfrak{L}}}^T(A) := \nu_T^\tau\big(\big(\mathfrak{L}(\lambda_1, a_1, b_1, c; s+i\tau), \dots, \mathfrak{L}(\lambda_m, a_m, b_m, c; s+i\tau)\big) \in A\big), \quad A \in \mathfrak{B}\big(H^m(D)\big).$$

We define the $H^m(D)$-valued random elements $\underline{\mathfrak{L}}(s, \underline{\omega})$ by

$$\underline{\mathfrak{L}}(s, \underline{\omega}) := \big(\mathfrak{L}(\lambda_1, a_1, b_1, c; s, \omega_1), \dots, \mathfrak{L}(\lambda_m, a_m, b_m, c; s, \omega_m)\big),$$

where

$$\mathfrak{L}(\lambda_l, a_l, b_l, c; s, \omega_l) := \sum_{n=0}^\infty \frac{e^{2\pi i \lambda_l n}\omega_l(n)}{(n+a_l)^{s-c}(n+b_l)^c}, \quad s \in D,\ \omega_l \in \Omega,\ 1 \leqslant l \leqslant m.$$

Let $P_{\underline{\mathfrak{L}}}$ be the distribution of the random element $\underline{\mathfrak{L}}(s, \underline{\omega})$.

Proposition 4.1. *Suppose $0 < a_l < 1$ are algebraically independent numbers and $0 < \lambda_l \leqslant 1$ for $1 \leqslant l \leqslant m$. The probability measure $P_{\underline{\mathfrak{L}}}^T$ converges weakly to $P_{\underline{\mathfrak{L}}}$ as $T \to \infty$.*

To prove this proposition, we prepare notations. For $\sigma_{1l} > 1/2$, $1 \leqslant l \leqslant m$, we define functions

$$\mathfrak{L}_r(\lambda_l, a_l, b_l, c; s) := \sum_{n=0}^\infty \frac{e^{2\pi i \lambda_l n} v_l(n, r)}{(n+a_l)^{s-c}(n+b_l)^c},$$

$$L_r(\lambda_l, a_l, s) := \sum_{n=0}^\infty \frac{e^{2\pi i \lambda_l n} v_l(n, r)}{(n+a_l)^s}, \quad v_l(n, r) := \exp\left\{-\left(\frac{n+a_l}{r+a_l}\right)^{\sigma_{1l}}\right\}.$$

Let $\{C_k\}$ be a sequence of compact subsets of D such that $\bigcup_{k=1}^\infty C_k$, $C_k \subset C_{k+1}$, and if C is a compact subset of D, then $C \subseteq C_k$ for some k.

Lemma 4.2. *We have*

$$\lim_{r\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s \in C_k} \big|\mathfrak{L}(\lambda_l, a_l, b_l, c; s+i\tau) - \mathfrak{L}_r(\lambda_l, a_l, b_l, c; s+i\tau)\big|\, d\tau = 0. \tag{4.1}$$

Proof. By the triangle inequality, we have

$$\begin{aligned}
&\big|\mathfrak{L}(\lambda_l, a_l, b_l, c; s+i\tau) - \mathfrak{L}_r(\lambda_l, a_l, b_l, c; s+i\tau)\big| \\
&\quad \leqslant \big|\mathfrak{L}(\lambda_l, a_l, b_l, c; s+i\tau) - L(\lambda_l, a_l, s+i\tau) + L_r(\lambda_l, a_l, s+i\tau) - \mathfrak{L}_r(\lambda_l, a_l, b_l, c; s+i\tau)\big| \\
&\qquad + \big|L(\lambda_l, a_l, s+i\tau) - L_r(\lambda_l, a_l, s+i\tau)\big| \\
&\quad := \big|f_r(\lambda_l, a_l, b_l, c; s+i\tau)\big| + \big|g_r(\lambda_l, a_l, b_l, c; s+i\tau)\big|,
\end{aligned}$$

say. We have

$$\lim_{r\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in C_k}\bigl|g_r(\lambda_l, a_l, b_l, c; s+i\tau)\bigr|\,d\tau = 0, \tag{4.2}$$

by [4, Lemma 5.2.11]. Hence we have to show

$$\lim_{r\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in C_k}\bigl|f_r(\lambda_l, a_l, b_l, c; s+i\tau)\bigr|\,d\tau = 0.$$

By the binomial theorem, we have

$$1-\left(\frac{n+b_l}{n+a_l}\right)^c = O\bigl(n^{-1}\bigr).$$

Hence, for some positive constant K, we have

$$\begin{aligned}
&\sup_{s\in C_k}\bigl|f_r(\lambda_l, a_l, b_l, c; s+i\tau)\bigr| \\
&\quad\leqslant \sum_{n=0}^{\infty}\frac{1}{(n+a_l)^{\Re(s-c)}(n+b_l)^{\Re(c)}}\left|1-\left(\frac{n+b_l}{n+a_l}\right)^c\right|\left|1-\exp\left\{-\left(\frac{n+a_l}{r+a_l}\right)^{\sigma_{1l}}\right\}\right| \\
&\quad\leqslant K\sum_{n=0}^{\infty}\frac{1}{(n+a_l)^{3/2}}\left|1-\exp\left\{-\left(\frac{n+a_l}{r+a_l}\right)^{\sigma_{1l}}\right\}\right| := M_r(\sigma_{1l}),
\end{aligned}$$

say. Therefore we obtain

$$\begin{aligned}
&\lim_{r\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in C_k}\bigl|f_r(\lambda_l, a_l, b_l, c; s+i\tau)\bigr|\,d\tau \\
&\quad\leqslant \lim_{r\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T M_r(\sigma_{1l})\,d\tau = \lim_{r\to\infty} M_r(\sigma_{1l}) = 0. \qquad \square
\end{aligned} \tag{4.3}$$

Proof of Proposition 4.1. We modify the proof of [6, Theorem 1]. The only point which is different from the proof of [6, Theorem 1] is to use (4.1) instead of (4.2). Therefore by Lemma 4.2 we can prove Proposition 4.1. □

Proof of Theorem 1.5. Similarly to the argument of the first part of the proof of Theorem 1.3, we have to check that the set of all convergent series

$$\sum_{n=0}^{\infty}\left(\frac{e^{2\pi i\lambda_1 n}\alpha_{1,n}}{(n+a_1)^{s-c}(n+b_1)^c}, \dots, \frac{e^{2\pi i\lambda_m n}\alpha_{m,n}}{(n+a_l)^{s-c}(n+b_m)^c}\right), \quad \underline{\alpha}_n \in \Pi^m,\ n\in\mathbb{N}\cup\{0\},$$

is dense in $H^m(D)$. Hence we will confirm the assumptions of Theorem 3.2. Let μ_l, $1 \leqslant l \leqslant m$, be complex measures on $(\mathbb{C}, \mathfrak{B}(\mathbb{C}))$ with compact supports contained in D such that

$$\sum_{n=0}^{\infty} \left| \int_{\mathbb{C}} \sum_{l=1}^{m} \frac{e^{2\pi i \lambda_l n}}{(n+a_l)^{s-c}(n+b_l)^c} \, d\mu_l \right| < \infty.$$

By the same argument as in [5, (12)], we have

$$(n+a_l)^{-s} = n^{-s} + Bn^{-1-\sigma}|s|e^{B|s|}$$

where B is a positive constant. Hence the above formula is equivalent to

$$\sum_{n=1}^{\infty} \left| \int_{\mathbb{C}} \sum_{l=1}^{m} \frac{e^{2\pi i \lambda_l n}}{n^s} \, d\mu_l \right| < \infty.$$

Therefore we can easily confirm all assumptions by modifying the proof of Theorem 1.3. Hence we obtain Theorem 1.5 by the same argument as in the proof of [5, Theorem 1]. □

5. Non-denseness lemma

In this section, we reconsider the proofs of Theorems 1.3 and 1.5, especially Remark 3.3. The next theorem is a kind of counter-proposition for Lemma 2.5.

Theorem 5.1 *(Non-denseness lemma). Let $\{\underline{x}_n\}$ be a sequence in a Hilbert space X^m satisfying the following condition*:

(a) *There exists a non-zero $\underline{x} \in X^m$ such that $\sum_{n=1}^{\infty} |\langle \underline{x}_n, \underline{x} \rangle| < \infty$.*

Then the set of all convergent series

$$\sum_{n=1}^{\infty} a_n \underline{x}_n, \quad |a_n| = 1, \ n \in \mathbb{N},$$

is not dense in X^m.

Proof. Firstly, we consider the case of $\sum_{n=1}^{\infty} |\langle \underline{x}_n, \underline{x} \rangle| = 0$. We take an $\underline{x}$ which satisfies this condition. By the assumption, we have

$$\left\| \underline{x} - \sum_{n=1}^{\infty} a_n \underline{x}_n \right\|^2 = \|\underline{x}\|^2 + \left\| \sum_{n=1}^{\infty} a_n \underline{x}_n \right\|^2 \geqslant \|\underline{x}\|^2.$$

Hence in this case, the set of all convergent series $\sum_{n=1}^{\infty} a_n \underline{x}_n$ is not dense in X^m. Next we consider the case of $0 \neq \sum_{n=1}^{\infty} |\langle \underline{x}_n, \underline{x}\rangle| < \infty$. We take an $\underline{x}$ which satisfies this condition and choose $b_n \in \mathbb{C}$ so that $|b_n| = 1$ and

$$\begin{cases} b_n \langle \underline{x}_n, \underline{x}\rangle = -|\langle \underline{x}_n, \underline{x}\rangle| & \text{if } \langle \underline{x}_n, \underline{x}\rangle \neq 0, \\ b_n = 1 & \text{if } \langle \underline{x}_n, \underline{x}\rangle = 0. \end{cases}$$

We can assume that $|\langle \underline{x}_1, \underline{x}\rangle| \neq 0$ without loss of generality. Let M be a sufficiently large integer which satisfies

$$\left| 2\Re\left(\sum_{n=M+1}^{\infty} a_n \langle \underline{x}_n, \underline{x}\rangle \right) - \left\| \sum_{n=M+1}^{\infty} a_n \underline{x}_n \right\|^2 \right| < \frac{|\langle \underline{x}_1, \underline{x}\rangle|}{2}.$$

By the trigonometric inequality, we have

$$\begin{aligned} \left\| 2\sum_{n=1}^{M} b_n \underline{x}_n - \sum_{n=1}^{\infty} a_n \underline{x}_n \right\| &= \left\| \sum_{n=1}^{M} 2b_n \underline{x}_n - \underline{x} + \underline{x} - \sum_{n=1}^{\infty} a_n \underline{x}_n \right\| \\ &\geqslant \left| \left\| \sum_{n=1}^{M} (2b_n - a_n)\underline{x}_n - \underline{x} \right\| - \left\| \underline{x} - \sum_{n=M+1}^{\infty} a_n \underline{x}_n \right\| \right| := |A - B|, \end{aligned}$$

say. Then we obtain

$$\begin{aligned} A^2 - B^2 = \left\| \sum_{n=1}^{M} (2b_n - a_n)\underline{x}_n \right\|^2 &- 2\Re\left(\sum_{n=1}^{M} (2b_n - a_n)\langle \underline{x}_n, \underline{x}\rangle \right) \\ &+ 2\Re\left(\sum_{n=M+1}^{\infty} a_n \langle \underline{x}_n, \underline{x}\rangle \right) - \left\| \sum_{n=M+1}^{\infty} a_n \underline{x}_n \right\|^2. \end{aligned}$$

By the definition of $\underline{x}$ and b_n, we have

$$-\Re\left(\sum_{n=1}^{M} (2b_n - a_n)\langle \underline{x}_n, \underline{x}\rangle \right) \geqslant \sum_{n=1}^{M} \left|\langle \underline{x}_n, \underline{x}\rangle\right| \geqslant \left|\langle \underline{x}_1, \underline{x}\rangle\right|.$$

Therefore we have the inequality

$$\left\| 2\sum_{n=1}^{M} b_n \underline{x}_n - \sum_{n=1}^{\infty} a_n \underline{x}_n \right\| \geqslant A - B > \frac{|\langle \underline{x}_1, \underline{x}\rangle|}{2(A+B)}.$$

Hence the set of all convergent series $\sum_{n=1}^{\infty} a_n \underline{x}_n$ is not dense in X^m. □

Lemma 5.2. *If $\{\underline{x}_n\}$ is not dense in $H_2^m(D)$, then $\{\underline{x}_n\}$ is not dense in $H^m(D)$.*

Proof. We show the contraposition, that is, if $\{\underline{x}_n\}$ is dense in $H^m(D)$, then $\{\underline{x}_n\}$ is dense also in the Hardy space $H_2^m(D)$. By Lemma 2.7, we have

$$\|\underline{f}-\underline{g}\|^2=\sum_{l=1}^{m}\int\limits_{\partial D}(f_l-g_l)\,d\mu_{(f_l-g_l)}\leqslant c_2\sup_{1\leqslant l\leqslant m}\sup_{s\in\overline{D}}|f_l-g_l|^2.$$

This implies the contraposition. □

Proposition 5.3. *Suppose* $0<a_l<1$ *are algebraically independent numbers and* $0<\lambda_l\leqslant 1$ *for* $1\leqslant l\leqslant m$. *If at least two of* λ_l*'s are equal, it holds that* $S_{P_{\underline{L}_0}}\neq H^m(D)$ (*see Remark* 3.3).

Proof. Similarly to the argument of the first part of the proof of Theorem 1.3, we have to check that

$$\sum_{n=0}^{\infty}\left(\frac{e^{2\pi i\lambda_1 n}\omega_n}{(n+a_1)^s},\dots,\frac{e^{2\pi i\lambda_m n}\omega_n}{(n+a_m)^s}\right),\quad \omega_n\in\gamma,\ n\in\mathbb{N}\cup\{0\},$$

is not dense in $H^m(D)$. First, we consider the case of $m=2$, $\lambda:=\lambda_1=\lambda_2$. Let μ_1 and μ_2 be complex measures on $(\mathbb{C},\mathfrak{B}(\mathbb{C}))$ with compact supports contained in D such that

$$\sum_{n=0}^{\infty}\left|\int\limits_{\mathbb{C}}\sum_{l=1}^{2}\frac{e^{2\pi i\lambda n}}{(n+a_l)^s}\,d\mu_l\right|<\infty.$$

By the same argument as in [5, (12)], the above formula is equivalent to

$$\sum_{n=1}^{\infty}\left|\int\limits_{\mathbb{C}}\sum_{l=1}^{2}\frac{e^{2\pi i\lambda n}}{n^s}\,d\mu_l\right|<\infty. \tag{5.1}$$

If $0\neq\mu_1=-\mu_2$, then we see that the left-hand side of (5.1) is equal to zero, hence the measures satisfy condition (5.1). Applying Lemma 2.4 with $x=x_n=e^{2\pi i\lambda n}n^{-s}$ and

$$f:x_n\longmapsto\int\limits_{\mathbb{C}}x_n\,d\mu_l,$$

we can rewrite (5.1) in terms of inner products. Because of Lemma 5.1, the set of all convergent series $\sum_{n=0}^{\infty}\underline{f}_n(s,\omega_n)$ with $\omega_n\in\gamma$ is not dense in the Hardy space $H_2^2(D)$. Therefore the set of all convergent series is not dense in $H^2(D)$ by Lemma 5.2. If $m\geqslant 3$, we can put $\lambda:=\lambda_1=\lambda_2$ without loss of generality. In this case, we take $0\neq\mu_1=-\mu_2$, $0=\mu_3=\cdots=\mu_m$. □

6. Examples of non-existence of universality

In this section, we will show three examples which imply the non-existence of joint universality for Lerch zeta functions and generalized Lerch zeta functions. We remark that the parameters $a_1,\dots,a_m$ of the next two examples are not algebraically independent.

Proposition 6.1. *If we put* $a_1 = 1$ *and* $a_2 = 1/2$*, then there exist an* $\varepsilon > 0$ *and analytic functions* $f_l(s)$ *on* K_l*, for which there does not exist* τ *satisfying*

$$\sup_{1\leqslant l\leqslant m}\sup_{s\in K_l}\left|\zeta(a_l, s+i\tau) - f_l(s)\right| \leqslant \varepsilon.$$

Proof. Let $K := \{s; |s-3/4| \leqslant R\}$, $0 < R < 1/4$. We put $\varepsilon = 1/3$, $f_1(s) \equiv 1$ and $f_2(s) \equiv 8$. Suppose

$$\sup_{s\in K}\left|\zeta(s+i\tau) - 1\right| < \frac{1}{3}. \tag{6.1}$$

For every τ satisfying (6.1), by the well-known formula

$$\zeta(1/2, s) = \left(2^s - 1\right)\zeta(s), \tag{6.2}$$

we have

$$\begin{aligned}\sup_{s\in K}\left|\zeta(1/2, s+i\tau) - 8\right| &= \sup_{s\in K}\left|\left(2^{s+i\tau}-1\right)\left(\zeta(s+i\tau)-1\right) + 2^{s+i\tau} - 9\right| \\ &\geqslant \sup_{s\in K}\left|\left|\left(2^{s+i\tau}-1\right)\left(\zeta(s+i\tau)-1\right)\right| - \left|2^{s+i\tau}-9\right|\right| \\ &\geqslant \sup_{s\in K}|1-7| = 6. \qquad \square\end{aligned}$$

This proposition implies that the set of Hurwitz zeta functions does not necessarily have the joint t-universality. Proposition 6.1 is a rather obvious example, but we can observe that the key of the proof is the functional relation (6.2). By using another functional relation, we can show the following result.

Proposition 6.2. *Let* a *be a positive number and* λ *be a real number. If we put* $\lambda_n = \lambda + n/m$, $a_n = ma$ *for* $0 \leqslant n \leqslant m-1$*, and* $\lambda_m = m\lambda$, $a_m = a + j/m$ $(0 \leqslant j \leqslant m-1)$*, then there exist an* $\varepsilon > 0$ *and analytic functions* $f_l(s)$ *on* K_l*, for which there does not exist* τ *satisfying*

$$\sup_{0\leqslant l\leqslant m}\sup_{s\in K_l}\left|L(\lambda_l, a_l, s+i\tau) - f_l(s)\right| \leqslant \varepsilon.$$

Proof. We define ω_m^j by

$$\omega_m^j := \exp(2\pi i j/m), \quad j, m \in \mathbb{N},\ 0 \leqslant j \leqslant m-1.$$

By using the inversion formula [8, Lemma 2.1] (see also [1, Theorem 2.1])

$$L\left(m\lambda, a + \frac{j}{m}, s\right) = m^{s-1}e^{-2\pi i\lambda j}\sum_{n=0}^{m-1}\omega_m^{-jn}L\left(\lambda + \frac{n}{m}, ma, s\right),$$

and modifying the proof of Proposition 6.1, we can show this proposition. $\square$

Remark 6.3. These propositions show that the existence of functional relations implies the non-existence of joint t-universality. Therefore we can see that joint t-universality is essentially more difficult than single t-universality (for example [2, p. 111, Theorem 1.1]) because of its connection with functional relations. These facts should be compared with Theorem 3.4 concerning functional independence, deduced by joint t-universality Theorem 1.3.

In the case of $a_1 = \cdots = a_m$, we have the following non-existence of joint t-universality for $\mathfrak{L}(\lambda_l, a, b, c; s)$.

Proposition 6.4. *If at least two of λ_l's are equal, then there exist an $\varepsilon > 0$ and analytic functions $f_l(s)$ on K_l, for which there does not exist τ satisfying*

$$\sup_{1\leqslant l\leqslant m} \sup_{s\in K_l} \bigl|\mathfrak{L}(\lambda_l, a, b_l, c; s + i\tau) - f_l(s)\bigr| \leqslant \varepsilon. \tag{6.3}$$

Proof. We assume $m = 2$ and $\lambda := \lambda_1 = \lambda_2$ without loss of generality. For some positive constants C_1 and C_2, we have

$$\begin{aligned}
&\bigl|\mathfrak{L}(\lambda, a, b_1, c; s + i\tau) - \mathfrak{L}(\lambda, a, b_2, c; s + i\tau)\bigr| \\
&\quad = \left|\sum_{n=0}^{\infty} \frac{e^{2\pi i\lambda n}}{(n+a)^{s+i\tau-c}(n+b_1)^c}\left(1 - \frac{(n+b_1)^c}{(n+b_2)^c}\right)\right| \\
&\quad \leqslant C_1 \sum_{n=0}^{\infty} \frac{1}{n(n+a)^{\Re(s)}} \leqslant C_2.
\end{aligned} \tag{6.4}$$

Let $K_1 = K_2 = K := \{s\colon\ |s - 3/4| \leqslant 1/5\}$. We put $\varepsilon = 1/3$, $f_1(s) \equiv 1$ and $f_1(s) \equiv C_2 + 1$. Suppose

$$\sup_{s\in K} \bigl|\mathfrak{L}(\lambda_l, a, b_1, c; s + i\tau) - 1\bigr| \leqslant 1/3.$$

For every τ satisfying the above formula, we have

$$\sup_{s\in K} \bigl|\mathfrak{L}(\lambda_l, a_1, b_2, c; s + i\tau) - (C_2 + 1)\bigr| > 1/3.$$

Hence we have (6.3) in this case. □

In the case when a is transcendental, we obtain another proof of Proposition 6.4 by using Theorem 5.1. Firstly we show the limit theorem for $\mathfrak{L}(\lambda, a, b, c; s)$. Denote on $(H^m(D), \mathfrak{B}(H^m(D)))$ the probability measure

$$P_{\underline{\mathfrak{L}}_0}^T(A) := \nu_T^\tau\bigl(\bigl(\mathfrak{L}(\lambda_1, a, b_1, c; s + i\tau), \ldots, \mathfrak{L}(\lambda_m, a, b_m, c; s + i\tau)\bigr) \in A\bigr), \quad A \in \mathfrak{B}\bigl(H^m(D)\bigr).$$

We define the $H^m(D)$-valued random element $\underline{\mathfrak{L}}_0(s, \underline{\omega})$ by

$$\underline{\mathfrak{L}}_0(s, \underline{\omega}) := \bigl(\mathfrak{L}(\lambda_1, a, b_1, c; s, \omega_1), \ldots, \mathfrak{L}(\lambda_m, a, b_m, c; s, \omega_m)\bigr),$$

where

$$\mathfrak{L}_0(\lambda_l, a, b_l, c; s, \omega_l) := \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda_l n} \omega_l(n)}{(n+a)^{s-c}(n+b_l)^c}, \quad s \in D,\ \omega_l \in \Omega,\ 1 \leqslant l \leqslant m.$$

Let $P_{\underline{\mathfrak{L}}_0}$ stand for the distribution of the random element $\underline{\mathfrak{L}}_0(s, \underline{\omega})$.

Proposition 6.5. *Suppose* $0 < a < 1$ *is a transcendental number and* $0 < \lambda_l \leqslant 1$ *for* $1 \leqslant l \leqslant m$. *The probability measure* $P^T_{\underline{\mathfrak{L}}_0}$ *converges weakly to* $P_{\underline{\mathfrak{L}}_0}$ *as* $T \to \infty$.

Proof. We can prove this theorem by modifying [3, Theorem 3] and using Lemma 4.2. □

Similarly to the argument of Proposition 5.3, we can check

$$\sum_{n=0}^{\infty} \left(\frac{e^{2\pi i \lambda_1 n} \alpha_n}{(n+a)^{s-c}(n+b_1)^c}, \ldots, \frac{e^{2\pi i \lambda_m n} \alpha_n}{(n+a)^{s-c}(n+b_m)^c} \right), \quad \alpha_n \in \gamma,\ n \in \mathbb{N} \cup \{0\},$$

is not dense in $H^m(D)$, since

$$\sum_{n=0}^{\infty} \left| \int_{\mathbb{C}} \sum_{l=1}^{2} \frac{e^{2\pi i \lambda n}}{(n+a)^{s-c}(n+b_l)^c} \, d\mu_l \right| < \infty$$

is equivalent to (5.1).

Suppose that functions $F_l(s)$ for $1 \leqslant l \leqslant m$ can be continued analytically to the whole D. Denote by V_k the set of $\underline{g} \in H^m(D)$ such that

$$\sup_{1 \leqslant l \leqslant m} \sup_{s \in K_l} \left| g_l(s) - F_l(s) \right| < (k+1)\varepsilon, \quad k = 1, 2.$$

We recall that the support S_P consists of all $\underline{f} \in H^m(D)$ such that for every neighborhood V of $\underline{f}$ the inequality $P(V) > 0$ is satisfied. Since the support of the random element $\underline{\mathfrak{L}}_0(s, \omega)$ is not whole $H^m(D)$, there exist a set of analytic functions $f_l(s)$ and its neighborhood V_2 satisfying $P_{\underline{\mathfrak{L}}_0}(V_2) = 0$. Since $\overline{V_1} \subset V_2$, we have $P_{\underline{\mathfrak{L}}_0}(\overline{V_1}) = 0$. Let P_n and P be probability measures defined on $(S, \mathfrak{B}(S))$. It is well known that P_n converges weakly to P as $n \to \infty$ if and only if

$$\limsup_{n \to \infty} P_n(C) \leqslant P(C)$$

for all closed sets C. The set of $\overline{V_1}$ is closed, hence by Lemma 2.1, we obtain

$$\limsup_{T \to \infty} \nu_T^{\tau} \left\{ \sup_{1 \leqslant l \leqslant m} \sup_{s \in K_l} \left| \mathfrak{L}(\lambda_l, a, b_l, c; s + i\tau) - f_l(s) \right| \leqslant 2\varepsilon \right\} \leqslant P_{\underline{\mathfrak{L}}_0}(\overline{V_1}) = 0.$$

This formula yields the assertion of non-existence of joint t-universality.

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