**"INTEGER-MAKING" THEOREMS**

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Let $X = \{x_1, x_2, \ldots\}$ be a finite set and associate to every $x_i$ a real number $\alpha_i$. Let $f(n)$ [$g(n)$] be the least value such that given any family $\mathcal{F}$ of subsets of $X$ having maximum degree $n$ [cardinality $\pi$], one can find integers $a_i$, $i = 1, 2, \ldots$ so that $|a_i - \alpha_i| < 1$ and

\[
\left| \sum_{x_i \in E} a_i - \sum_{x_i \in E} \alpha_i \right| \leq f(n) \quad \left[ \sum_{x_i \in E} a_i - \sum_{x_i \in E} \alpha_i \right] \leq g(n)
\]

for all $E \in \mathcal{F}$. We prove

\[ f(n) \leq n - 1 \quad \text{and} \quad g(n) \leq c(n \log n)^{1/2}. \]

1. Introduction

A well-known theorem of Zs. Baranyai [1] states that the complete $r$-uniform hypergraph of $n$ vertices, $K'_n$, is $1$-factorizable if $r$ is a divisor of $n$. His fundamental lemma was the following: Let $[\alpha_{ij}]$ be an $k \times l$ matrix of real elements, then there exists a $k \times l$ integer matrix $[a_{ij}]$ such that

\[ |a_{ij} - \alpha_{ij}| < 1 \quad \text{for all } i, j, \]

\[ \left| \sum_i a_{ij} - \sum_i \alpha_{ij} \right| < 1 \quad \text{for all } j, \]

\[ \left| \sum_j a_{ij} - \sum_j \alpha_{ij} \right| < 1 \quad \text{for all } i, \quad \text{and} \]

\[ \left| \sum_i \sum_j a_{ij} - \sum_i \sum_j \alpha_{ij} \right| < 1. \]

We shall investigate the following general problem. Let $X = \{x_1, x_2, \ldots\}$ be a finite set and let us associate a real number $\alpha_i$ to $x_i$, $i = 1, 2, \ldots$. What is the smallest value $f(n)$, such that, given any family $\mathcal{F}$ of subsets of $X$ having maximum degree $n$, one can find integers $a_i$, $i = 1, 2, \ldots$ so that $|a_i - \alpha_i| < 1$ and
\[ \sum_{i \in E} a_i - \sum_{x \in E} a_i \leq f(n) \quad \text{for all } E \in \mathcal{F}. \]

(A point in a hypergraph has degree \( d \) if the number of sets containing this point is \( d \) and the maximum degree is the maximum for all points.)

We shall consider also a second function \( g(n) \) that arises when we modify our original question such that the family \( \mathcal{F} \) has the cardinality \( n \), i.e. by asking for the smallest value \( g(n) \) such that, for every family \( \mathcal{F} \) having cardinality \( n \), one can find integers \( a_i, i=1,2,... \) so that \( |a_i - a_j| < 1 \) and for each \( E \in \mathcal{F}, |\sum_{x \in E} a_i - \sum_{x \in E} a_j| \leq g(n) \).

Finally we mention the following balancing problem of J. Olson and J. Spencer [3]: What is the smallest value \( h(n) \) such that, given any \( n \) finite sets \( E_1, ..., E_n \) one can find two disjoint sets \( A \) and \( B \) so that \( A \cup B = E_1 \cup ... \cup E_n \) and for all \( i, |E_i \cap A| - |E_i \cap B| \leq h(n) \). This is essentially the problem of determining \( g(n) \) under the additional condition that all \( a_i = \frac{1}{2} \).

Clearly
\[ \frac{1}{2} h(n) \leq g(n) \leq f(n). \tag{1} \]

J. Olson and J. Spencer [3] proved \((\frac{1}{2} - o(1)) n^{1/2} < h(n)\), thus from (1) follows
\[ (\frac{1}{6} - o(1)) n^{1/2} < g(n) \leq f(n). \]

We conjecture, but cannot prove, that \( f(n) < c n^{1/2} \). We shall prove, however, that \( f(n) \leq n - 1 \) and \( g(n) < c(n \log n)^{1/2} \).

**Theorem 1.** \( f(n) \leq n - 1 \) for \( n \geq 2 \).

Baranyai's theorem and several questions concerning coloring shows, that it would be important to know the exact value of \( f(n) \) for small values of \( n \). K3, the simple triangle shows that \( f(2) \geq 1 \), so our theorem implies \( f(2) = 1 \). We do not know the exact value of \( f(3) \), only the inequalities \( \frac{1}{2} \leq f(3) < 2 \). (A lower bound can be derived by considering the finite geometry on 7 points.) We conjecture that \( f(n) \leq \frac{1}{2} n \) is true even for small values of \( n \).

**Theorem 2.** \( g(n) < 2(2n \log 2n)^{1/2} \) for \( n \geq 5 \).

By (1) and Theorem 2 we obtain
\[ h(n) < 8(2n \log 2n)^{1/2} \quad \text{for } n \geq 5, \tag{2} \]

which is an improvement of the upper bound of J. Olson and J. Spencer [3] (they showed \( h(n) < (2n)^{1/2} \log n) \). We shall prove the following more general

**Theorem 3.** Given \( n \geq 5 \) finite sets \( E_1, ..., E_n \) it is possible to partition their union into \( r \) parts \( A_1, ..., A_r \) for any positive integer \( r \) in such a way, that, for each \( i, j \) and \( k \),
\[ |E_i \cap A_j| - |E_i \cap A_k| < 24(2n \log 2n)^{1/2}. \]
2. Proof of Theorem 1

We begin with a rough proof that \( f(n) \leq n \). Let \( \mathcal{F} \) be a family of subsets of \( X \) having maximal degree \( n \) and assume that to each point has been associated a real number which we call its weight. Assume, without loss of generality, that all weights lie in \([0, 1]\). Call a point fixed if its weight is zero or one; otherwise, variable. Call a set \( E \in \mathcal{F} \) unsafe if it has more than \( n \) variable points; otherwise, safe.

Now we adjust the weights. We require that the total weight of an unsafe set remain constant. All weights must remain in \([0, 1]\) and fixed points do not change their weights. Under these conditions, each unsafe set gives a linear equation and each variable point gives a variable. As each variable point is in at most \( n \) unsafe sets and each unsafe set has at least \( n + 1 \) variable points, there are more variables than equations. The solution set therefore contains a line through the current weight vector; we move along that line until one of the weights becomes zero or one. This concludes the weight adjustment.

We apply this adjustment repeatedly (each application fixes at least one more point) until all points are fixed. Our "integer-making" is complete: What has happened to the total weight of a particular \( E \in \mathcal{F} \). As long as \( E \) was unsafe its total weight remained constant. When \( E \) became safe it contained at most \( n \) variable points so that its total weight could change by at most \( n \).

In the remainder of this section we formalize and finetune the above argument. Let \( X = \{x_1, \ldots, x_s\} \). Without loss of generality we can assume \( 0 \leq \alpha_i \leq 1 \) for \( i = 1, \ldots, s \). We shall define a sequence \( \alpha^0, \alpha^1, \ldots, \alpha^p \) of \( s \)-dimensional vectors \( \alpha^k = (\alpha^k_1, \ldots, \alpha^k_s) \) and a sequence \( Y_k \) of subsets of \( X \) with the following properties.

1. \( \alpha^0_i = \alpha_i \) for \( i = 1, \ldots, s \).
2. \( 0 \leq \alpha_j^k \leq 1 \), \( j = 1, \ldots, s, k = 0, 1, \ldots, p \).
3. \( Y_k \subset X \) is the set of points \( x_i \), for which \( \alpha^k_i \) is not an integer.
4. \( Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_p = \emptyset \).
5. \( \alpha^j_i = \alpha^k_i \) for \( j = k + 1, \ldots, p \), whenever \( \alpha^k_i \) is an integer.
6. \( \sum_{x_i \in Y_k} \alpha^k_i = \sum_{x_i \in E} \alpha^k_i - 1 \) if \( |E \cap Y_k| > n \).
7. If \( |E \cap Y_k| = n \) and \( \sum_{x_i \in E} \alpha^k_i \neq \sum_{x_i \in E} \alpha^{k+1}_i \), then
   \[
   E \cap Y_{k+1} = \emptyset \quad \text{and} \quad \left| \sum_{x_i \in E} \alpha^{k+1}_i - \sum_{x_i \in E} \alpha^k_i \right| \leq \frac{1}{2} n.
   \]

According to (3) these sequences are finished, if the final vector \( \alpha^p \) has only 0, 1 coordinates. Choosing \( \alpha_i = \alpha^p_i \), it follows from the above relations, that for all \( E \in \mathcal{F} \) either \( |\sum_{x_i \in E} \alpha_i - \sum_{x_i \in E} \alpha_i| \leq \frac{1}{2} n \) or \( |\sum_{x_i \in E} \alpha_i - \sum_{x_i \in E} \alpha_i| < n - 1 \), and this is a bit more sharp than Theorem 2.

We construct the sequence \( \alpha^k \) by induction. Supposing \( \alpha^k \) is defined and \( Y_k \) is not void. Let

\[
G_k = \{ E \in \mathcal{F} : |E \cap Y_k | \geq n \}.
\]

If \( G_k \) is void, then put \( p = k + 1 \), \( \alpha^{k+1}_i = \alpha_i \) for \( x_i \in Y_k \) and \( \alpha^{k+1}_i = 0 \) or \( 1 \) arbitrarily for
If \( G_k \) isn't void, then because of maximum degree \( n \) there are two cases to consider.

**Case (a).** \( |G_k| < |Y_k| \).

**Case (b).** \( |G_k| = |Y_k| \) and \( |E \cap Y_k| = n \), if \( E \cap Y_k \neq \emptyset \) and \( x_i \in Y_k \) has degree \( n \) in \( G_k \) (that is \( G_k \) is an \( n \)-uniform, \( n \)-regular hypergraph on \( Y_k \)).

In Case (a) let us associate a real variable \( \beta_i \) for every \( i = 1, \ldots, s \) and consider the following linear system of equations:

\[
\sum_{x_i \in E \cap Y_k} \beta_i = 0, \quad E \in G_k, \\
\beta_i = 0, \quad x_i \notin Y_k.
\]

A nontrivial solution \( \{\beta_i\}_{i=1}^s \) exists, because there are more variables than equations.

Now let \( t_0 \) be the greatest positive value for which

\[
0 \leq \alpha_i^k + t_0 \beta_i \leq 1 \quad \text{for all } i \in Y_k
\]

holds, and put

\[
\alpha_i^{k+1} = \alpha_i^k + t_0 \beta_i \quad \text{for } i = 1, \ldots, s.
\]

Because of the maximality of \( t_0 \), \( Y_{k+1} \subseteq Y_k \). It is easy to check, that

\[
\sum_{x_i \in E} \alpha_i^k = \sum_{x_i \in E} \alpha_i^{k+1} \quad \text{for all } E \in G_k.
\]

In Case (b) let \( \alpha_i^{k+1} \) be the integer closest to \( \alpha_i \) for \( x_i \in Y_k \). Clearly \( |\alpha_i^{k+1} - \alpha_i| \leq \frac{1}{2} \) for \( x_i \in Y_k \), so

\[
\left| \sum_{x_i \in E} \alpha_i^{k+1} - \sum_{x_i \in E} \alpha_i \right| \leq \frac{1}{2} n.
\]

Because of (5) \( \alpha_i^{k+1} - \alpha_i^k = \alpha_i \) for \( x_i \in Y_k \). It is easy to check that in each case the relations (1), (2), \ldots, (7) hold, and this completes the proof.

3. **Proof of Theorem 2**

Let \( X = \{x_1, \ldots, x_s\} \) and \( n \geq 5 \). We may assume that \( 0 \leq \alpha_i \leq 1 \) for \( i = 1, \ldots, s \).

Firstly we construct a vector \( \alpha^* = (\alpha_1^*, \ldots, \alpha_s^*) \) so that \( \alpha_i^* \) has value 1 or 0, except \( n \) suffix \( j_1, \ldots, j_n \) \((1 \leq j_1 < \cdots < j_n \leq s)\) for which \( 0 \leq \alpha_i^* \leq 1, \ i = 1, \ldots, n \) and

\[
\sum_{x_i \in E} \alpha_i^* - \sum_{x_i \in E} \alpha_i = 0 \quad \text{for all } E \in \mathcal{F}.
\]

We shall define a sequence \( \alpha^0, \alpha^1, \ldots, \alpha^p \) of \( s \)-dimensional vectors \( \alpha^k = (\alpha_1^k, \ldots, \alpha_s^k) \) and a sequence \( Y_k \) of subsets of \( X \) with the following properties.
"Integer-making" theorems

(1) $a_i^0 = a_i$ for $i = 1, \ldots, s$.
(2) $0 \leq a_i^k \leq 1$, $i = 1, \ldots, s$, $k = 0, 1, \ldots, p$.
(3) $Y_k \subseteq X$ is the set of points $x_i$, for which $\alpha_i^k$ is not an integer.
(4) $Y_0 \supsetneq Y_1 \supsetneq \ldots \supsetneq Y_p$ and $|Y_p| \leq n$.
(5) $\alpha_i^j = \alpha_i^{k+1}$ for $j = k+1, \ldots, p$, whenever $\alpha_i^k$ is an integer.
(6) $\sum_{x_i \in E} \alpha_i^k = \sum_{x_i \in E} \alpha_i^{k+1}$ for all $E \in \mathcal{F}$.
 Choosing $a_1^* = a_1^p$ we obtain the desired vector $a^* = (a_1^*, \ldots, a_s^*)$. If $|Y_k| > n$, then let us associate a real variable $\beta_i$ for every $i = 1, \ldots, s$ and consider the following linear system of equations:

$$
\sum_{x_i \in E \cap Y_k} \beta_i = 0, \quad E \in \mathcal{F},
$$
$$
\beta_i = 0, \quad x_i \notin Y_k.
$$

A nontrivial solution $\{\beta_i\}_{i=1}^s$ exists, because there are more variables than equations.

Now let $t_0$ be the greatest positive value for which

$$
0 \leq \alpha_i^k + t_0 \beta_i \leq 1 \quad \text{for } x_i \in Y_k
$$

holds, and put

$$
\alpha_i^{k+1} = \alpha_i^k + t_0 \beta_i \quad \text{for } i = 1, \ldots, s.
$$

Because of the maximality of $t_0$, $Y_{k+1} \supsetneq Y_k$. It is easy to check that

$$
\sum_{x_i \in E} \alpha_i^k = \sum_{x_i \in E} \alpha_i^{k+1} \quad \text{for all } E \in \mathcal{F},
$$

which was to be proved.

We remark that the existence of $a^*$ is almost trivial by the following geometric argument.

Let $c_E = (c_{E,1}, \ldots, c_{E,s})$ where $c_{E,i} = 1$ if $x_i \in E$ and $c_{E,i} = 0$ if $x_i \notin E$. Let $V \subseteq \mathbb{R}^s$ be the subspace generated by $\{c_E : E \in \mathcal{F}\}$ and let $V^\perp = \{v \in \mathbb{R}^s : v \perp V\}$, i.e. $V^\perp$ is the orthogonal complement of $V$ in $\mathbb{R}^s$. Since $0 \in V^\perp$ and $\dim V^\perp \geq s - n$, therefore one of the $n$-dimensional sides of the rectangular parallelepiped

$$
P = \prod_{i=1}^s [-\alpha_i, 1 - \alpha_i] \subset \mathbb{R}^s
$$

has a common point with $V^\perp$, i.e. there exists a $v^* = (v_1^*, \ldots, v_s^*)$ such that $v_i^*$ has value $1 - \alpha_i$ or $-\alpha_i$, except $n$ suffices $j_1, \ldots, j_n$ for which $-\alpha_{j_i} \leq v_{j_i}^* < 1 - \alpha_{j_i}$, $i = 1, \ldots, n$. Let $p_i (0 \leq p_i \leq 1)$ be defined by the equation

$$
v_i^* = p_i (1 - \alpha_i) + (1 - p_i) (-\alpha_i), \quad i = 1, \ldots, n.
$$

We define the vector $a^* = (a_1^*, \ldots, a_s^*)$ as follows: if $k \notin \{j_1, \ldots, j_n\}$, then put $a_k^* = 1$ or $0$ according as $v_k^* = 1 - \alpha_k$ or $-\alpha_k$; if $k = j_i$ then put $a_k^* = p_i$. Observe that

$$
\sum_{x_i \in E} a_i^* - \sum_{x_i \in E} \alpha_i = v^* c_E = 0 \quad \text{for all } E \in \mathcal{F}.
$$
Since $\sum_{x_i \in E} a_i^* - \sum_{x_i \in E} a_i = 0$ for all $E \in \mathcal{F}$, it suffices to give integers $a_{j_1}^*, \ldots, a_{j_n}^*$, where $a_{j_i}^* = 1$ or $0$, so that
\[ \left| \sum_{j=1}^n (a_{j_i}^* - a_{j_i}) c_{E,j_i} \right| < 2(2n \log 2n)^{1/2} \text{ for each } E \in \mathcal{F}. \tag{7} \]

(we remind the reader that $c_{E,i} = 1$ or $0$ according as $x_i \in E$ or not).

Indeed, Theorem 2 follows from (7), since choosing $a_{j_i}^* = a_i^*$ for $i \in \{j_1, \ldots, j_n\}$ we obtain
\[ \left| \sum_{x_i \in E} a_i^* - \sum_{x_i \in E} a_i \right| = \left| \left( \sum_{x_i \in E} a_i^* - \sum_{x_i \in E} a_i \right) + \sum_{i=1}^n (a_{j_i}^* - a_{j_i}^*) c_{E,j_i} \right| \]
\[ = \left| \sum_{i=1}^n (a_{j_i}^* - a_{j_i}^*) c_{E,j_i} \right| < 2(2n \log 2n)^{1/2} \text{ for each } E \in \mathcal{F}. \]

Let us return to (7). Since $a_{j_i}^* - a_{j_i}^*$ has value $1 - a_{j_i}^*$ or $-a_{j_i}^*$, it suffices to prove

**Lemma.** Let $p_1, \ldots, p_n, 0 \leq p_i \leq 1 (n \geq 5)$ be real numbers and let $C = [c_{ij}]$ be an $n$ by $n$ $0$-$1$-matrix, then there is a vector $w = (w_1, \ldots, w_n)$, $w_i = 1 - p_i$ or $-p_i$, such that
\[ \left| \sum_{j=1}^n w_j c_{ij} \right| < 2(2n \log 2n)^{1/2} \text{ for all } j = 1, \ldots, n. \]

Using the "probabilistic method" it is easy to prove the lemma, however, we shall give an algorithm for constructing $w$ (see [2, 3]).

**Proof.** We construct a vector $w = (w_1, \ldots, w_n)$ as follows. Having assigned values to $w_j, j \leq i - 1$ we assign to $w_i (i \geq 1)$ that value $1 - p_i$, $-p_i$ which minimizes the quantity
\[ Q_i = \sum_{k=1}^n \left\{ \prod_{j \leq i} (1 + \beta w_j c_{jk}) + \prod_{j \leq i} (1 - \beta w_j c_{jk}) \right\} \]
where $\beta$ will be fixed later. (Here we interpret the empty product as $1$). For notational convenience we let
\[ Q_i(y) = \sum_{k=1}^n \left\{ (1 + \beta y c_{ik}) \prod_{j \leq i} (1 + \beta w_j c_{jk}) + (1 - \beta y c_{ik}) \prod_{j \leq i} (1 - \beta w_j c_{jk}) \right\}. \]
\[ R_{i,k}^+ = \prod_{j \leq i} (1 \pm \beta w_j c_{jk}). \]

By our construction
\[ Q_i(1 - p_i) - Q_i - 1 = \beta (1 - p_i) \sum_{k=1}^n c_{ik} (R_{i-1,k}^+ - R_{i-1,k}^-), \]
\[ Q_i(-p_i) - Q_i - 1 = \beta (-p_i) \sum_{k=1}^n c_{ik} (R_{i-1,k}^+ - R_{i-1,k}^-). \]
Hence, by $Q_i = \min\{Q_i(l - p_i), Q_i(-p_i)\}$, we get

$$Q_i - Q_{i-1} \leq p_i\{Q_i(l - p_i) - Q_{i-1}\} + (1 - p_i)\{Q_i(-p_i) - Q_{i-1}\} = 0.$$ 

Since $Q_0 = 2n$, it follows that $Q_n \leq 2n$. Thus, for each $j$,

$$\prod_{i=1}^n (1 + \beta w_i c_{ij}) \leq 2n \quad (0 \leq \beta < 1).$$

Using the inequality $1 + x > \exp(x - 2x^2)$ if $|x| < \frac{1}{4}$, we get

$$2n \geq \prod_{i=1}^n (1 + \beta w_i c_{ij})$$

$$\geq \exp\left\{\beta \left(\sum_{i=1}^n w_i c_{ij}\right) - 2\beta^2 \left(\sum_{i=1}^n w_i^2 c_{ij}^2\right)\right\}$$

$$\geq \exp\left\{\beta \left(\sum_{i=1}^n w_i c_{ij}\right) - 2\beta^2 n\right\} \quad \text{if } \beta < \frac{1}{4}.$$ 

Hence

$$\sum_{i=1}^n w_i c_{ij} < 2n\beta + \frac{\log 2n}{\beta} \quad \text{if } \beta < \frac{1}{4}. \quad (8)$$

The right side of (8) is minimised by our value $\beta = (\log 2n/2n)^{1/2}$, and we get

$$\sum_{i=1}^n w_i c_{ij} < 2(2n \log 2n)^{1/2}$$

if $n \geq 5$. Similar argument shows that

$$\sum_{i=1}^n w_i c_{ij} > -2(2n \log 2n)^{1/2}.$$ 

thus for each $j$ we obtain

$$\left|\sum_{i=1}^n w_i c_{ij}\right| < 2(2n \log 2n)^{1/2} \quad \text{if } n \geq 5.$$ 

This proves the lemma, and thereby proves Theorem 2.

4. Proof of Theorem 3

Theorem 3 follows trivially from the following statement: Given $n$ ($n \geq 5$) finite sets $E_1, \ldots, E_n$, it is possible to partition their union into $r$ parts ($r \geq 2$) $A_1, \ldots, A_r$ in such a way that, for each $i, j$,

$$\left|E_i \cap A_j - \frac{|E_i|}{r}\right| < \left(12 - \frac{4}{r-1}\right)(2n \log 2n)^{1/2}.$$
We prove it by induction on $r$. For $r = 2$ the statement is valid by (2). Assume that $r > 2$, and the statement is true for every $2 \leq r' \leq r - 1$. It will be convenient to have the following brief notation

$$
\gamma_r(n) = \left(12 - \frac{4}{r-1}\right)(2n \log 2n)^{1/2}.
$$

Let $r = 2r'$ or $2r' + 1$ according as $r$ is even or odd, and let us associate $r'/r$ to every point of $E_1 \cup \cdots \cup E_n$. By Theorem 2, one can find two disjoint sets $A$ and $B$ so that $A \cup B = E_1 \cup \cdots \cup E_n$,

$$
\left|E_i \cap A\right| - \frac{r'}{r} \left|E_i\right| < 2(2n \log 2n)^{1/2},
$$

(1)

$$
\left|E_i \cap A\right| - \frac{r''}{r} \left|E_i\right| < 2(2n \log 2n)^{1/2},
$$

(2)

for all $i = 1, \ldots, n$. Since $\max\{r', r''\} \leq r - 1$, thus, by the induction hypothesis, one can partition the sets $A$ and $B$ into $r'$ and $r - r'$ parts $A_1, \ldots, A_r$ and $B_1, \ldots, B_{r-r'}$, respectively, so that

$$
\left|E_i \cap A_j\right| - \frac{r'}{r'} \left|E_i\right| \leq \gamma_{r'}(n) \quad \text{for all } i, j;
$$

(3)

$$
\left|E_i \cap A_j\right| - \frac{r''}{r-r'} \left|E_i\right| \leq \gamma_{r-r'}(n) \quad \text{for all } i, j.
$$

(4)

By (1), (2), (3) and (4) it suffices to check

$$
\max\left\{\gamma_{r'}(n) + \frac{2(2n \log 2n)^{1/2}}{r'}, \gamma_{r-r'}(n) + \frac{2(2n \log 2n)^{1/2}}{r-r'}\right\} \leq \gamma_r(n).
$$

(5)

Simple computation shows that (5) is true, which completes the proof of the statement.

References

