Characters and Super-Characters of Discrete Series Representations for Orthosymplectic Lie Superalgebras

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and a bracket product is defined as

\[[x, y] = xy - (-1)^{\ell(x)\ell(y)}yx\]

for homogeneous elements \(x, y \in \mathfrak{gl}(V)\).

In general a Lie superalgebra is a direct sum of two vector spaces indexed by \(\mathbb{Z}_2, g = g_0 \oplus g_1\), with bracket product \([, ,]\) which satisfies

1. \([g_i, g_j] \subset g_{i+j}\), i.e., \(g\) is a graded algebra.
2. For homogeneous \(x, y \in g\), it holds that \([x, y] = (-1)^{\ell(x)\ell(y)}[y, x]\).
3. For homogeneous \(x, y, z \in g\), super Jacobi identity holds:

\[\begin{align*}
(-1)^{\ell(x)\ell(z)}[x, [y, z]] + (-1)^{\ell(y)\ell(x)}[y, [z, x]]
+ (-1)^{\ell(z)\ell(y)}[z, [x, y]] &= 0.
\end{align*}\]

Note that \(g_0\) is a usual Lie algebra. We call a Lie superalgebra homomorphism of \(g\) into \(\mathfrak{gl}(W_c)\) a representation of \(g\) on a complex super-space \(W_c\).

Let \(\mathfrak{osp}(2n, M; \mathbb{R})\) be a sub Lie superalgebra of \(\mathfrak{gl}(V)\) which consists of the elements leaving \(\langle \cdot, \cdot \rangle\) invariant. Recall that we say \(x \in \mathfrak{gl}(V)\) leaves \(\langle \cdot, \cdot \rangle\) invariant if \(x\) is a linear combination of homogeneous elements \(y \in \mathfrak{gl}(V)\) which satisfies

\[\langle yv, w \rangle + (-1)^{\ell(y)\ell(v)}\langle v,yw \rangle = 0\]  for homogeneous \(v, w \in V\).

The algebra \(\mathfrak{osp}(2n, M; \mathbb{R})\) is called an orthosymplectic algebra and has attracted much attention from mathematicians and physicists recently (for example, see [3]). As a mathematical object, \(\mathfrak{osp}(2n, M; \mathbb{R})\) is a simple Lie superalgebra ([8], in that paper V. Kac classified all the finite dimensional simple Lie superalgebras) and it is fundamental to study its representations.

In [11], we constructed oscillator representations for \(\mathfrak{osp}(2n, M; \mathbb{R})\), using Heisenberg Lie superalgebras. It is super unitary (see Definition 1.1 for a precise definition) and has very important properties. In fact, "almost all" the irreducible super unitary representations of \(\mathfrak{osp}(2n, M; \mathbb{R})\) can be realized in some oscillator representation of a larger orthosymplectic algebra \(\mathfrak{osp}(2nL, ML; \mathbb{R})\) [12]. In low rank cases, for instance for \(\mathfrak{osp}(2,1; \mathbb{R})\) or \(\mathfrak{osp}(2,2; \mathbb{R})\), we obtained all the irreducible super unitary representations in this manner [10; 12, Theorem 3.5].

In the present paper, we study unitary representations of \(\mathfrak{osp}(2n, 2m; \mathbb{R})\) in several directions. First, we try to realize super unitary representations
as induced representations from a parabolic subalgebra. This type of induction is popular and studied by many people (for example, see [9, Sect. 2; 4, Sect. 2.4]). Although all the unitary representations are not treated here, "generic" representations can be realized as induced representations from a parabolic with reductive part compact (Theorem 3.3). We call these representations discrete series for \( \mathfrak{osp}(2n, 2m; \mathbb{R}) \). The name "discrete series" comes from several reasons. Among those, three major reasons are

(i) Eigenvalues of Laplace–Casimir operators are discretely distributed.

(ii) They are super unitary lowest weight modules.

(iii) They have character formulas almost the same as those of holomorphic discrete series for \( \mathfrak{sp}(2n; \mathbb{R}) \times \mathfrak{so}(2m) \).

Second, results concerned with (iii) in the above are discussed. We calculate out characters and super-characters of these discrete series representations (Propositions 4.2 and 4.3). These characters are expressed like Weyl's character formula, i.e., using Weyl groups and (super) denominators. We think that these formulas are interesting from the combinatorial point of view.

Third, decompositions of discrete series as a representation of the even part, which is a usual Lie algebra, are discussed. We use a character formula to prove a general decomposition theorem (Theorem 6.1).

Let us explain each section briefly. After preparing notations and definitions in Section 1, we summarize results on Laplace–Casimir operators of \( \mathfrak{osp}(2n, 2m; \mathbb{R}) \) in Section 2 after F. A. Berezin. In Section 3, we define discrete series for \( \mathfrak{osp}(2n, 2m; \mathbb{R}) \) as induced representations from a parabolic subalgebra (Definition 3.1). Using the theory of Laplace–Casimir operators, it can be proved that generically discrete series are irreducible super unitary representations (Theorem 3.3).

In Section 4, we develop the theory of characters and super-characters. In Section 4.1, (super-)characters of discrete series are given (Propositions 4.2 and 4.3), which reveals a strong resemblance with characters of holomorphic discrete series of a semisimple Lie group of Hermitian symmetric type. After that, in Section 4.2, we also give (super-)characters of oscillator representations (Propositions 4.2 and 4.3). Section 5 is devoted to the case of \( \mathfrak{osp}(2, 2; \mathbb{R}) \) and we get a complete list of characters of irreducible super unitary representations.

The last section, Section 6, deals with the decomposition of a discrete series as a representation of the even part, which is isomorphic to \( \mathfrak{sp}(2n, \mathbb{R}) \times \mathfrak{so}(2m) \). A general decomposition theorem (Theorem 6.1) and explicit decompositions (Corollaries 6.2 and 6.3) are obtained.
1. **Preliminaries**

1.1. **Super Unitary Representation**

Let \( g = g_0 \oplus g_1 \) be a Lie superalgebra over \( \mathbb{R} \).

**Definition 1.1.** We call a representation \((\rho, F)\) of \( g \) super unitary (or simply unitary in this paper), if the superspace \( F = F_0 \oplus F_1 \) admits a non-degenerate super Hermitian form \((\cdot, \cdot)\) which satisfies

1. The form \((\cdot, \cdot)\) is homogeneous of degree zero, i.e., \((a, b) \neq 0\) only if \( \varepsilon(a) + \varepsilon(b) = 0 \) (\( a, b \in F \), homogeneous), where \( \varepsilon(a) \) denotes the degree of \( a \).

2. The operators \( \{\rho(X) | X \in g\} \) leave \((\cdot, \cdot)\) invariant:

\[
(\rho(X) v, w) + (-1)^{\varepsilon(X)\varepsilon(v)} (v, \rho(X) w) = 0
\]

for homogeneous \( X \in g, v \) and \( w \in F \). Here we call \((\cdot, \cdot)\) the super Hermitian form when \((a, b) = (-1)^{\varepsilon(a)\varepsilon(b)} (b, a)\) holds.

From now on we do not declare that elements \( X \in g, a, b \in F \), and so on are homogeneous, but it should be understood that elements are homogeneous whenever their degrees appear. Of course non-homogeneous elements are said to have some property if their homogeneous parts have that property.

Assume that \( g \) is a classical simple Lie superalgebra over \( \mathbb{R} \) (for a definition, see [8, Sect. 2]). Then \( g \) is a real form of a complex simple Lie superalgebra \( g_c = g \otimes_{\mathbb{R}} \mathbb{C} \). In this case, the even part \((g_0)_c\) of 
\( g_c \) is always reductive (loc. cit.).

Let \( f \) be a maximal compact subalgebra of \( g_0 \). We always assume that \( f \) contains the center of \( g_0 \).

**Definition 1.2.** A representation \((\rho, F)\) of \( g \) is called admissible, if the representation \((\rho |_{f}, F)\) of \( f \) is completely reducible and its irreducible components are all finite dimensional with multiplicities finite.

1.2. **Orthosymplectic Algebras**

Let \( V = V_0 \oplus V_1 \) be a super space over \( \mathbb{R} \) (or \( \mathbb{C} \)). We call a bilinear form \( b \) on \( V \) super skew symmetric if \( b \) satisfies

\[
b(v, w) = -(-1)^{\varepsilon(v)\varepsilon(w)} b(w, v) \quad (v, w \in V: \text{homogeneous}),
\]
where \( \varepsilon (v) \) denotes the degree of \( v \in V \). For a super skew symmetric form \( b \) on \( V \), we denote by \( \mathfrak{osp}(b; V) \) an orthosymplectic algebra in \( \mathfrak{gl}(V) \):

\[
\mathfrak{osp}(b; V) = \{ X \in \mathfrak{gl}(V) | X \text{ leaves } b \text{ invariant} \}.
\]

Remark that the meaning of "invariance" in the above expression must be considered in the sense of superalgebras:

\[
b(Xv, w) + (-1)^{\varepsilon(X)\varepsilon(v)} b(v, Xw) = 0 \quad \text{for } X \in \mathfrak{gl}(V), v \text{ and } w \in V, \tag{2}
\]

where \( \varepsilon(X) \) denotes the degree of a linear transformation \( X \).

Suppose that \( b \) is non-degenerate and homogeneous of degree zero. Then \( \dim V_0 \) is necessarily even and we put \( \dim V_0 = 2n, \dim V_1 = M \). Moreover we assume that, after some normalization of the basis in \( V \), \( b \) is expressed by a matrix of the form

\[
B = \begin{bmatrix}
0 & 1_n & 0 \\
-1_n & 0 & 0 \\
0 & 0 & 1_M
\end{bmatrix}.
\]

We write this algebra \( \mathfrak{osp}(b; V) = \mathfrak{osp}(2n, M; \mathbb{R}) \) (or \( \mathfrak{osp}(2n, M; \mathbb{C}) \) if \( V \) is a vector space over \( \mathbb{C} \)). Hereafter we call \( \mathfrak{osp}(2n, M; \mathbb{R}) \) an orthosymplectic algebra (over \( \mathbb{R} \)). It is known that only the real form \( \mathfrak{osp}(2n, M; \mathbb{R}) \) of \( \mathfrak{osp}(2n, M; \mathbb{C}) \) has non-trivial admissible super unitary representations (see [12]). Moreover we have

**Proposition 1.3.** Irreducible admissible super unitary representations of \( \mathfrak{osp}(2n, M; \mathbb{R}) \) are lowest weight modules (respectively highest weight modules) if the associated constant is \(-1\) (respectively 1).

## 2. Laplace–Casimir Operators for Orthosymplectic Algebras

From now on we put \( M = 2m \) and treat only the orthosymplectic algebras \( g = \mathfrak{osp}(2n, 2m; \mathbb{R}) \). Let \( U(g_C) \) be the enveloping algebra of \( g_C \). We call elements of the center of \( U(g_C) \) Laplace–Casimir operators (or simply Laplace operators). In this section, we collect results on the center \( Z \) of \( U(g_C) \) after F. A. Berezin and V. Kac [1, 9, 7].

Let \( S(g_C) \) be a symmetric superalgebra of \( g_C \). That is, \( S(g_C) \) is a quotient algebra of the tensor algebra \( T(g_C) \) with respect to a two-sided ideal generated by the elements of the form

\[
x \otimes y - (-1)^{\varepsilon(x)\varepsilon(y)} y \otimes x \quad (x, y \in g_C).
\]
One can easily see that $S(g_c) \simeq S((g_0)_c) \otimes \wedge (g_1)_c$, where $S((g_0)_c)$ is a usual symmetric algebra of $(g_0)_c$ and $\wedge (g_1)_c$ is an exterior algebra. Since the Killing form of $g_c$ is nondegenerate, we can identify $S(g_c)$ with the space of super symmetric polynomial functions on $g_c$. Let $I(g_c)$ be a homogeneous ideal of $S(g_c)$ consisting of all the invariant elements with respect to the adjoint action of $g_c$. We define super symmetrization map of $S(g_c)$ to $U(g_c)$ by

$$
\delta(x_1, x_2, \ldots, x_k) = \frac{1}{k!} \sum_{i=0}^{k} \epsilon_i x_{i_1} x_{i_2} \cdots x_{i_k} \quad (x_1, x_2, \ldots, x_k \in g),
$$

where $\epsilon_i$ is a sign $\pm 1$ determined by the number of transposition of odd elements in $x_1, x_2, \ldots, x_k$.

**Lemma 2.1.** Under the super symmetrization map $\delta$, $S(g_c)$ and $U(g_c)$ are isomorphic as super spaces. Moreover, we have $\delta(I(g_c)) = 3$.

**Proof.** This is similar to the Lie algebra case. Q.E.D.

Let $h_c$ be a Cartan subalgebra of $g_c$. Here a Cartan subalgebra $h_c$ of $g_c$ means that of the even part $(g_0)_c$. We call $\alpha \in h_c^*$ a root of $(g_c, h_c)$ if $\alpha$ is a non-zero simultaneous eigenvalue of $\text{ad} h_c$. Let $\Delta$ be the roots of $(g_c, h_c)$. If a non-zero eigenvector $X_\alpha$ of $\alpha$, which we call a root vector, is contained in $(g_0)_c$ (respectively $(g_1)_c$), then $\alpha$ is called even (respectively odd). Let $\Delta_0$ (respectively $\Delta_1$) be the collection of all the even roots (respectively odd roots). In the case of orthosymplectic algebras, we can define the notion of positivity (see Section 3 and, for example, [8]). We denote positive roots by $\Delta^+$ and put $\Delta^+_i = \Delta^+ \cap \Delta_i (i = 0, 1)$.

The restriction map $j : S(g_c) \to S(h_c)$ brings $I(g_c)$ into $S(h_c^*)^W$, where $W$ is the Weyl group of $(g_0)_c \simeq \text{sp}(2m; \mathbb{C}) \times \text{so}(2m; \mathbb{C})$ and $S(h_c^*)^W$ denotes the subspace of $W$-invariant polynomials in $S(h_c)$. In general it holds that $j(I(g_c)) \subseteq S(h_c^*)^W$. Moreover we have

**Proposition 2.2** [1, p. 296, Corollary]. Put $R = \prod \beta (\beta \in \Delta^+_i)$. Then we have $R \cdot S(h_c^*)^W \subseteq j(I(g_c))$. As a consequence, we get $S(h_c^*)^W [1/R] = j(I(g_c)) [1/R]$.

Let $g^+$ (respectively $g^-$) be a direct sum of the positive (respectively negative) root spaces. Then $g_c$ decomposes as $g_c = g^+ \oplus h_c \oplus g^-$. By the Poincaré–Birkhoff–Witt theorem, it is known that

$$
3 = g^+ U(g_c) g^- \oplus U(h_c).
$$
We denote by $p$ a projection $p : \mathfrak{z} \to U(\mathfrak{h}_c)$ along this decomposition. Put

$$\rho = \rho_0 - \rho_1; \quad \rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha, \quad \rho_1 = \frac{1}{2} \sum_{\beta \in \Delta_1^+} \beta.$$ 

Successive application of $p$ and translation by $\rho$ is denoted by $\tilde{p}$, i.e., $\tilde{p}(z)(\lambda) = p(z)(\lambda + \rho)$ ($\lambda \in \mathfrak{h}_c^*$), where we identify an element of $U(\mathfrak{h}_c) = S(\mathfrak{h}_c)$ with a polynomial on $\mathfrak{h}_c^*$. Then $\tilde{p} : \mathfrak{z} \to S(\mathfrak{h}_c^*)^W$ is an algebra homomorphism (see, for example, [1, p. 302, Theorem 3.2]). For semisimple Lie algebras, it is well-known that $\tilde{p}(\mathfrak{z}) = j(I(\mathfrak{g}_c)) = S(\mathfrak{h}_c^*)^W$. However, for simple Lie superalgebras, this equality fails to hold. In the case of $\mathfrak{g}_c = \mathfrak{osp}(2n, 2m; \mathbb{C})$, Berezin proved

**Proposition 2.3** [1, p. 333, Theorem 4.5]. Let $\mathfrak{g}_c = \mathfrak{osp}(2n, 2m; \mathbb{C})$. Then there exists a subalgebra $CS$ of $\tilde{p}(\mathfrak{z}) \cap j(I(\mathfrak{g}_c))$ such that

1. The element $R$ is contained in $CS$.
2. Any element $P \in S(\mathfrak{h}_c^*)^W$ can be expressed as $P = Q/R^k$ for some $Q \in CS$ and $k \in \mathbb{Z}_+$.

**Remark 1.** In [1, Chap. II-4], $CS$ is explicitly given.

**Remark 2.** In [7] it is declared that $\tilde{p} \circ \delta = j$, hence $\tilde{p}(\mathfrak{z}) = j(I(\mathfrak{g}_c))$. However, the equality $\tilde{p} \circ \delta = j$ is not valid even in semisimple Lie algebra cases as shown in [6, Sect. 23.3].

### 3. Discrete Series for $\mathfrak{osp}(2n, 2m; \mathbb{R})$

We fix a compact Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{osp}(2n, 2m; \mathbb{R})$:

$$\mathfrak{h} = \begin{cases} h = \begin{bmatrix} 0 & A & 0 \\ -A & 0 & 0 \\ 0 & 0 & B \end{bmatrix} & A = \text{diag}(a_1, a_2, \ldots, a_n), \\ B = \text{diag}(b_1 u, b_2 u, \ldots, b_m u), a_i, b_j \in \mathbb{R}, u = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \end{cases} \quad (3)$$

We define $e_i \in \mathfrak{h}_c^*$ ($1 \leq i \leq n$) and $f_j \in \mathfrak{h}_c^*$ ($1 \leq j \leq m$) by putting

$$e_i(h) = \sqrt{-1} a_i, \quad f_j(h) = \sqrt{-1} b_j.$$
for \( h \in \mathfrak{h} \) of the form in (3). Then roots are given as

\[
\Delta^+_C = \{ e_i - e_j | 1 \leq i < j \leq n \} \cup \{ f_i + f_j | 1 \leq i < j \leq m \}
\]

: the set of positive compact roots,

\[
\Delta^+_n = \{ e_i + e_j | 1 \leq i < j \leq n \}
\]

: the set of positive non-compact roots,

\[
\Delta^+_e = \Delta^+_C \cup \Delta^+_n
\]

: the set of positive even roots,

\[
\Delta^+_o = \{ e_i \pm f_j | 1 \leq i \leq n, 1 \leq j \leq m \}
\]

: the set of positive odd roots.

Denote by \( \mathfrak{f} \) a subalgebra of \( g \) corresponding to \( \Delta_C \), a maximal compact subalgebra of \( g_0 \). Let \( g(\pm 1) \) be a direct sum of root spaces corresponding to \( \Delta^+_1 \) and \( g(\pm 2) \) a direct sum of those corresponding to \( \Delta^+_n \). Then \( g_C \) has a natural \( \mathbb{Z} \)-graded algebra structure

\[
g_C = g(-2) \oplus g(-1) \oplus g(0) \oplus g(1) \oplus g(2),
\]

with \( g(0) = \mathfrak{f}_C \). We put \( q = g(-2) \oplus g(-1) \oplus \mathfrak{f}_C \), a parabolic subalgebra with reductive part \( \mathfrak{f}_C \).

If \( \lambda \in \mathfrak{h}_C^* \) is of the form

\[
\lambda = \sum_{1 \leq i \leq n} \lambda_i e_i + \sum_{1 \leq i \leq m} \mu_i f_i,
\]

then we write \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n; \mu_1, \mu_2, \ldots, \mu_m) \) and call it a coordinate expression of \( \lambda \). Now we can state Proposition 1.3 more precisely.

**Proposition 3.1.** Let the associated constant be \(-1\). Then an irreducible admissible super unitary representation of \( \mathfrak{osp}(2n, 2m; \mathbb{R}) \) is a lowest weight module with the lowest weight

\[
\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n; \mu_1, \mu_2, \ldots, \mu_m)
\]

which satisfies

\[
0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n; \mu_1 \leq \mu_2 \leq \ldots \leq -|\mu_m|; |\mu_1| \leq \lambda_1
\]

and

\[
\lambda_i - \lambda_j \in \mathbb{Z}; \quad \mu_i \pm \mu_j \in \mathbb{Z} (i \neq j).
\]

**Remark 1.** For \( \mathfrak{osp}(2n, 2m + 1; \mathbb{R}) \), the lowest weight \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n; \mu_1, \mu_2, \ldots, \mu_m) \) of a super unitary representation satisfies

\[
0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n; \mu_1 \leq \mu_2 \leq \ldots \leq \mu_m \leq 0; |\mu_1| \leq \lambda_1,
\]
Remark 2. It is possible that there is no super unitary representation for a given lowest weight $\lambda$ satisfying the above conditions. However, in almost all the cases, an integral $\lambda$ of the above form gives a super unitary representation. For example, if $\lambda_i \geq n$, then $\lambda$ is the lowest weight of a super unitary representation (see Theorem 3.3 and [12, Theorem 3.5]).

Take a lowest weight $\lambda \in \mathfrak{h}_c^*$ for a (limit of) holomorphic discrete series representation of $Sp(2n, \mathbb{R}) \times SO(2m)$. Moreover we assume that $\lambda$ satisfies the condition in Proposition 3.1. Then necessarily we have

(a) $\lambda_i (1 \leq i \leq n)$ and $\mu_j (1 \leq j \leq m)$ are all integers.

(b) It holds that $n \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n; -\lambda_1 \leq \mu_1 \leq \mu_2 \leq \cdots \leq -|\mu_m|.$

Since $\lambda$ is also the lowest weight for a finite dimensional representation of $\mathfrak{q}_c$, we denote it by $\tau(\lambda)$. Extend $\tau(\lambda)$ trivially on $\mathfrak{g}(-2) \oplus \mathfrak{g}(-1)$ and get a representation of $\mathfrak{q}$ denoted by the same symbol $\tau(\lambda)$.

Definition 3.2. Put

$$D(\lambda) = \text{Ind}_q^\mathfrak{g} \tau(\lambda)$$

and call it a discrete series for $\mathfrak{osp}(2n, 2m; \mathbb{R})$ with parameter $\lambda$.

The name "discrete series" will be justified in the sequel.

At first, we study the irreducibility of $D(\lambda)$.

Theorem 3.3. If $R(\lambda - \rho) \neq 0$, then $D(\lambda)$ is an irreducible admissible super unitary representation of $\mathfrak{osp}(2n, 2m; \mathbb{R})$. Its associated constant is $\delta = -1$.

Proof. Admissibility is obvious by standard arguments. Super unitarity follows from [12, Theorem 3.5] once the irreducibility of $D(\lambda)$ is established. So here it is enough to prove the irreducibility. Let $N \subseteq D(\lambda)$ be a non-trivial subrepresentation. Then $N$ has a lowest weight vector $v$. Denote its weight by $v$. Since a Laplace operator $z \in \mathfrak{z}$ acts on $D(\lambda)$ as a scalar multiplication by $p(z)(\lambda)$, it holds that $zv = p(z)(\lambda) v$. On the other hand, $z$ acts as a scalar $p(z)(v)$, on a lowest weight vector with weight $v$. So we get

$$p(z)(\lambda) = p(z)(v) \quad (z \in \mathfrak{z}).$$
By Proposition 2.3, we conclude that for any \( f \in \mathfrak{S} \), \( f(\lambda - \rho) = f(\nu - \rho) \) holds. Note that \( R(\lambda - \rho) = R(\nu - \rho) \neq 0 \) in particular. By the same proposition, any \( g \in \mathfrak{S}(\mathfrak{h}_\mathfrak{c})^W \) can be expressed as \( g = f/R^k \) for some \( f \in \mathfrak{S} \). So we have \( g(\lambda - \rho) = g(\nu - \rho) \) for any \( g \in \mathfrak{S}(\mathfrak{h}_\mathfrak{c})^W \), which in turn means

\[
\lambda - \rho = w(\nu - \rho)
\]  

(4)

for some \( w \in W \). Let us prove that \( \lambda = \nu \), using (4). Then it is a contradiction that \( N \) is a proper subrepresentation and we prove the proposition.

Let \( \nu = (\nu_1, \nu_2, \ldots, \nu_n; \omega_1, \omega_2, \ldots, \omega_m) \) be a coordinate expression of \( \nu \). Since \( \nu \) is a lowest weight vector of an admissible representation, \( \nu \) is a lowest weight for a finite dimensional representation of \( \mathfrak{h}_\mathfrak{c} \). Therefore \( \nu \) satisfies

\[
n \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_n; \quad \omega_1 \leq \omega_2 \leq \cdots \leq -|\omega_m|.
\]

Coordinate expressions of \( \lambda - \rho \) and \( \nu - \rho \) become

\[
\lambda - \rho = (\lambda_1 - n + m, \lambda_2 - n + m + 1, \ldots, \lambda_n + m - 1; \\
\mu_1 - m + 1, \mu_2 - m + 2, \ldots, \mu_m),
\]

\[
\nu - \rho = (\nu_1 - n + m, \nu_2 - n + m + 1, \ldots, \nu_n + m - 1; \\
\omega_1 - m + 1, \omega_2 - m + 2, \ldots, \omega_m).
\]

From this we see that

\[
0 < \lambda_1 - n + m < \lambda_2 - n + m + 1 < \cdots < \lambda_n + m - 1,
\]

\[
0 < \nu_1 - n + m < \nu_2 - n + m + 1 < \cdots < \nu_n + m - 1,
\]

hence \( w = 1 \) on symplectic part and \( \lambda_i = \nu_i \) (1 \( \leq i \leq n \)). Note that \( \nu \) is expressed on \( \mathfrak{h} \) as

\[
\nu = \lambda + \sum_{i < j} s_{ij}(e_i - e_j) + \sum_{i \leq j} t_{ij}(e_i + e_j)
\]

\[
+ \sum_{i,j} u_{ij}(e_i - f_j) + \sum_{i,j} v_{ij}(e_i + f_j),
\]

where \( s_{ij}, t_{ij}, u_{ij}, v_{ij} \) are all non-negative integers. Since \( \lambda_i = \nu_i \) (1 \( \leq i \leq n \)), we can conclude all \( s_{ij}, t_{ij}, u_{ij}, v_{ij} \) are zero. Now we have \( \lambda = \nu \). \( \text{Q.E.D.} \)

Remark. As we mentioned in Proposition 1.3, a super unitary representation is a lowest weight module if the associated constant \( \delta \) is \(-1\). For \( \delta = 1 \), we can define discrete series representations in a similar manner,
using a parabolic $q' = g(0) \oplus g(1) \oplus g(2)$ instead of $q$. They are highest weight representations. All the results stated in this article are valid for those highest weight modules after appropriate modifications of statements.

4. Characters and Super-Characters of Admissible Super Unitary Representations

First of all, we define a character and a super-character of an admissible super unitary representation of $g$.

**Definition 4.1.** Let $(\omega, F)$ be an admissible super unitary representation of $g$. For an element $h$ of a Cartan subalgebra $\mathfrak{h}$, put

$$Ch(F)(h) = Ch((\omega, F))(h) = \text{trace}\{\exp \omega(h)\}$$

and

$$s-Ch(F)(h) = s-Ch((\omega, F))(h) = s-\text{trace}\{\exp \omega(h)\},$$

where $s$-trace means supertrace. We call $Ch(F)$ the character of $(\omega, F)$, and $s-Ch(F)$ the super-character.

4.1. Discrete Series Representations

Characters and super-characters are formal power series on $\mathfrak{h}$, and analytic on negative Weyl chambers if $\omega$ is a lowest weight representation. Let us calculate the characters and super characters of discrete series representations.

By the definition, we have

$$D(\lambda) = \text{Ind}_{\mathfrak{g}}^{g} \tau(\lambda)$$

$$= U(g(2) \oplus g(1)) \otimes_{C} \tau(\lambda)$$

$$\approx S(g(2)) \otimes_{C} \bigwedge g(1) \otimes_{C} \tau(\lambda).$$

Therefore it becomes

$$Ch(D(\lambda)) = Ch(\tau(\lambda)) \cdot Ch(U(\mathfrak{g}(1) \oplus g(2))).$$

By Weyl's classical character formula, the character of $\tau(\lambda)$ is given by

$$Ch(\tau(\lambda)) = \sum_{w \in W(\mathfrak{a}_{c})} \det(w) \exp w(\lambda - \rho_{c}) \frac{\exp(-\rho_{c}) \prod_{\alpha \in \Delta^{+}_{c}} (1 - \exp \alpha)}{\exp(-\rho_{c}) \prod_{\alpha \in \Delta^{+}_{c}} (1 - \exp \alpha)}.$$
On the other hand, we have

\[ \text{Ch}(U(g(2) \oplus g(1))) = \text{Ch}(g(2)) \cdot \text{Ch}\left( \bigwedge^1 g(1) \right) = \frac{1}{\prod_{x \in \Delta^*_+} (1 - \exp x)} \cdot \prod_{\beta \in \Delta^*_+} (1 + \exp \beta), \]

hence

\[ \text{Ch}(D(\lambda)) = \frac{\sum_{w \in W(\Delta_c)} \det(w) \exp w(\lambda - \rho_c)}{\exp(-\rho_c) \prod_{x \in \Delta^*_+} (1 - \exp x)} \cdot \prod_{\beta \in \Delta^*_+} (1 + \exp \beta). \]

Remark that \( \rho - \rho_c = ((n+1)/2 - m, (n+1)/2 - m, (n+1)/2 - m; 0, 0, ..., 0) \) is invariant under the compact Weyl group \( W(\Delta_c) \). Now we proved

**Proposition 4.2.** The character of a discrete series representation \( D(\lambda) \) is given as

\[ \text{Ch}(D(\lambda)) = \frac{\sum_{w \in W(\Delta_c)} \det(w) \exp w(\lambda - \rho)}{\exp(-\rho) \prod_{x \in \Delta^*_+} (1 - \exp x)} \cdot \prod_{\beta \in \Delta^*_+} (1 + \exp \beta). \]

Similarly we can calculate the super-character of \( D(\lambda) \).

**Proposition 4.3.** The super-character of a discrete series representation \( D(\lambda) \) is given as

\[ \text{s-Ch}(D(\lambda)) = \frac{\sum_{w \in W(\Delta_c)} \det(w) \exp w(\lambda - \rho)}{\exp(-\rho) \prod_{x \in \Delta^*_+} (1 - \exp x)} \cdot \prod_{\beta \in \Delta^*_+} (1 - \exp \beta). \]

**Definition 4.4.** We call

\[ \exp(-\rho) \prod_{x \in \Delta^*_+} (1 - \exp x) \cdot \prod_{\beta \in \Delta^*_+} (1 + \exp \beta)^{-1} \]

the Weyl denominator and

\[ \exp(-\rho) \prod_{x \in \Delta^*_+} (1 - \exp x) \cdot \prod_{\beta \in \Delta^*_+} (1 - \exp \beta)^{-1} \]

the Weyl super denominator.

The algebraic construction of holomorphic discrete series representations for semisimple Lie groups of Hermitian symmetric type is studied very well by many mathematicians. First of all is Harish-Chandra [5] and a very concrete presentation is given by Varadarajan [14, Proposition 2.3.5]. Since those works are archetypes for our definition of \( D(\lambda) \), there is a
strong resemblance between characters of (holomorphic) discrete series of 
Lie algebras and Lie superalgebras. Namely they differ only on the 
appearance of the Weyl denominator.

Now we can justify the term "discrete series for orthosymplectic algebra." 
They have those distinct properties which are

(a) They are parametrized by lowest weights in a positive part of the 
integral lattice of $\mathfrak{h}_C$. Eigenvalues of Laplace operators are discretely 
distributed for this series of representations.

(b) They are irreducible at generic parameters.

(c) They are unitarizable generically. At least their irreducible 
quotients are unitarizable.

(d) Similar character formulas hold for discrete series of $\mathfrak{osp}$ and 
holomorphic discrete series of $\mathfrak{osp}_0 \simeq \mathfrak{sp} \times \mathfrak{so}$.

(e) Restriction of $D(\lambda)$ to the even part $\mathfrak{osp}_0$ decomposes into (limits 
of) holomorphic discrete series representations for $\mathfrak{osp}_0$. For this, see 
Theorem 6.1.

(f) The induced module of $\tau(\lambda)$ from $\mathfrak{g}(0) + \mathfrak{g}(-2)$ to $\mathfrak{osp}_0$ is a (limit 
of) holomorphic discrete series representation for $\mathfrak{osp}_0$.

4.2. Oscillator Representations

In this subsection, we calculate the character and the super-character 
of the oscillator representation, which is a unitary representation of 
orthosymplectic algebra but does not belong to the discrete series. For 
definitions and detailed properties, see [11].

Let $\mathcal{S}$ be the oscillator representation of $\mathfrak{osp}(2n, 2m; \mathbb{R})$. Then $\mathcal{S}$ can be 
realized as follows. Its representation space is

$$F = \mathbb{C}[z_i | 1 \leq i \leq n] \otimes \mathfrak{S}(r_j | 1 \leq j \leq m),$$

where $\mathbb{C}[z_i | 1 \leq i \leq n]$ is a polynomial ring of variables $\{z_i | 1 \leq i \leq n\}$ and 
$\mathfrak{S}(r_j | 1 \leq j \leq m)$ is a Clifford algebra generated by $\{r_j | 1 \leq j \leq m\}$ with relations

$$r_j^2 = 1, \quad r_i r_j + r_j r_i = 0 \quad (i \neq j).$$

An element $h \in \mathfrak{h}$ in (3) acts on monomials in $F$ as

$$\mathcal{S}(h) z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n} r_1^{l_1} r_2^{l_2} \cdots r_m^{l_m}$$

$$= \left\{ \sqrt{-1} \sum_{i=1}^{n} a_i \left( z_i \frac{\partial}{\partial z_i} + \frac{1}{2} \right) + \sqrt{-1} \sum_{j=1}^{m} b_j s_j \right\}$$

$$\times z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n} r_1^{l_1} r_2^{l_2} \cdots r_m^{l_m}$$
\( \alpha \) is an automorphism of \( \mathfrak{g}(r_j \mid 1 \leq j \leq m) \) which sends \( r_i \) to \( r_i \) \((i \neq j)\) and \( r_j \) to \(-r_j)\),

\[
\left\{ \sqrt{-1} \sum_{i=1}^{n} a_i \left(k_i + \frac{1}{2}\right) + \frac{\sqrt{-1}}{2} \sum_{j=1}^{m} b_j (-1)^j \right\} \\
\times z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n} r_1^{l_1} r_2^{l_2} \ldots r_m^{l_m}.
\]

From this, using the notations \( \{e_i, f_i\} \subseteq h^\ast \) in Section 3, we have

\[
\text{Ch}(\Xi) = \prod_{i=1}^{n} (1 - \exp e_i)^{-1} \exp \frac{1}{2} e_i \prod_{i=1}^{m} \left( \exp \frac{1}{2} f_i + \exp(-\frac{1}{2} f_i) \right)
\]

and

\[
\text{s-Ch}(\Xi) = \prod_{i=1}^{n} (1 - \exp e_i)^{-1} \exp \frac{1}{2} e_i \prod_{i=1}^{m} \left( \exp \frac{1}{2} f_i - \exp(-\frac{1}{2} f_i) \right).
\]

However, \( \Xi \) is not irreducible but has two irreducible components \( \Xi^+ \) and \( \Xi^- \). Their representation spaces are given by

\[
F^+ = C[z_i \mid 1 \leq i \leq n]^+ \otimes \mathfrak{g}(r_j \mid 1 \leq j \leq m)^+ \\
+ C[z_i \mid 1 \leq i \leq n]^+ \otimes \mathfrak{g}(r_j \mid 1 \leq j \leq m)^-,
\]

\[
F^- = C[z_i \mid 1 \leq i \leq n]^+ \otimes \mathfrak{g}(r_j \mid 1 \leq j \leq m)^- \\
+ C[z_i \mid 1 \leq i \leq n]^+ \otimes \mathfrak{g}(r_j \mid 1 \leq j \leq m)^+,
\]

where \( C[z_i \mid 1 \leq i \leq n]^+ \) (respectively \( \mathfrak{g}(r_j \mid 1 \leq j \leq m)^+ \)) is a subspace of \( C[z_i \mid 1 \leq i \leq n] \) (respectively \( \mathfrak{g}(r_j \mid 1 \leq j \leq m) \)) generated by monomials of even degree. Of course \( C[z_i \mid 1 \leq i \leq n]^+ \) (respectively \( \mathfrak{g}(r_j \mid 1 \leq j \leq m)^- \)) is a subspace generated by monomials of odd degree. Put

\[
\Theta^\pm = \text{Ch}(C[z_i \mid 1 \leq i \leq n]^\pm)
\]

and

\[
\Psi^\pm = \text{Ch}(\mathfrak{g}(r_j \mid 1 \leq j \leq m)^\pm).
\]

Then we have \( \text{Ch}(\Xi^+) = \Theta^+ \Psi^+ + \Theta^- \Psi^- \) and \( \text{Ch}(\Xi^-) = \Theta^+ \Psi^- + \Theta^- \Psi^+ \). Let us calculate \( \text{Ch}(\Xi^+) \), for example. Notice that

\[
2\Theta^+ = \exp \nu \cdot \left\{ \prod_{i=1}^{n} (1 - \exp e_i)^{-1} + \prod_{i=1}^{n} (1 + \exp e_i)^{-1} \right\}
\]

\[
= \exp \nu \cdot \frac{\prod_{i=1}^{n} (1 - \exp e_i) + \prod_{i=1}^{n} (1 + \exp e_i)}{\prod_{i=1}^{n} (1 - \exp 2e_i)}, \tag{5}
\]
where \( v = \sum_{1 \leq i \leq n} e_i/2 \). We put \( D_n^+ = \{ e_i \pm e_j \mid i < j \} \) and consider it as a positive system of a root system of type \( D \). Similarly we put \( C_n^+ = D_n^+ \cup \{ 2e_i \mid 1 \leq i \leq n \} \). Then by Weyl's character formula it holds that

\[
\prod_{\alpha \in D_n^+} (1 - \exp \alpha) = \sum_{w \in W(D_n)} \det w \exp(\rho(D_n) - w\rho(D_n)).
\] (6)

Here \( W(D_n) \) is a Weyl group of \( D_n \) and \( \rho(D_n) = \sum_{i=1}^{n} (n-i) e_i \). For the time being, we use a tentative notation

\[
\Omega(A) = \prod_{\alpha \in A} (1 - \exp \alpha)
\]

for a subset \( A \) of \( h_\mathbb{C}^* \). Multiply (5) by the formula (6) and we see

\[
\Omega(C_n^+ \Theta^+) = \frac{1}{2} \exp v \left\{ \prod_{i=1}^{n} (1 - \exp e_i) + \prod_{i=1}^{n} (1 + \exp e_i) \right\} \\
\times \sum_{w \in W(D_n)} \det w \exp(\rho(D_n) - w\rho(D_n))
\]

\[
= \frac{\exp v}{n!} \sum_{s \in W(D_n)} \exp(v - sv) \sum_{w \in W(D_n)} \det w \exp(\rho(D_n) - w\rho(D_n))
\]

\[
= \frac{1}{n!} \sum_{s, w \in W(D_n)} \det w \exp(\rho(D_n) + 2v - w(\rho(D_n) + sv)).
\]

It is easy to see that, if \( sv \neq v \), then

\[
\sum_{w \in W(D_n)} \det w \exp(\rho(D_n) + v - w(\rho(D_n) + sv)) = 0.
\]

Now the above formula becomes

\[
\Omega(C_n^+ \Theta^+) = \sum_{w \in W(D_n)} \det w \exp(\rho(D_n) + 2v - w(\rho(D_n) + v)).
\]

Since \( \Theta^- = \mathrm{Ch}(\mathbb{C} [z_i \mid 1 \leq i \leq n]) - \Theta^+ \), we get

\[
\Omega(C_n^+ \Theta^-) = - \sum_{w \in W(C_n) \setminus W(D_n)} \det w \exp(\rho(D_n) - 2v - w(\rho(D_n) + v)).
\]

Let \( D_m^+ = \{ f_i \pm f_j \mid i < j \} \) be a positive system of a root system of type \( D \). Weyl's character formula tells us

\[
\Omega(D_m^+ \Psi^+) = \sum_{w \in W(D_m)} \det w \exp(\rho(D_m) - w(\rho(D_m) + \mu)),
\]
where \( \mu = \sum_{j=1}^{m} \frac{1}{2} f_j \) and \( \rho(D_m) = \sum_{j=1}^{m} (m-j) f_j \). Let \( C^+ = D^+ \cup \{2f_j | 1 \leq j \leq m\} \) be a positive root system of type \( C \). Then we obtain similarly

\[
\Omega(D^+_m) \Psi^- = \sum_{w \in W(C^+) \setminus W(D_m)} \det w \exp(\rho(D_m) - w(\rho(D_m) + \mu)).
\]

At last we can calculate \( \text{Ch}(\Xi^+) \):

\[
\text{Ch}(\Xi^+) = \Theta^+ \Psi^+ + \Theta^- \Psi^-
\]

\[
= \Omega(A_0^+)^{-1} \exp(\rho(D_n) + \rho(D_m) + 2v)
\]

\[
\times \left\{ \sum_{w \in W(D_n \cup m)} \det w e^{-w(\rho(D_n) + v)} \sum_{s \in W(D_n)} \det s e^{-s(\rho(D_m) + \mu)}
\right. 
\]

\[
+ \sum_{w \in W(C_n) \setminus W(D_n)} \det w e^{-w(\rho(D_n) + v)}
\]

\[
\times \sum_{s \in W(C_m) \setminus W(D_m)} \det s e^{-s(\rho(D_m) + \mu)} \right\}
\]

\[
= \Omega(A_0^+)^{-1} e^{\rho(D_n \cup m) + 2v}
\]

\[
\times \left\{ \sum_{w \in W(D_n \cup m)} \det w e^{-w(\rho(D_n \cup m) + v + \mu)}
\right. 
\]

\[
+ \sum_{w \in W(D_n \cup m)} \det w e^{-w(\rho(D_n \cup m) + v - \epsilon_n + \mu - f_m)} \right\}
\]

\[
= \Omega(A_0^+)^{-1} e^{\rho(D_n \cup m) + 2v}
\]

\[
\times \left\{ \sum_{w \in W(D_n \cup m)} \det w e^{-w(\rho(D_n \cup m) + v + \mu)} (1 + e^{w(\epsilon_n + f_m)}) \right\}.
\]

Here we denote \( D_{n \cup m} = D_n \cup D_m \) for simplicity and put \( \rho(D_{n \cup m}) = \rho(D_n) + \rho(D_m) \). Similar calculations lead us to \( \text{Ch}(\Xi^-) \):

\[
\text{Ch}(\Xi^-) = \Omega(A_0^+)^{-1} e^{\rho(D_n \cup m) + 2v}
\]

\[
\times \sum_{w \in W(D_n \cup m)} \det w e^{-w(\rho(D_n \cup m) + v + \mu - \epsilon_n)} (1 + e^{w(\epsilon_n + f_m)}).
\]

We summarize these results into

**Proposition 4.5.** The characters of irreducible components of the oscillator representations \( \Xi^+ \) and \( \Xi^- \) are given by

\[
\text{Ch}(\Xi^+) = \frac{\sum_{w \in W(D_n \cup m)} \det w \exp(\rho - w(\rho + \lambda)) (1 + \exp w(\epsilon_n + f_m))}{\prod_{a \in A_0^+} (1 - \exp a)}.
\]
and

$$\text{Ch}(\Xi^-) = \sum_{w \in W(D_0)} \det w \exp(\rho - w(\rho + \lambda))(1 - \exp(w(e_n + f_m))) \prod_{\alpha \in \mathcal{A}_0^+}(1 - \exp \alpha),$$

where $\lambda = \gamma \mu = \sum_{i=1}^n (1/2) e_i + \sum_{j=1}^m (1/2) f_j$. 

**Proof.** Note that $\rho(D_{n+m}) + 2\gamma = \rho$ and $W(D_0) \setminus W(D_{n+m}) = sW(D_{n+m})$, where $s \in W(D_0)$ sends $e_i$ to $e_i$ ($i \neq n$), $e_n$ to $-e_n$, and $f_j$ to $f_j$ ($1 \leq j \leq m$). Now it is easy to make the formulas before the proposition into the desired form. Q.E.D.

Similarly we get

**Proposition 4.6.** The super-characters of irreducible oscillator representations $\Xi^+$ and $\Xi^-$ are given by

$$\text{s-Ch}(\Xi^+) = \sum_{w \in W(D_{n+m})} \det w \exp(\rho - w(\rho + \lambda))(1 - \exp(w(e_n + f_m))) \prod_{\alpha \in \mathcal{A}_0^+}(1 - \exp \alpha),$$

and

$$\text{s-Ch}(\Xi^-) = \sum_{w \in W(D_0) \setminus W(D_{n+m})} \det w \exp(\rho - w(\rho + \lambda))(1 - \exp(w(e_n + f_m))) \prod_{\alpha \in \mathcal{A}_0^+}(1 - \exp \alpha).$$

### 5. Characters and Super-Characters of the Unitary Representations of $\mathfrak{osp}(2, 2; \mathbb{R})$

In this section, we determine characters and super-characters of all the irreducible unitary representations of $\mathfrak{osp}(2, 2; \mathbb{R})$ which can be integrated up to the representations of $\text{Sp}(2, \mathbb{R}) \times \text{SO}(2)$ as representations of the even part. We call these representations integrable unitary representations.

In [12, Theorem 4.5], we determined all the irreducible integrable unitary representations of $\mathfrak{osp}(2, 2; \mathbb{R})$. To prove Theorem 4.5 in [12], we used some character formulas. Here we will give these formulas. We don’t know a priori if the representations treated here exhaust all the irreducible integrable unitary representations; however, a posteriori, they indeed exhaust all the irreducible integrable unitary representations by Theorem 4.5 in [12].

Let $V(\lambda; \mu)$ be an irreducible lowest weight module of $\mathfrak{osp}(2, 2; \mathbb{R})$ with the lowest weight $(\lambda; \mu) = \lambda e_1 + \mu f_1$. We assume that $\lambda$ and $\mu$ are both integers which satisfy $\lambda \geq |\mu|$. Since $\rho = (0; 0)$, $V(\lambda; \mu)$ coincides with a discrete series representation $D(\lambda; \mu)$ if and only if $\lambda > |\mu|$. Let $x = \exp e_1$ and $y = \exp f_1$. 
THEOREM 5.1. Let \( V(\lambda; \mu) \) be an irreducible lowest weight module of \( \mathfrak{osp}(2, 2; \mathbb{R}) \) with lowest weight \((\lambda; \mu)\). Assume that \( \lambda \) and \( \mu \) are both integers satisfying \( \lambda \geq |\mu| \).

(1) If \( \lambda > |\mu| \), then it holds that

\[
\text{Ch } V(\lambda; \mu) = x^\lambda y^\mu \frac{(1 + xy)(1 + xy^{-1})}{1 - x^2},
\]

and

\[
\text{s-Ch } V(\lambda; \mu) = x^\lambda y^\mu \frac{(1 - xy)(1 - xy^{-1})}{1 - x^2}.
\]

(2) If \( \lambda = |\mu| \), then it holds that

\[
\text{Ch } V(\lambda; \mu) = x^\lambda y^\mu \frac{1 + xy^{-\text{sngn } \mu}}{1 - x^2},
\]

and

\[
\text{s-Ch } V(\lambda; \mu) = x^\lambda y^\mu \frac{1 - xy^{-\text{sngn } \mu}}{1 - x^2}.
\]

**Proof.** Part (1) is clear from Propositions 4.5 and 4.6, since \( \text{Ch } V(\lambda; \mu) = \text{Ch } D(\lambda; \mu) \).

Let us prove (2). Note that \( V(\lambda; \mu) \) is an irreducible quotient of \( D(\lambda; \mu) \). In the case of \( \mathfrak{osp}(2, 2; \mathbb{R}) \), we have

\[
D(\lambda; \mu) \simeq \bigwedge \mathfrak{g}(1) \otimes S(\mathfrak{g}(2)) \quad \text{(as vector spaces)}
\]

because \( \tau(\lambda; \mu) \) is one-dimensional. Let \( X(1; \pm 1) \) be non-zero root vectors corresponding to odd roots \( e_i \pm f_1 \), respectively and \( X(2; 0) \) be that corresponding to an even root \( 2e_i \). Then a vector \( v \) in \( D(\lambda; \mu) \) can be expressed like

\[
v = F_1 + F_2 X(1; 1) + F_3 X(1; -1) + F_4 X(1; -1) X(1; 1),
\]

where the \( F_i \) (\( 1 \leq i \leq 4 \)) are polynomials in \( X(2; 0) \).

As in the proof of Theorem 3.3, a vector \( v \) is annihilated by negative root vectors only if its weight \((v; \omega)\) satisfies \( R((v; \omega) - \rho) = 0 \). Note that \( R((v; \omega) - \rho) = (v - \omega)(v + \omega) \).

Assume \( \lambda = \mu \). Then it is impossible that \( v + \omega = 0 \). So a primitive vector \( v \) must have a weight \((v; v)\). Now it is easy to conclude that \( v \) is of the form \( v = X(1; 1) 1 \) \((1 \in \tau(\lambda; \lambda))\). Explicit calculation tells us that indeed \( v = X(1; 1) 1 \) is primitive. Therefore \( v = X(1; 1) 1 \) generates a submodule

\[
\{ F_2 X(1; 1) + F_4 X(1; -1) X(1; 1) | F_2 \text{ and } F_4 \text{ are polynomials in } X(2; 0) \}
\]
in \( D(\lambda; \lambda) \). Its irreducible quotient
\[
\{ F_1 + F_3 X(1; -1) | F_1 \text{ and } F_3 \text{ are polynomials in } X(2; 0) \}
\]
is equivalent to \( V(\lambda; \lambda) \). Now it is easy to deduce the desired character formula.

The case where \( \lambda = -\mu \) can be treated similarly. Q.E.D.

6. An Application of the Character Formulas

Again we return to the case of \( \mathfrak{osp}(2n, 2m; \mathbb{R}) \) and keep to the notations in Sections 1–4. In this section we decompose \( D(\lambda) \) explicitly as a representation of the even part in two extreme cases. In one of the cases, \( \lambda \) is "very regular" and, in the other, \( \lambda \) is "very singular."

Let \( \pi(\lambda) \) be a finite dimensional irreducible representation of \( \mathfrak{f}_C \) with lowest weight \( \lambda \) as in Section 3. Since the adjoint action of \( \mathfrak{f}_C = \mathfrak{g}(0) \) leaves \( \mathfrak{g}(1) \) invariant, \( \bigwedge \mathfrak{g}(1) \) gives a representation of \( \mathfrak{f}_C \). Decompose \( \pi(\lambda) \otimes \bigwedge \mathfrak{g}(1) \) as a representation of \( \mathfrak{f}_C \):
\[
\pi(\lambda) \otimes \bigwedge \mathfrak{g}(1) = \sum_{v \in T(\lambda)} m(\lambda, v) \pi(v),
\]
where \( m(\lambda, v) \) is a multiplicity. Put \( T(\lambda) = \{ v \in \mathfrak{h}_C^* | m(\lambda, v) \neq 0 \} \).

For \( \lambda \in \mathfrak{h}_C^* \) satisfying the conditions (a) and (b) in Section 3, we denote by \( \mathcal{H} \mathcal{D}(\lambda) \) a (limit of) holomorphic discrete series representation of \( \text{Sp}(2n, \mathbb{R}) \times \text{SO}(2m) \) with lowest weight \( \lambda \).

**Theorem 6.1.** If \( \lambda \in \mathfrak{h}_C^* \) is a parameter of the discrete series representations satisfying \( R(\lambda - \rho) \neq 0 \), then a discrete series representation \( D(\lambda) \) of \( \mathfrak{osp}(2n, 2m; \mathbb{R}) \) is decomposed as a representation of the even part \( g_0 = \mathfrak{sp}(2n, \mathbb{R}) \times \mathfrak{so}(2m) \) as
\[
D(\lambda) = \sum_{v \in T(\lambda)} m(\lambda, v) \mathcal{H} \mathcal{D}(v).
\]

**Proof.** Since \( D(\lambda) \) is a lowest weight module for \( \mathfrak{g}_0 \), its subquotient as a \( g_0 \)-module is a lowest weight module for \( g_0 \). By Theorem 3.3, if \( R(\lambda - \rho) \neq 0 \) then \( D(\lambda) \) is super unitary, so it is infinitesimally unitary for \( g_0 \). The classification of unitarizable lowest weight modules for \( \mathfrak{sp}(2n, \mathbb{R}) \times \mathfrak{so}(2m) \) [2, Theorem 8.4] tells us that \( D(\lambda) \) is a direct sum of (limits of) holomorphic discrete series representations of \( g_0 \). Characters of holomorphic discrete series representations are linearly independent on a compact Cartan subalgebra (see [13], for example), the only thing to do
is to decompose the character of $D(\lambda)$ as a sum of characters of (limits of) holomorphic discrete series representations of $g_0$.

Using the notations in Section 4, the character $\text{Ch} \mathcal{H} \mathcal{D}(v)$ is given by

$$\text{Ch} \mathcal{H} \mathcal{D}(v) = \text{Ch}(\tau(v)) \prod_{x \in \mathcal{A}_n^+} (1 - \exp \alpha)^{-1}.$$  

Now it is easy to see

$$\text{Ch} D(\lambda) = \text{Ch}(\tau(\lambda) \otimes \wedge g(1)) \prod_{x \in \mathcal{A}_n^+} (1 - \exp \alpha)^{-1}$$

$$= \sum_{v \in T(\lambda)} m(\lambda, v) \text{Ch}(\tau(v)) \prod_{x \in \mathcal{A}_n^+} (1 - \exp \alpha)^{-1}$$

$$= \sum_{v \in T(\lambda)} m(\lambda, v) \text{Ch} \mathcal{H} \mathcal{D}(v). \quad \text{Q.E.D.}$$

In the following, we give two explicit decompositions, which are typical in the meaning that they give the most singular and the most regular cases. Both can be obtained as corollaries of Theorem 6.1.

Case 1. Let $\lambda = (a, a, ..., a; b, b, ..., b)$ be a parameter of discrete series such that $R(\lambda - \rho) \neq 0$. So integers $a$ and $b$ must satisfy

$$a + b \leq n, \quad b \leq 0.$$  

**Corollary 6.2.** Let $\lambda = (a, a, ..., a; b, b, ..., b)$ be as above. Moreover, we assume that $b \leq -2nm$. Then a discrete series $D(\lambda)$ is decomposed as a $g_0$-module as

$$D(\lambda) = \sum_{s = 0}^{2nm} \sum_{t = 0}^{[\sqrt{2}] - 1} \sum_{u = 0}^{t - 2t} \left( \begin{array}{c} m + t - 1 \\ t \end{array} \right)$$

$$\times \mathcal{H} \mathcal{D}(a, ..., a, a + s; b - u, b, ..., b, b + s - 2t - u).$$

**Proof.** Corresponding decomposition of $\tau(\lambda) \otimes \wedge g(1)$ can be easily obtained. Q.E.D.

Case 2. Take a parameter $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n; \mu_1, \mu_2, ..., \mu_m)$ of a discrete series representation and assume that

$$\lambda_i + 2nm \leq \lambda_{i+1} \quad (1 \leq i \leq n-1), \quad |\mu_1| + 2nm \leq \lambda_1,$$

$$|\mu_j + 1| + 2nm \leq |\mu_j| \quad (1 \leq j \leq m),$$

where we put $\mu_{m+1} = 0$. Then $\lambda$ automatically satisfies $R(\lambda - \rho) \neq 0$.  

For a subset $A$ of roots $\Lambda$, we set $\sum A = \sum_{x \in A} x$. Then as above, we can prove

**Corollary 6.3.** Let $\lambda$ be as above. Then a discrete series $D(\lambda)$ is decomposed as a $g_0$-module as

$$D(\lambda) = \bigoplus_{A \subset \Lambda^+} \mathcal{H}D \left( \lambda + \sum A \right).$$

**References**

10. K. Nishiyama, Decomposing oscillator representations of $osp(2n; \mathbb{R})$ by a super dual pair $osp(2/1; \mathbb{R}) \times so(n)$, *Comp. Math.*, to appear.