Versal Deformations and Normal Forms for Reversible and Hamiltonian Linear Systems

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The problem of this article is the characterization of equivalence classes and their versal deformations for reversible and reversible Hamiltonian matrices. In both cases the admissible transformations form a subgroup $G$ of $\text{Gl}(m)$. Therefore the $\text{Gl}(m)$-orbits of a given matrix may split into several $G$-orbits. These orbits are characterized by signs. For each sign we have a normal form and a corresponding versal deformation. The main tool in the characterization is reduction to the semi simple case.

1. Introduction

Any study of the neighbourhood of a special solution of a differential equation, e.g. a stationary point or a (quasi) periodic solution, starts with examining a linear system. As an example, consider the differential equation

$$\dot{x} = Ax + F(x) \quad (1)$$

on $\mathbb{R}^m$ where 0 corresponds to the special solution. $A$ is an $m \times m$ matrix and $F$ is such that $F(0) = 0$ and $DF(0) = 0$. In principle, (1) with $F \equiv 0$, determines the stability of the special solution. In further analysis, for example, in a normal form approximation or unfolding problems, the linear system plays a prominent part. See [11], [18] concerning the Floquet matrix of periodic solutions and [6], [7] for Floquet matrices of quasi-periodic solutions, or [2] for yet another class of problems. Since linear differential equations, under a change of basis transform as linear maps, we consider these maps in their own right. The main part of the analysis of the linear system is to find a change of basis so that the matrix
of $A$ takes a particularly simple form called normal form. Here we are especially interested in the classification of such normal forms and their versal deformations. A versal deformation of a system is a systematic way to explore its neighbourhood. This is important for both perturbed and parameter dependent systems.

The main goal of this paper is to classify normal forms as well as their versal deformations, of reversible Hamiltonian linear systems. Our approach is to classify reversible and Hamiltonian systems separately. The classification of reversible Hamiltonian systems then follows almost automatically. The essential step of our method for constructing normal forms and versal deformations, is a reduction to the semisimple case. Therefore our results are to a large extent coordinate free.

1.1. Group Actions, Equivalence Classes, and Structure

The transformation of a matrix $A \in \mathfrak{gl}(m)$ under a change of basis gives rise to the $GL(m)$ action $A \mapsto g^{-1}Ag$ where $g \in GL(m)$. This similarity transformation naturally defines an equivalence relation. Thus the $GL(m)$-orbit of $A$ in $\mathfrak{gl}(m)$ defines the equivalence class of $A$. The terms “orbit” and “class” will be used interchangeably in this paper. Since any member of the class of $A$ can be taken as a representative, it is not a priori clear which one should be called the normal form. For example, in the literature there is no agreement on what is the Jordan normal form.

In many cases a dynamical system respects some additional structure. For example, the system may be Hamiltonian or invariant under some symmetry group. This means that the matrix $A$ in (1) lies in some subset $U$ of $\mathfrak{gl}(m)$. For several reasons this structure should be preserved under transformations. The admissible transformations form the largest subgroup $G$ of $GL(m)$ that preserve the structure of $U$. This $U$ is not just any subset of $\mathfrak{gl}(m)$ but in many cases, in particular for reversible and Hamiltonian systems, it is a linear subspace. The equivalence class of $A$ is now the $G$-orbit of $A$ in $U$. In general, a $G$-orbit is a submanifold of a $GL(m)$-orbit. Thus there might be different $G$-orbits in the same $GL(m)$-orbit. The latter phenomenon will be called splitting of $GL(m)$-orbits. This phenomenon is nicely illustrated by the following well known example.

**Example 1.** Let (1) be a four dimensional Hamiltonian system, so $A \in \mathfrak{sp}(4)$. The appropriate transformation group is the symplectic group $Sp(4)$. Suppose $A$ has two equal imaginary eigenvalues. It turns out that there are two different $Sp(4)$-classes within the same $GL(4)$-class. The $Sp(4)$-classes can be discriminated by a sign, associated to the eigenvalues. Two normal forms, each representing one of the $Sp(4)$-classes, are called $1:1$ and $1:-1$ resonance. They have quite different unfoldings. At $1: -1$
resonance generically a codimension 1 Hamiltonian-Hopf bifurcation occurs (see van der Meer [22]), whereas the 1:1 resonance has a high codimension (see Cotter [12] and Duistermaat [13]).

The general problem is the following. Let \( U = \mathfrak{gl}(m) \) be a submanifold which is preserved under the similarity action of a transformation group \( \mathcal{G} \subset \text{Gl}(m) \). We wish to characterize the \( \mathcal{G} \)-orbits in \( U \). Moreover we like to know how they relate to the corresponding \( \text{Gl}(m) \)-orbits. That is, we want to know how a \( \text{Gl}(m) \)-orbit splits up into \( \mathcal{G} \)-orbits when intersected with \( U \). We also want to label the \( \mathcal{G} \)-orbits. This problem becomes even more interesting when two or more subgroups of \( \text{Gl}(m) \) interact. In section 4 we encounter an example of two interacting groups. Namely the groups that leave invariant the structure of reversible and Hamiltonian matrices.

### 1.2. Characterization of \( \mathcal{G} \)-Classes

An important tool in the characterization problem is the Jordan–Chevalley decomposition which says that a matrix \( A \) can uniquely be written as the sum of a nilpotent part \( N \) and a semisimple part \( S \) with \([N, S] = 0\). This decomposition holds in \( \mathfrak{gl}(m) \) and in any Lie subalgebra of \( \mathfrak{gl}(m) \) if the corresponding Lie group is algebraic see [17]. Since we did not require that \( U \) be a Lie algebra, we have to check that the Jordan–Chevalley decomposition is compatible with \( U \).

We now briefly review the essentials of the characterization of \( \text{Gl}(m) \)-classes in \( \mathfrak{gl}(m) \), since this will be our starting point for the classification of reversible and reversible Hamiltonian matrices. Let \( A = N + S \) be in \( \mathfrak{gl}(m) \) and let \( \lambda \) be an eigenvalue of \( A \). Let \( \lambda \) be fixed. For each \( \lambda \) there are real indecomposable \( A \)-invariant subspaces \( V_\lambda \). An \( A \)-invariant subspace is called indecomposable if it cannot be written as the direct sum of two proper \( A \)-invariant subspaces. Now consider the restriction of \( A \) to \( V_\lambda \).

Since \( N \) is nilpotent there is an \( n \), called the height of \( N \), such that \( N^j = 0 \) for \( j < n \) and \( N^n \neq 0 \) on \( V_\lambda \). Only if the heights \( n \) are different, the \( V_\lambda \) are unique. For the \( V_\lambda \) the following holds.

- \( \text{(a1) Real eigenvalues: } \lambda = \alpha \in \mathbb{R} \text{ and } (A - \alpha)^n = 0 \). Then \( \dim(V_\lambda) = n \) and for all \( v \in V_\lambda \) we have \( Sv = \alpha v \).
- \( \text{(a2) Complex eigenvalues: } \lambda = \alpha + i \beta \text{ with } \alpha, \beta \in \mathbb{R} \text{ and } \beta \neq 0 \). Then \( ((A - \lambda)^2 + \beta^2)^n = 0 \) and \( \dim(V_\lambda) = 2n \). For all \( v \in V_\lambda \) we have \( (S - \alpha)^2 v = -\beta^2 v \).

(b) The splitting \( \mathbb{R}^m = \bigoplus \lambda V_\lambda \) is unique (for different heights only) up to permutations.

This implies that we know the \( \text{Gl}(m) \)-classes precisely if we know the classification on the subspaces \( V_\lambda \). We can even further restrict to a
subspace $W_*$ of $V_*$ by the following reduction lemma due to Burgoyne and Cushman [9].

**Lemma 1 (Reduction lemma).** There is an $S$-invariant complement $W_*$ of $NV_*$ in $V_*$ such that $V_* = W_* \oplus NV_*$, $W_* \oplus NW_* \oplus \ldots \oplus N^{n-1}W_*$. If $S$ is given on $W_*$, then $A$ is determined on $V_*$ up to similarity.

Thus we recover the fact that the $Gl(m)$-orbit of $A$ in $gl(m)$ is characterized by the eigenvalue $\lambda$ of $A$ on $V_*$ and the height of the nilpotent part $N$ on $V_*$. Furthermore by this reduction lemma the problem of finding the $Gl(m)$-orbit of $A$ on $V_*$ is reduced to finding the $Gl(m)$-orbit of the semisimple part $S$ of $A$ on $W_*$. As we will see in the next section the reduction lemma is also very useful for the problem of finding versal deformations.

In general, for the characterization of the $G$-classes in $u$ we use the following scheme. First we have to check whether the Jordan–Chevalley decomposition $A = N + S$ is compatible with $u$. Next we find the compatible indecomposable $A$-invariant subspaces $V$ and a compatible $S$-invariant complement $W$ of $NV$ in $V$. The meaning of “compatible” depends on the structure of $u$. For example for reversible systems, a compatible $A$-invariant subspace must also be invariant under the reversing transformations. The final step is the classification of semisimple matrices $S$ on $W$ which involves only a small number of low dimensional cases.

1.3. Deformations of $G$-Classes

As mentioned before a deformation of $A$ is a means to explore the neighbourhood of $A$. Since we consider classes i.e. $G$-orbits, of matrices, we are interested in the $G$-orbits near the $G$-orbit of $A$. Therefore one only considers the neighbourhood transversal to the $G$-orbit of $A$. Transverse sections at different points of the $G$-orbit of $A$ are equivalent by the similarity action of $G$. Thus we take this section at a point where the computations are easiest. Usually this will be the normal form of $A$. A smooth parametrization of a section at $A$ transverse to the $G$-orbit of $A$ is called a versal deformation of $A$, see Arnold [3]. We now address to the problem of finding a basis for such a transversal section.

Let $u$ be a linear subspace of $gl(m)$ which is preserved under the similarity action of a subgroup $G$ of $Gl(m)$. We assume that $u$ is compatible with the Jordan–Chevalley decomposition, that is if $A \in u$ then also $S$ and $N$ in $u$. In order to express transversality in $u$ we define an inner product on $u$ as follows: $(A, B) = \text{trace}(A'B)$ for $A, B \in u$. With this inner product we characterize a transverse section of the $G$-orbit of $A$ as the orthogonal complement of the tangent space of the orbit at $A$. The tangent space of the $G$-orbit of $A$ can be found as follows. Let $U$ be in the Lie algebra of $G$.
Then the tangent, in the \( U \)-direction, of the \( G \)-orbit of \( A \) at \( A \) is \( (d/dt) \exp(-tU) A \exp(tU) \mid_{t=0} = A U - U A = [A, U] = \text{ad}_A(U) \). A computation gives the following. The linear space \( \{ A + B \in \mathfrak{u} | \text{ad}_A(B) = 0 \} \) is a transverse section at \( A \) of the \( G \)-orbit. Since \( \text{ad}_A(B) = \text{ad}_A(B') \) the above boils down to finding a basis for \( \{ B' \in \mathfrak{u} | \text{ad}_A(B) = 0 \} \).

The two main ingredients of the construction of the versal deformation are the Jordan–Chevalley decomposition and the reduction lemma above. Let \( A = N + S \in \mathfrak{gl}(m) \) be a given matrix and let \( V \) be an \( A \)-invariant subspace. Then \( \text{ad}_A = \text{ad}_N + \text{ad}_S \) is the Jordan–Chevalley decomposition of \( \text{ad}_A \). Therefore \( \text{Ker}(\text{ad}_A) = \text{Ker}(\text{ad}_N) \cap \text{Ker}(\text{ad}_S) \). By the reduction lemma, we can first find a basis for \( \text{Ker}(\text{ad}_S) \) on a smaller subspace \( W \) of \( V \). In fact, we only have to find this basis for a few cases where \( S \) is low dimensional. Then it is rather straightforward to construct a basis of \( \{ B : W \rightarrow W | \text{ad}_S(B) = 0 \} \).

Let us now define the spaces relevant to our purposes. Suppose \( V \) is an \( A \)-invariant subspace. \( A \) restricted to \( V \) has only one real eigenvalue \( \lambda = \alpha \) or two complex conjugate eigenvalues \( \lambda = \alpha \pm i\beta \), \( \beta \neq 0 \). Let \( V = V_1 \oplus \cdots \oplus V_r \), where each \( V_i \) is an indecomposable \( A \)-invariant complement of \( N V_i \) in \( V \). The restriction \( S_i \) of \( S \) to \( W_i \) does not depend on \( i \). Thus all \( W_i \) have the same dimension. Let the restriction of \( N \) to \( V_i \) have height \( n_i \) and assume for simplicity that \( n_1 \geq n_2 \geq \cdots \geq n_r \). Let \( W = W_1 \oplus \cdots \oplus W_r \), then \( V = W \oplus N V_i \). Furthermore let \( \langle e_1, \ldots, e_r \rangle \) be a basis of \( W \). With the following proposition we reduce the problem of finding a basis of \( \mathfrak{b}_W \) to the problem of finding a basis of \( \{ B : W_i \rightarrow W_i | \text{ad}_S(B) = 0 \} \).

**Proposition 2.** (a) Let \( \langle B^{(1)}, \ldots, B^{(l)} \rangle \) be a basis of \( \mathfrak{b}_{W_i} \). Define \( B^{(k)}_W : W \rightarrow W \) as follows:

\[
B^{(k)}_W = \begin{cases} B^{(k)} &: W_i \rightarrow W_j \\ 0 &: W_i \rightarrow W_j \end{cases} \quad (i', j') \neq (i, j)
\]

Then \( \{ B^{(k)}_W | i, j = 1, \ldots, r \text{ and } k = 1, \ldots, l \} \) forms a basis of \( \mathfrak{b}_W \).

(b) Suppose \( \langle B_1, \ldots, B_p \rangle \) is a basis of \( \mathfrak{b}_W \). Define \( \tilde{B}_i \) on \( V \) as follows:

\[
\tilde{B}_i N e_l = N^k \tilde{B}_i e_l \quad \text{for } l = 1, \ldots, k \text{ and } j = 0, \ldots, n_j - 1 \text{ if } e_l \in W_q. \quad \text{Then the } N^k \tilde{B}_i \text{ for } i = 1, \ldots, p \text{ and values of } k \text{ such that } N^k \tilde{B}_i \neq 0, \text{ form a basis of } \mathfrak{b}.
\]

The formulation of the proposition seems needlessly general, but it facilitates future generalizations. From Proposition 2 the following corollary follows immediately. We drop the tilde.
Corollary 3. Using the basis of part b in the proposition above, the matrix $A(\mu) = A + \sum_{ik} \mu_{ik} (N^k B_i)'$ is a versal deformation of $A$.

Remark 1. The dimension of $b$ is the codimension of the orbit of $A$. This is the minimal number of parameters needed for a versal deformation. However, if the isotropy group $\mathcal{H} = \{ h \in G | hA = Ah \text{ and } hA' = A'h \}$ of $b$, acts non trivially on $b$, then it is in some cases possible to reduce the number of parameters. We will encounter this phenomenon in the example of Section 5.

Proof of Proposition 2. Let us begin with part a. Since the $W_i$ are $S$-invariant, $S$ can be put into block diagonal form. Let $S_i$ be the restriction of $S$ to $W_i$, then $S_i = S_1$ for all $i$ because the eigenvalues of $S$ on $W_i$ do not depend on $i$. Let $\tilde{B}$ be matrix in $b_W$. Partition $\tilde{B}$ in the same way as $S$. Then every block in $\tilde{B}$ must commute with $S_1$. Thus the $\tilde{B}_i$ are linearly independent and span $b_W$.

We prove part b only for real eigenvalues. The proof for complex conjugate eigenvalues is similar. The proof proceeds in three steps. First we show that $N^k \tilde{B}_i \in b$. Next we show that the $N^k \tilde{B}_i$ are linearly independent and finally we show that they span $b$. From the definition of $N^k \tilde{B}_i$ it is immediately clear that $N^k \tilde{B}_i \in b$. To prove that the $N^k \tilde{B}_ij$ are linearly independent we use the numbering of part a. Note that $N^k \tilde{B}_ij : V_i \rightarrow N^k V_j$. Since the $\tilde{B}_i$ have maximal rank, the $N^k \tilde{B}_i$ are linearly independent. The proof that the $N^k \tilde{B}_i$ span $b$ follows from the observation that the number of independent vectors $N^k \tilde{B}_i$ is equal to the number of linearly independent solutions of a special case. Suppose $N$ is the “standard” form with only ones on the upper (or lower) co-diagonal. The number of independent solutions of $\text{ad}_\nu(N) = 0$ is $\sum_{i=1}^{r+1} (2i - 1)n_i$ which is equal to the number of vectors $N^k \tilde{B}_i$, see Arnold [3] and Gantmacher [15].

Now we use this construction for a linear subspace $u$ of $\mathfrak{gl}(m)$. Assume that $u$ is compatible with the Jordan–Chevalley decomposition. Let $V, V', W$ and $W'$ be the compatible subspaces of $A \in u$. As mentioned earlier the meaning of compatibility depends on the structure of $u$. In general compatible $A$-invariant spaces for $A \in u$ will be larger than the corresponding ones for $A \in \mathfrak{gl}(m)$. Using the fact that the action of $G$ on $u$ maps a transverse section into a transverse section (not preserving orthogonality), we obtain the following lemma.

Lemma 4 (Deformation Lemma). Let $A \in u$. Then the construction of proposition 2 yields a versal deformation of $A$ on $V$ in $\mathfrak{gl}(m)$.
The remaining problem is to find a basis of $b \cap u$. In other words, we have to find a new basis of $b$ such that part of the new basis spans $b \cap u$. How this should be done depends on $u$. Fortunately, for our specific examples there is a decomposition lemma. That is, there is a decomposition of vectors into a part in $b \cap u$ and a part into the orthogonal complement.

1.4. Summary of the Results

We will now summarize our results of splitting of $\text{Gl}(m)$-orbits for reversible matrices and reversible Hamiltonian matrices. For a precise formulation we refer to the appropriate sections and for an example we refer to Section 5. In order to present a coherent picture we also mention the results for Hamiltonian matrices from Burgoyne and Cushman [8]. In all cases we distinguish purely real, purely imaginary, complex and zero eigenvalues of the matrix $A$. The results below hold for the restriction of $A$ to an indecomposable $A$-invariant subspace $V$.

For reversible matrices only the $\text{Gl}(m)$-class of zero eigenvalues splits up into two new $\text{Gl}(m)_d$-classes. The $\text{Gl}(m)_d$-classes can be characterized by the reversible sign of such a class, see also Sevryuk [27, p. 1669]. Geometrically, the reversible sign indicates to which eigenspace of the reversing map an eigenvector with zero eigenvalue belongs, see Section 2. The codimension of the $\text{Gl}(m)_d$-orbits is roughly half the codimension of the $\text{Gl}(m)$-orbits.

In the case of Hamiltonian matrices, a $\text{Gl}(m)$-class of purely imaginary eigenvalues splits into two parts, as we have seen in example 1. This also holds for the $\text{Gl}(m)$-class of eigenvalues zero. We can associate a symplectic sign to each class. Indeed, the symplectic form $\omega$ defines a preferred direction of rotation. The symplectic sign indicates the direction of rotation for imaginary eigenvalues, relative to the preferred direction, see Section 3. Again the codimension of the $\text{Gl}(m)_{2n}$-orbits is roughly half the codimension of the $\text{Gl}(2n)$-orbits.

Finally, for reversible Hamiltonian matrices almost all $\text{Gl}(m)$-classes split. The splittings of the classes of purely real complex eigenvalues are new. The splitting of the class of purely imaginary eigenvalues is the same as for Hamiltonian matrices. Thus no further splitting occurs. The new classes can be characterized by the reversible symplectic sign. The class of zero eigenvalues splits into four new classes. These classes can be distinguished by a pair of signs: the reversible symplectic and the reversible sign. The reversible symplectic sign is closely related to the symplectic sign. Moreover it takes into account the orientation with respect to the eigenspaces of the reversing map, see section 4. The simplest example showing the relevance of all signs is a $4 \times 4$ matrix with four eigenvalues zero and two Jordan blocks of equal size. We illustrate our results by this matrix in Section 5.
Normal forms and versal deformations for reversible matrices can also be found in Sevryuk [26, 27] and in Shih [10]. Also see Palmer [25] for some results on normal forms. The problem of finding normal forms and versal deformations for reversible Hamiltonian matrices was treated by Wan [28]. However, our method reveals splittings of Gl(m)-orbits which Wan seems to have overlooked; specifically the splittings for real and complex eigenvalues. Furthermore, our method gives a constructive procedure for finding normal forms and deformations starting from the semisimple reduced matrix, from which the normal form or deformation for the full matrix can easily be reconstructed. For related results on Hamiltonian matrices see for example Burgoyne & Cushman [8], Galin [14] and Koçak [20]. Melbourne [23] and Melbourne & Dellnitz [24] consider a similar problem for symmetric Hamiltonian matrices. Related results for maps can be found in Bridges & Furter [4] and Bridges & Cushman [5]. More general results on conjugacy classes can be found in Burgoyne & Cushman [9].

2. Reversible Matrices

Reversible dynamical systems are determined by a linear map \( R \) with \( R^{-1} = R \) (and \( R \neq \pm I \)), called an involution. The differential equation

\[
\dot{x} = F(x) \tag{2}
\]

is called \( R \)-reversible if \( RF(x) = -F(Rx) \). In other words for every solution \( g(t) \) of (2), \( Rg(-t) \) also is a solution. For the matrix of the linear part of (2) this amounts to the following. A matrix \( A \in \mathfrak{gl}(m) \) is called \( R \)-reversible if \( AR + RA = 0 \). A general reference for reversible systems is Sevryuk [26], for more recent results and references also see Broer et al. [7]. We note that Sevryuk [26, 27] uses the term “infinitesimally reversible” instead of our term “reversible”. The set of all \( R \)-reversible matrices is a linear subspace of \( \mathfrak{gl}(m) \) and will be denoted \( \mathfrak{gl}_{R}(m) \), following the notation of [7]. Observe that \( \mathfrak{gl}_{R}(m) \) is not a Lie algebra. The group of all transformations that leave \( u = \mathfrak{gl}_{R}(m) \) invariant, is \( \text{Gl}_{R}(m) = \{ g \in \text{Gl}(m) | gR = Rg \} \), the group of \( R \)-equivariant transformations. The following properties of \( R \)-reversible systems are frequently used.

(a) \( R \) is semisimple.

(b) The eigenvalues of \( R \) are 1 and \(-1\) with multiplicities \( n_+ \) and \( n_- \), respectively. Thus \( \mathbb{R}^m \) is the direct sum of the eigenspaces \( E_+ = \{ a \in \mathbb{R}^m | Ra = a \} \) and \( E_- = \{ b \in \mathbb{R}^m | Rb = -b \} \), with \( \dim E_+ = n_+ \) and \( \dim E_- = n_- \).
(c) If \( A \in \mathfrak{gl}_n(m) \) then \( AE_+ \subseteq E_+ \) and \( AE_- \subseteq E_- \). If \( \langle a_1, ..., a_n \rangle \) is a basis of \( E_+ \) and \( \langle b_1, ..., b_n \rangle \) of \( E_- \) then \( \langle a_1, ..., a_n, b_1, ..., b_n \rangle \) is a basis of \( \mathbb{R}^m \). With respect to this basis the matrix of \( A \) is of “antiblock-diagonal” form

\[
\begin{pmatrix}
0 & A_1 \\
A_2 & 0
\end{pmatrix}.
\]

(d) The characteristic polynomial \( p(t) \) of \( A \) contains the following factors

\[
t^{n_+ - n_-} \cdot t^2 \cdot t^2 - \alpha^2 \cdot t^2 + \beta^2 \cdot [(t - \alpha)^2 + \beta^2] \cdot [(t + \alpha)^2 + \beta^2].
\]

Remark 2. Let us consider a reversible dynamical system. The eigenspaces \( E_+ \) and \( E_- \) provide us with a natural splitting of the phase space \( \mathbb{R}^m \). Moreover the eigenspaces \( E_+ \) and \( E_- \) have dynamical properties. Indeed, orbits of (2) can have only transversal intersections with \( E_+ \). Thus orbits contained in \( E_+ \) necessarily must be stationary points. Such stationary points are usually called symmetric since \( \text{Fix}(R) = E_+ \). The linearization at a symmetric stationary point of the dynamical system is again an \( R \)-reversible matrix. A nonsymmetric stationary point, say \( x \), always has a stationary point \( Rx \) to match. Locally such nonsymmetric stationary points are indistinguishable from stationary points in general systems. In the next sections we will always use the splitting \( \mathbb{R}^m = E_+ \oplus E_- \) and take a basis of the form \( \langle a_1, ..., a_n, b_1, ..., b_n \rangle \). Vectors in \( E_+ \) will be indicated by \( a \) and vectors in \( E_- \) by \( b \).

Remark 3. \( R \)-reversible matrices with \( n_+ \neq n_- \) always have at least \( |n_+ - n_-| \) zero eigenvalues. Consequently, these zero eigenvalues have codimension zero in \( \mathfrak{gl}_n \), quite unlike the general or the Hamiltonian case. Thus in the linear setting such eigenvalues are rather uninteresting. Therefore we assume \( n_+ = n_- \). However, see Sevryuk [26] and Broer & Huitema [7] for the nonlinear case.

Remark 4. \( R \)-reversible systems with \( R^2 \neq I \) are called weakly reversible by Sevryuk [26]. So let \( A \) be weakly \( R \)-reversible. In general, such systems have more structure since now \( A \) is also \( R^2 \)-equivariant (only if \( R^2 = -I \) there is no additional structure in the linear case). Let us impose the rather natural restriction that the reversor \( R \) is still an isometry. As is easily checked, imaginary eigenvalues have geometric multiplicity greater than one. If \( R^2 \neq -I \), then also real eigen values have geometric multiplicity greater than one. In two dimensions, for example, this means that \( A = 0 \).

Since we do not want to restrict our set of matrices any further we will not
consider weakly reversible systems. For more information on such systems see Lamb & Quispel [21].

2.1. Equivalence Classes of Reversible Matrices

We now classify the \( \mathfrak{g} \)-equivalence classes of \( R \)-reversible matrices. Here \( \mathfrak{g} = \text{Gl}(m) \) is the group of \( R \)-equivariant transformations. Our argument boils down to finding the normal form of semisimple \( R \)-reversible matrices. Our proof of the classification is constructive, starting from the \( \text{Gl}(m) \) classification. In the appendix a normal form for the nilpotent case will be reconstructed from the semisimple case.

The classification of reversible matrices starting from the \( \text{Gl}(m) \) classification, consists of three steps. Let \( A \in \mathfrak{gl}_-(\mathfrak{g}) \) be an \( R \)-reversible matrix. Since \( \mathfrak{gl}_-(\mathfrak{g}) \) is not a Lie algebra, we have to check first that the Jordan–Chevalley decomposition \( A = N + S \) is compatible with \( \mathfrak{gl}_-(\mathfrak{g}) \). However, it is obvious that both \( N \) and \( S \) are in \( \mathfrak{gl}_-(\mathfrak{g}) \). The second step is to construct the compatible indecomposable \( A \)-invariant subspaces \( V \) from the subspaces \( V'_j \) and find a compatible \( S \)-invariant complement \( W \) of \( NV \) in \( V \). The last step is the classification of semisimple \( R \)-reversible matrices on \( V \).

Let us now construct the compatible indecomposable \( A \)-invariant subspaces. It is obvious that for \( R \)-reversible matrices, \( A \)-invariant spaces must be \( R \)-invariant. An \( A \), \( R \)-invariant subspace \( V \) is called indecomposable if \( V \) cannot be written as the direct sum of two proper \( A \), \( R \)-invariant subspaces of \( V \). Let \( V_j' \) be an indecomposable \( A \)-invariant subspace as in Section 1.2 and let \( W_j' \) be an \( S \)-invariant complement of \( NV_j \) in \( V_j \). From the definition of \( R \)-reversible matrices it is easily seen that \( RV'_j = V'_j \). Now \( V_j' \cap V'_{-j} = \{0\} \) if \( j \neq -j \) and \( V_j' \cap V'_{-j} = V_j' \) if \( j = -j \). Therefore the indecomposable \( A \), \( R \)-invariant subspace \( V \) is \( V_j \oplus V'_{-j} \) if \( j \neq -j \) and \( V_j \) if \( j = -j \). For the complement \( W \) we find similar relations. We are now in a position to state the normal form theorem for semisimple \( R \)-reversible matrices.

**Theorem.** Let \( S \in \mathfrak{gl}_-(\mathfrak{g}) \) be semisimple on the indecomposable \( S \), \( R \)-invariant space \( W \). The normal forms of \( S \) are listed in Table V.

From Table V we see that in the reversible case the \( \text{Gl}(m) \)-orbits do not split up for nonzero eigenvalues, not even for purely imaginary eigenvalues. As we shall see in Section 3, this is different from the Hamiltonian case. For zero eigenvalues, however, the \( \text{Gl}(m) \) equivalence classes do split up into two distinct classes. We label these classes by the *reversible sign* \( \rho \) of the eigenvector of \( S \). If \( e \in E_+ \) then we define \( \rho(e) = +1 \) while \( \rho(e) = -1 \) if \( e \in E_- \). The occurrence of two different normal forms for zero eigenvalues...
was also noted by Iooss [19] (page 2) in a specific example, also see Table VI. Our example in Section 5 has four zero eigenvalues in two Jordan blocks. Thus this matrix is characterized by two reversible signs, one for each block.

**Proof.** We know that on any basis \( \langle a_1, ..., a_k, b_1, ..., b_k \rangle \) of \( W \), \( S \) takes the form

\[
S = \begin{pmatrix} 0 & S_1 \\ S_2 & 0 \end{pmatrix}.
\]

For the details we proceed case by case, \( \lambda \) is an eigenvalue of \( S \) restricted to \( W \).

(a) **Real eigenvalues:** \( \lambda = \alpha \in \mathbb{R} \), \( \alpha \neq 0 \). Then \( W = W_j \oplus W_{-j} \) is \( S \), \( R \)-invariant and \( S^2 - \alpha^2 = 0 \) on \( W \). Hence \( W \) is two dimensional. Take any nonzero vector \( e \in W_j \), then \( e + Re \neq 0 \) since \( RW_j = W_{-j} \) and \( W_j \cap W_{-j} = \{0\} \). Let \( a = e + Re \) and \( b = (1/\alpha) Sa \), then \( a \in E_+ \), \( b \in E_- \). Thus \( W = \langle a, b \rangle \) and \( Sa = \alpha b, Sb = \alpha a \).

(b) **Imaginary eigenvalues:** \( \lambda = i\beta \in i\mathbb{R} \), \( \beta \neq 0 \). Then \( W = W_j \) is \( S \), \( R \)-invariant and \( S^2 + \beta^2 = 0 \) on \( W \). Hence \( W \) is two dimensional. Take any nonzero vector \( e \in W \). First we construct a nonzero vector \( a \in E_+ \). If \( e + Re \neq 0 \) let \( a = e + Re \). If \( e + Re = 0 \) then \( e \in E_- \). Let \( a = Se \), then \( a \in E_+ \) is a nonzero vector. Finally let \( b = -(1/\beta) Sa \). Then \( b \in E_- \), \( W = \langle a, b \rangle \) and \( Sa = -\beta b, Sb = \beta a \).

(c) **Complex eigenvalues:** \( \lambda = \alpha + i\beta \in \mathbb{C} \), \( \alpha \neq 0 \), \( \beta \neq 0 \). Then \( W = W_j \oplus W_{-j} \) is \( S \), \( R \)-invariant and \( [(S - \alpha)^2 + \beta^2] \cdot [(S + \alpha)^2 + \beta^2] = 0 \) on \( W \). Hence \( W \) is four dimensional. Take any nonzero vector \( e \in W_j \) and let \( f = -(1/\beta)(S - \alpha) e \). Then \( f \) is a nonzero vector in \( W_j \) and \( W_j = \langle e, f \rangle \).

Since \( W_{-j} \cap RW_j \) we have \( W_{-j} = \langle Re, Rf \rangle \). So \( W = \langle e, f, Re, Rf \rangle = \langle a_1, a_2, b_1, b_2 \rangle \) if \( a_1 = e + Re, a_2 = f + Re, b_1 = e - Re \) and \( b_2 = f - Re \).

Thus we arrive at \( Sa_1 = \alpha b_1 - \beta b_2, Sa_2 = \alpha b_2 + \beta b_1 \) and similarly for \( b_1 \) and \( b_2 \).

(d) **Zero eigenvalues:** \( \lambda = 0 \). There is only one vector \( e \) with \( Se = 0 \), so either \( Re = e \) or \( Re = -e \). Indeed, suppose there is a vector \( e \notin E_+ \), \( E_- \) and \( Se = 0 \). Then \( W = \langle e, Re \rangle = \langle e + Re, e - Re \rangle \) is decomposable.

It is still not clear what the signs of \( \alpha \) and \( \beta \) should be. The following argument shows that we can always assume \( \alpha > 0 \) and \( \beta > 0 \). Since \( R \in \text{GL}_p(m) \) and \( RSR = -S \), \( \alpha \) and \( \beta \) can be taken positive in the first two
cases ($\lambda = \alpha$ and $\lambda = i\beta$). In the third case we again use $RSR = -S$. Furthermore if

$$R = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

and

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$$

with $P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

then $P \in \mathfrak{gl}_d(m)$ with $P^{-1}S_{\alpha,\beta}P = S_{\alpha,\beta}$. This shows that $S_{\alpha,\beta}$, $S_{-\alpha,-\beta}$ and $S_{\alpha,-\beta}$ are in the same $\mathfrak{g}$-orbit.

Remark 5. As we have seen above, one preserves the structure of $R$-reversible matrices by allowing only $R$-equivariant transformations. Thus we obtain the $\mathfrak{g}$-orbit $\{(g^{-1}Ag | g \in \text{Gl}(m), gR = Rg)\}$ of an $R$-reversible matrix $A$. We can also consider this structure as a pair $(A, R)$ with $AR + RA = 0$ and $R^2 = I$. In this way the structure is preserved by the whole group $\text{Gl}(m)$ if we define the $\text{Gl}(m)$-orbit of the pair $(A, R)$ as $\{(g^{-1}Ag, g^{-1}Rg) | g \in \text{Gl}(m)\}$. Of course the classification of $\mathfrak{g}$-orbits of $R$-reversible matrices is equivalent to the classification of the $\text{Gl}(m)$-orbits of pairs $(A, R)$. Because of the nice relations $AE_+ \subseteq E_-$ and $AE_- \subseteq E_+$ for $R$-reversible matrices $R$, we think it is most convenient to first normalize $R$ and then classify the $\mathfrak{g}$-orbits of $A$ (for symmetric systems these relations are even nicer). Similarly we can consider the structure of Hamiltonian matrices as pairs $(A, J)$ with $JA + AJ = 0$ and $J^2 = -I$ (see Section 3). Again this structure is preserved by the group $\text{Gl}(m)$ if we define the $\text{Gl}(m)$-orbit of the pair $(A, J)$ as $\{(g^{-1}Ag, g^{-1}Jg) | g \in \text{Gl}(m)\}$. Unfortunately we do not have nice relations for the eigenspaces of $J$. Therefore we have to do some more work to find the simultaneous normal form of $A$ and $J$.

2.2. Deformations of Reversible Matrices

To construct the deformations of reversible matrices we use the deformation Lemma 4. The problem that remains to be solved is the construction of a basis of $\{B | \text{ad}_A(B) = 0\} \cap \mathfrak{gl}_d(m)$. In view of this problem the following lemma is very useful.

**Lemma 6 (Reversible Decomposition Lemma).** Let $R$ be an involution on $\mathbb{R}^n$. Then $\mathfrak{gl}(m) = \mathfrak{gl}_d(m) \oplus \mathfrak{gl}_m(m)$. The projections $\mathfrak{R}_+ : \mathfrak{gl}(m) \to \mathfrak{gl}_d(m)$ and $\mathfrak{R}_- : \mathfrak{gl}(m) \to \mathfrak{gl}_m(m)$ are given by $\mathfrak{R}_+(A) = \frac{1}{2}(A + RAR)$ and $\mathfrak{R}_-(A) = \frac{1}{2}(A - RAR)$ respectively. Moreover the decomposition is orthogonal with respect to the inner product $(A, B) = \text{trace}(A'B)$ on $\mathfrak{gl}(m)$.

Before stating the theorem on deformations of reversible matrices, we recall some notation. Let $A = N + S$ be an $R$-reversible matrix. Let
The space $V_i$ is indecomposable. $R$-invariant subspace. For each $V_i$, there is a splitting $V_i = W_i \oplus N V_i$, where $W_i$ is an $S$, $R$-invariant complement of $N V_i$ in $V_i$. The restriction $S_i$ of $S$ to $W_i$ does not depend on $i$. So all $W_i$ have the same dimension. The restriction of $N$ to $V_i$ has height $n_i$. Let $W_i = W_1 \oplus \cdots \oplus W_r$, then $V = W \oplus N V$.

**Theorem 7.** The versal deformations of $A \in \mathfrak{gl}_{-R}(m)$ are listed in Table VII.

**Proof.** We only prove the case of zero eigenvalues because in that case the reversible sign plays a part. The other cases are treated similarly. The proof merely follows the construction of Proposition 2. Let $\lambda = 0$. Using the reduction lemma we restrict to $W$. Then $A = S = \text{diag}(S_1, \ldots, S_r)$, with $S_i = 0$ and $R = \text{diag}(\rho_1, \ldots, \rho_r)$ where $\rho_i$ is the reversible sign of $i$th eigenvector of $S$ on $W$. A basis of $b_{W_i} = \{ B : W_i \to W_i \mid \text{ad}_B(B) = 0 \} = B = 1$. Then the $B_{ij}$ form a basis of $b_{W_i}$, if

$$B_{ij} = \begin{cases} B : W_i \to W_j \\ 0 : W_i \to W_{i'}, (i', j') \neq (i, j) \end{cases}$$

Since $B_{ij} = B_{ji} \pm \rho_i \rho_j B_{ij}$, each basis vector is either $R$-reversible or $R$-equivariant. For sake of simplicity we use the same notation for the extension of $B_{ij}$ to $V$. By the deformation lemma (4), $\{ N^l B_{ij} \mid i, j = 1, \ldots, r \}$ and $l = 0, \ldots, \min(n_i, n_j) - 1$ is a basis of $b$. Therefore $N^l B_{ij}$ if $\rho_i \rho_j = -1$ and $N^l B_{ij}$ if $\rho_i \rho_j = 1$ form a basis of $b \cap \mathfrak{gl}_{-R}$.

**3. Hamiltonian Matrices**

Hamiltonian systems on $\mathbb{R}^m$ are defined with respect to a symplectic (nondegenerate and antisymmetric) form $\omega$. The differential equation

$$\dot{x} = F(x)$$

is called Hamiltonian if $\omega(F, \cdot)$ is a closed 1-form, see Abraham and Marsden [1]. The conditions on $\omega$ force the phasespace to be even dimensional: $m = 2n$. Since we are interested in linear Hamiltonian systems we assume that $\omega$ is a bilinear form on $\mathbb{R}^m$. Let $A$ be the matrix of the linear part of (3). Since (3) is Hamiltonian, $A$ is a Hamiltonian matrix, that is $\omega(Ax, y) + \omega(x, Ay) = 0$ for all $x, y \in \mathbb{R}^m$. Some authors use the term “infinitesimally symplectic matrix” instead of “Hamiltonian matrix.”
The characteristic polynomial \( p(t) \) of a Hamiltonian matrix contains the following factors:

\[
t^2, t^2 - \alpha^2, t^2 + \beta^2, [(t - \alpha)^2 + \beta^2] [(t + \alpha)^2 + \beta^2].
\]

The set \( \mathfrak{u} = \mathfrak{gl}_n(2n) \) of all Hamiltonian matrices is a Lie algebra. The transformation group that preserves \( \mathfrak{gl}_n(2n) \) is the corresponding Lie group \( \mathcal{G} = \text{Gl}_n(2n) = \{ g \in \text{Gl}(2n) \mid g^* \omega = \omega \} \). \( \text{Gl}_n(2n) \) is called the group of symplectic transformations.

Sometimes it is convenient to have a matrix representation of \( \omega \). Let \( \langle \cdot, \cdot \rangle \) denote the standard inner product on \( \mathbb{R}^{2n} \), then there is a skew-symmetric matrix \( J \) such that \( \omega(x, y) = \langle x, Jy \rangle \) for all \( x, y \in \mathbb{R}^{2n} \). Thus the Hamiltonian matrices and the symplectic transformations can be written as \( \mathfrak{h}_n(2n) = \{ A \in \mathfrak{gl}(2n) \mid JA + A^T J = 0 \} \) and \( \text{Gl}_n(2n) = \{ g \in \text{Gl}(2n) \mid g^* J g = J \} \), respectively.

### 3.1. Equivalence Classes of Hamiltonian Matrices

We use the classification of Hamiltonian matrices from [8]. This classification follows the general scheme of Section 1.2. We repeat some constructions inherent to Hamiltonian systems from [8], since we will need them again in the next section.

The Jordan–Chevalley decomposition clearly holds in this case because the set of Hamiltonian matrices is a Lie subalgebra of \( \mathfrak{gl}(2n) \) and the corresponding Lie group is algebraic. In the Hamiltonian case the compatible \( A \)-invariant subspaces are very similar to those in the reversible case. Namely, for Hamiltonian matrices the \( A \)-invariant spaces \( V \) must be such that the restriction of \( \omega \) to \( V \) is nondegenerate, see [8]. This means that the \( A \)-invariant subspaces must be \( J \)-invariant too. To find the \( A \), \( J \)-invariant spaces first note that \( A' \) and \( A \) have the same invariant spaces and also the same characteristic polynomial \( p(t) \). From the definition of Hamiltonian matrices we deduce that \( JV_\lambda = V_\lambda \). By the same reasoning as in Section 2, the indecomposable \( A \), \( J \)-invariant subspace \( V \) is \( V_\lambda \oplus V_{-\lambda} \) if \( \lambda \neq -\lambda \) and \( V_\lambda \) if \( \lambda = -\lambda \). The \( A \), \( J \)-invariant spaces \( V \) corresponding to different eigenvalues are \( \omega \)-orthogonal. Two spaces \( U_1 \) and \( U_2 \) are \( \omega \)-orthogonal if \( \omega(u_1, u_2) = 0 \) for every \( u_1 \in U_1 \) and \( u_2 \in U_2 \). For choosing a “good” \( S \)-invariant complement \( W \) of \( NV \) in \( V \) we have to make a small digression. The following construction is taken from [8].

As already noted in Remark 5 we not only want to find the equivalence classes but also we want to put \( A \) and \( J \) in a “nice” form. In other words we want to find a simultaneous normal form for the pair \( (A, J) \). To achieve this we use the following lemma.
LEMMA 8. By the Symplectic Gramm–Schmidt process we can choose an $S$-invariant complement $W$ of $NV$ in $V$ such that the spaces $W$, $NW$, $\ldots$, $N^{n-1}$ are mutually $o$-orthogonal except for the pairs $NW$, $N^{n-1}$ for $i = 0, \ldots, [n/2]$.

Before introducing the Symplectic Gramm–Schmidt process we first consider a bilinear form $\tau$ on $W$, where $W$ is any $S$-invariant complement to $NV$ in $V$. Define $\tau$ by

$$\tau(x, y) = \omega(x, N^{n-1}y) \quad \text{for all } x, y \in W.$$

We can always write $\tau(x, y) = (x, Ty)$ for some linear map $T$. The following properties of $\tau$ easily follow from the definition.

(a) $\tau$ is nondegenerate on $W$ or equivalently, $T$ is invertible.

(b) $\tau(x, y) = (-1)^{n} \tau(y, x)$ or equivalently, $T^{t} = (-1)^{n} T$.

(c) $\tau(Sx, y) = -\tau(x, Sy)$ or equivalently, $TS + S'T = 0$.

Here $x$ and $y$ are vectors in $W$. Thus we see that if $n$ is odd, $\tau$ is symplectic and if $n$ is even $\tau$ is symmetric on $W$. We will now use the freedom we have in choosing $W$ to make $W, NW, \ldots, N^{n-1}$ mutually $o$-orthogonal in the sense of Lemma 8. The first observation is.

LEMMA 9. If the bilinear forms $\tau_{j}(x, y) = \omega(x, N^{j}y)$ for $j = 1, \ldots, n - 1$ are identically zero on $W$, then the subspaces $N^{j}W$ are mutually $o$-orthogonal in the sense of Lemma 8.

By a procedure which resembles the Gramm–Schmidt process we can transform in finitely many steps, any $S$-invariant complement $W$ to $NV$ in $V$ into a $W$ such that all $\tau_{j}$ vanish identically on $W$ except for $j = n - 1$.

PROPOSITION 10 (Symplectic Gramm–Schmidt Process). Let $m = n - 1$ and $\tau(x, y) = \omega(x, N^{m}y)$ and $\tau_{j}(x, y) = \omega(x, N^{j}y)$, write $\tau(x, y) = (x, Ty)$ and $\tau_{j}(x, y) = (x, T_{j}y)$. Suppose $W_{k}$ is an $S$-invariant complement of $NV$ in $V$ such that $\tau_{m-k} = 0$ on $W_{k}$ for $j = 1, \ldots, k - 1$. Then $W_{k+1} = (I - \frac{1}{2}(N^{j-1} T^{t} T_{m-k}) W_{k}$ is an $S$-invariant complement of $NV$ in $V$ such that $\tau_{m-k} = 0$ on $W_{k+1}$ for $j = 1, \ldots, k$.

A nice property of this procedure is that it is $R$-equivariant, that is if $W_{k}$ is $R$-invariant then $W_{k+1}$ is also $R$-invariant. This will be very convenient in the next section. For proofs of the lemma and the proposition see [8].

Before stating a normal form theorem for semisimple Hamiltonian matrices we define the symplectic sign. The symplectic sign $\sigma$ of the $S,$
$T$-invariant space $W$, is defined as the signature of yet another bilinear form $\tau$.

$$\tau(x, y) = \begin{cases} \tau(x, y) & \text{if } \tau \text{ is symmetric} \\ \tau(Sx, y) & \text{if } \tau \text{ is symmetric} \end{cases}$$

for $x, y \in W$. Thus $\sigma$ is well defined provided $S \neq 0$. Fortunately we do not need $\sigma$ if $S = 0$ and $\tau$ symplectic. For a proof of the normal form theorem we again refer to [8] or [9].

Theorem 11. Let $T$ define a nondegenerate bilinear form on $W$. Let $S$ be semisimple on the indecomposable $S, T$-invariant space $W$. The normal forms of $S$ are listed in Table VIII.

Splitting of $\text{GL}(m)$-classes occurs for imaginary eigenvalues. The new classes can be discriminated by the symplectic sign, which we call the symplectic sign of the class. Splitting also occurs for zero eigenvalues if $\tau$ is symmetric. This happens for the matrix in Section 5 which has four zero eigenvalues in two Jordan blocks. Therefore two symplectic signs, one for each block, label the various cases.

4. REVERSIBLE HAMILTONIAN MATRICES

Reversible Hamiltonian matrices form the intersection of the linear subspace of reversible matrices and the Lie algebra of Hamiltonian matrices: $\mathfrak{gl}_d(2n) \cap \mathfrak{gl}_s(2n)$. In this case the structure preserving transformation group is the group $\text{GL}_d(2n) \cap \text{GL}_s(2n)$ of $R$-equivariant symplectic transformations. Usually, see Wan [28], one assumes that there is a relation between the symplectic form $\omega$ and the reversor $R$.

**Definition 1.** The symplectic form $\omega$ and the reversing map $R$ are called compatible if $R^*\omega = -\omega$.

**Remark 6.** In principle there need not be a relation between the symplectic form and the reversor. However, if they are not compatible in the sense of the above definition, then the $R$-reversible Hamiltonian matrices are also equivariant. Indeed, let $A$ be such a matrix, that is $A$ satisfies $A'J + JA = 0$ and $AR + RA = 0$, where $\omega(x, y) = (x, Jy)$. Then we use the relations $A' = JAJ$ and $A = -RAR$ to obtain $0 = A'J + JA = A'R'J + R'JA = AJR'JR - JR'JRA$. This means that $A$ is $JR'J$-equivariant. Now we do not want to restrict our set of matrices any further (cf. Remark 4), therefore we must have $JR'JR = sI$ or $R'JR = -sI$, with $s \in \mathbb{R}$. 
Taking the determinant on both sides we see that $s = \pm 1$. However if $s = -1$ then $A$ can have no real or imaginary eigenvalues. Thus we only consider compatible $\omega$ and $R$.

So from now on we assume that $R^*\omega = -\omega$. We call $R$ an antisymplectic transformation. For the matrix $J$ associated to $\omega$ the relation $R^*\omega = -\omega$ implies $R^*JR = -J$. Assuming that $R$ is in normal form, we can also interpret this relation as $J$ being $R$-reversible. From this last observation we conclude that $\dim(E_+) = \dim(E_-) = n$, since $J$ has no zero eigenvalues. Furthermore from the properties of $R$-reversible matrices (see Section 2) we infer that on any basis $\langle a_1, ..., a_n, b_1, ..., b_n \rangle$, $J$ takes the form

$$J = \begin{pmatrix} 0 & J_1 \\ J_2 & 0 \end{pmatrix},$$

with $J_2 = -J_1^t$ and $J_1J_1 = J_1J_1^t = I_n$.

Remark 7. In connection with Remark 5 we can consider the structure of $R$-reversible Hamiltonian matrices as a triple $(A, R, J)$ with $AR + RA = 0$, $JA + AJ = 0$ and $R^2 = I$, $J^t = J^{-1} = -J$ and $RJ + JR = 0$. The $\text{Gl}(2n)$-orbit of the triple $(A, R, J)$ is defined as $\{(g^{-1}Ag, g^{-1}Rg, g^tJg) \mid g \in \text{Gl}(2n)\}$. As in the $R$-reversible case we first normalize $R$. Then we use only $R$-equivariant transformations from $\text{Gl}(2n)$ to normalize both $A$ and $J$.

4.1. Equivalence Classes of Reversible Hamiltonian Matrices

The classifications of reversible matrices and Hamiltonian matrices contain almost all ingredients needed for the classification of reversible Hamiltonian matrices. We observe a further splitting of $\text{Gl}(m)$-classes. In fact, all $\text{Gl}(m)$-classes split into different $\mathfrak{g}$-classes. We give the normal form for semisimple reversible Hamiltonian matrices, starting from the normal form for semisimple reversible matrices. In the appendix, a normal form for the nilpotent case will be reconstructed from the semisimple case.

Although the set of reversible Hamiltonian matrices is not a Lie algebra, it is a linear subspace compatible with the Jordan–Chevalley decomposition. Next we have to find the compatible $A$-invariant subspaces $V$. Let $A$ be a reversible Hamiltonian matrix and $V_1$ be an indecomposable $A$-invariant subspace as in Section 1.2. From Sections 2 and 3 we know that $RV_1 = V_{-1}$ and $JV_1 = V_{-1}$. Thus an $A$, $R$-invariant space is also an $A$, $J$-invariant space. The indecomposable $A$, $R$, $J$-invariant subspaces $V$ are $V_1 \oplus V_{-1}$ if $\lambda \neq -\bar{\lambda}$ and $V_1$ if $\lambda = -\bar{\lambda}$. An $A$, $R$, $J$-invariant subspace $V$ is called indecomposable if $V$ can not be written as the direct sum of two proper $A$, $R$, $J$-invariant subspaces of $V$. Let $W$ be an $S$, $R$-invariant complement to $NV$ in $V$ (see Section 2). If we now apply the symplectic
Gramm–Schmidt process (see Section 3) to \( W \) then the new \( W \) is still \( S \), \( R \)-invariant and the \( W, NW, ..., N^{n-1}W \) are mutually \( \omega \)-orthogonal in the sense of Lemma 8. So we assume that this has been done.

It remains to be checked that the symplectic Gramm–Schmidt process is \( R \)-equivariant. Let \( W_k \) be an \( S, R \)-invariant complement to \( NV \) in \( V \). Let \( W_{k+1} \) and \( T \) be as in Proposition 10. Since \( R^*\omega = -\omega \) we have \( RT_j = (-1)^{j+1} T_j R \) and therefore \( RW_{k+1} = W_{k+1} \).

Now that we have the right complement \( W \) we can give the normal form and thus the classification of semisimple reversible Hamiltonian matrices \( S \) on \( W \).

**Theorem 12.** Let \( R \) be an involution on \( W \) and suppose that \( T \) defines a nondegenerate bilinear form on \( W \). Let \( S \) be a semisimple reversible Hamiltonian matrix on the indecomposable \( S, R, T \)-invariant space \( W \). The normal forms of \( S \) and \( T \) are listed in Table IX.

In the reversible Hamiltonian case all \( \text{Gl}(m) \)-classes split, except for the double zero eigenvalue when \( \tau \) is symplectic. The splittings in case of real and complex eigenvalues are new and have no reversible or Hamiltonian counterpart. The classes with purely imaginary eigenvalues split just as in the Hamiltonian case. The splitting of the class with a zero eigenvalue and \( \tau \) symmetric, is also new. Single zero eigenvalues already had a reversible sign and a symplectic sign. To label the new \( \mathcal{G} \)-classes we introduce the reversible symplectic sign \( \chi \). We define \( \chi \) with help of the symmetric bilinear form \( \nu \), see Section 3. If there is a vector \( a \in E_+ \cap W \) then \( \chi \) is the sign of \( \nu(a, a) \). If no such vector exists, we define \( \chi \) as the sign of \( \nu(b, b) \), with \( b \in E_- \cap W \). Since either \( E_+ \cap W \) or \( E_- \cap W \) is not just \( \{0\} \), \( \chi \) is well defined. The reversible symplectic sign coincides with the symplectic sign in those cases where the latter is relevant. Only in case of zero eigenvalues and \( \tau \) symmetric we also need the reversible sign to label the \( \mathcal{G} \)-classes. This will be illustrated in Section 5 for a matrix with four zero eigenvalues in two Jordan blocks.

**Proof.** Since \( S \) is a reversible matrix we can apply Theorem 5 to put \( S \) into normal form. We only have to show that we can also put \( T \) into normal form. First note that from the definition of \( \tau \) we have \( \tau(Rx, Ry) = \omega(\tau(Rx, N^{n-1}Ry)) = (-1)^n \omega(x, N^{n-1}y) = (-1)^n \tau(x, y) \). Thus if \( \tau \) is symmetric then \( RTR = T \) and therefore \( T \) is block diagonal on any basis \( \langle a_1, ..., a_k, b_1, ..., b_k \rangle \). If \( \tau \) is symplectic then \( RTR = -T \) which means that \( T \) is a reversible matrix and so is “antiblockdiagonal” on any basis \( \langle a_1, ..., a_k, b_1, ..., b_k \rangle \). For the details we proceed case by case.
(a) Real eigenvalues: \( \lambda = \pi \in \mathbb{R} \), \( W = \langle a, b \rangle \) with \( b = (1/\pi) S a \). If \( \tau \) is symmetric then \( \tau(a, b) = \tau(b, a) \). On the other hand \( \tau(a, b) = (1/\pi) \tau(a, Sa) = -(1/\pi) \tau(a, Sa) = -\tau(b, a) \) so \( \tau(a, b) = 0 \). Furthermore \( \tau(b, b) = (1/\pi^2) \tau(S a, Sa) = -(1/\pi^2) \tau(S^2 a, a) = -\tau(a, a) \). Since \( \tau \) is nondegenerate on \( W \), \( \tau(a, a) = \varepsilon \neq 0 \).

If \( \tau \) is symplectic then \( \tau(a, a) = \tau(b, b) = 0 \). Since \( \tau \) is nondegenerate on \( W \), \( \tau(a, b) = \varepsilon \neq 0 \).

(b) Imaginary eigenvalues. \( \lambda = \mathrm{i} \beta \in \mathbb{R} \). Again \( W = \langle a, b \rangle \), but now \( b = -(1/\beta) S a \). The proof is similar to that in (a) except when \( \tau \) is symmetric. Then \( \tau(b, b) = (1/\beta^2) \tau(S a, Sa) = -(1/\beta^2) \tau(S^2 a, a) = \tau(a, a) = \varepsilon \neq 0 \).

(c) Complex eigenvalues: \( \lambda = \pi + \mathrm{i} \beta \in \mathbb{C} \), \( W = \langle a_1, a_2, b_1, b_2 \rangle \). In this case we have to do slightly more work to put \( T \) into normal form.

Let us first suppose that \( \tau \) is symmetric. Then \( T_1' = T_1 \) and \( T_2' = T_2 \). From \( S'T + TS = 0 \) we readily obtain \( T_1 = -T_2 \). Now we look for a transformation \( g \) with \( g^{-1} S g = S \) and \( g^{-1} R g = R \) which puts \( T \) into normal form. Then \( g \) must be of the form

\[
g = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} \quad \text{with} \quad U^* = U^{-1}.
\]

Hence

\[
g' T g = \begin{pmatrix} U'T_1 U & 0 \\ 0 & -U'T_2 U \end{pmatrix}
\]

Since \( T_1' = T_1 \) we can find \( U \) such that \( U'T_1 U \) is diagonal. Furthermore, we have \( \tau(a_1, a_2) = \tau(a_1, a_1) \). So \( U'T_1 U = \varepsilon I_2 \) with \( \varepsilon = \tau(a_1, a_1) \). If \( \tau \) is symplectic the proof runs along the same lines, again \( T_1 = -T_2 \) and \( T_1' = T_1 \). From \( \tau(a_2, b_2) = \tau(a_2, b_1) = \varepsilon \) we conclude that \( U'T_1 U = \varepsilon I_2 \).

(d) Zero eigenvalues: \( \lambda = 0 \). Suppose \( x \in E_+ \), \( E_- \) and \( S x = 0 \). Then \( S(x + R x) = 0 \) and \( S(x - R x) = 0 \). So there are two linearly independent vectors \( a \in E_+ \) and \( b \in E_- \) with \( S a = S b = 0 \). If there is only one vector \( x \) with \( S x = 0 \) then either \( x \in E_+ \) or \( x \in E_- \).

Suppose there is only one vector \( a \in E_+ \) with \( S a = 0 \). Then \( W = \langle a \rangle \) so \( \tau \) can only be symmetric since \( \tau \) is nondegenerate on \( W \). Similarly for \( b \in E_- \).

Suppose now that \( a \in E_+ \) with \( S a = 0 \) and \( \tau \) is symplectic. Since \( \tau \) is nondegenerate on \( W \) there must be a vector \( x \) linear independent of \( a \) with \( S x = 0 \) and \( \tau(a, x) \neq 0 \). The vector \( x \) can uniquely be split \( x = x_+ + x_- \) with \( x_+ \in E_+ \) and \( x_- \in E_- \). Hence we have \( 0 \neq \tau(a, x) = \tau(a, x_+) + \tau(a, x_-) \).
Now \( \tau(a, x_+ - x_+) = \tau(Ra, Rx_+ - x_+) = -\tau(a, x_+) \) so \( \tau(a, x_+) = 0 \). Consequently \( x_+ \neq 0 \). Thus there is a \( b \in E_+ \) with \( Sb = 0 \) and \( W = \langle a, b \rangle \).

By scaling the vectors \( a \) and \( b \) we can always assume that \( \varepsilon^2 = 1 \). Concerning the signs of \( \alpha, \beta \) and \( \varepsilon \) we argue as follows. Recall that the equivalence class of \( S \) is \( \{ g^{-1}Sg \mid g^{-1}Rg = R, g^Tg = T \} \).

If \( \tau \) is symplectic we can always find a transformation \( g \) with \( gR = Rg \) and \( g^Tg = -T \). However, such a transformation also affects \( S \). In any event we can always achieve that \( g^{-1}Sg \) and \( S \) differ only by a sign. Now we fix the sign of \( S \). If \( \lambda = \alpha + i\beta \) then \( S_{\alpha, \beta} \) and \( S_{\alpha, -\beta} \) are equivalent (take \( g = P \) as in the proof of Theorem 5. Then \( PRP = R \), \( PTP = T \) and \( PS_{\alpha, \beta}P = S_{\alpha, -\beta} \)). Therefore we can assume \( \alpha > 0, \beta > 0 \) and the sign of \( \varepsilon \) labels inequivalent normal forms. If \( \lambda = 0 \) we take \( g = R \). Then \( g^Tg = T \) and \( g^{-1}Sg = RSR = -S \). But \( S = 0 \), so we can assume \( \varepsilon = 1 \).

If \( \tau \) is symmetric then there is no transformation \( g \) with \( gR = Rg \) and \( g^Tg = -T \). In this case however, \( S \) and \( -S \) are equivalent (take \( g = R \)). Also \( S_{\alpha, \beta} \) and \( S_{\alpha, -\beta} \) are equivalent (take \( g = P \)). So we can assume \( \alpha > 0 \) and \( \beta > 0 \). Again the sign of \( \varepsilon \) labels inequivalent normal forms.

4.2. Deformations of Reversible Hamiltonian Matrices

The construction of deformations of reversible Hamiltonian matrices is analogous to the construction for reversible matrices. Again the remaining problem is that of finding a basis of \( \{ B \mid \text{ad}_B(B) = 0 \} \cap \{ \mathfrak{gl}_{\pm}(2n) \cap \mathfrak{gl}(2n) \} \). Beside the reversible decomposition lemma there is also a Hamiltonian decomposition lemma.

**Lemma 13 (Hamiltonian Decomposition Lemma).** Let \( \omega \) be a symplectic form on \( \mathbb{R}^n \) and \( \omega(x, y) = (x, Jy) \). Then \( \mathfrak{gl}(2n) = \mathfrak{gl}_{-}(2n) \oplus \mathfrak{gl}_{+}(2n) \). The projections \( J_- : \mathfrak{gl}(m) \rightarrow \mathfrak{gl}_{-}(2n) \) and \( J_+ : \mathfrak{gl}(m) \rightarrow \mathfrak{gl}_{+}(2n) \) are given by \( J_-(A) = \frac{1}{2}(A - J^{-1}A^TJ) \) and \( J_+(A) = \frac{1}{2}(A + J^{-1}A^TJ) \) respectively. Moreover the decomposition is orthogonal with respect to the inner product \( (A, B) = \text{trace}(A^TB) \) on \( \mathfrak{gl}(2n) \).

Furthermore, since \( R^*\omega = -\omega \) implies \( JR + RJ = 0 \), the reversible and Hamiltonian projections \( J_- \) and \( J_+ \) commute. So \( \mathfrak{gl}(2n) \) admits the following orthogonal splitting

\[
\mathfrak{gl}(2n) = (\mathfrak{gl}_{-}(2n) \cap \mathfrak{gl}_{+}(2n)) \oplus (\mathfrak{gl}_{-}(2n) \cap \mathfrak{gl}_{-}(2n)) \\
\oplus (\mathfrak{gl}_{+}(2n) \cap \mathfrak{gl}_{-}(2n)) \oplus (\mathfrak{gl}_{+}(2n) \cap \mathfrak{gl}_{+}(2n)).
\]

Let us recall some notation. Let \( A = N + S \) be a reversible Hamiltonian matrix. Let \( V = V_1 \oplus \cdots \oplus V_r \), where each \( V_i \) is an indecomposable \( A, R \),
J-invariant subspace. For each \( V_i \) there is a splitting \( V_i = W_i \oplus N V_i \), where \( W_i \) is an \( S, R, J \)-invariant complement of \( N V_i \) in \( V_i \). The restriction \( S_i \) of \( S \) to \( W_i \) does not depend on \( i \). So all \( W_i \) have the same dimension. The restriction of \( N \) to \( V_i \) has height \( n_i \). Let \( W = W_1 \oplus \cdots \oplus W_r \) then \( V = W \oplus N V \).

**Theorem 14.** The deformations of \( A \in \mathfrak{gl}(2n) \cap \mathfrak{gl}(2n) \) are listed in Table XI.

**Proof.** We only prove the cases of real and zero eigenvalues, since the other cases are treated similarly. The proof merely follows the construction of Proposition 2.

(a) Real Eigenvalues. Let \( \lambda = \varepsilon \in \mathbb{R} \). Using the reduction lemma we restrict to \( W \). Then \( A = S = \text{diag}(S_1, \ldots, S_r) \) and \( R = \text{diag}(R_1, R_1) \) with

\[
S_1 = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} \quad \text{and} \quad R_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

On \( W \) the symplectic form \( \omega \) reduces to the bilinear form \( \tau \). The matrix \( T \) associated to \( \tau \) is \( T = \text{diag}(T_1, \ldots, T_1) \). From Table IX row 3 we read off \( T_i \),

\[
T_i = e_i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{if} \quad n_i \text{ even}, \quad \text{and} \quad T_i = e_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{if} \quad n_i \text{ odd}.
\]

In order that \( T \) and \( RTR \) are well defined on \( W \), the \( n_i \) are either even or odd for all \( i \). First we find a basis of \( \mathfrak{b}_W \), the matrices commuting with \( S_1 \). As is easily seen the matrices

\[
B^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B^{(2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

will do. According to Proposition 2, \( B^{(1)} \) and \( B^{(2)} \) form a basis of \( \mathfrak{b}_W \). For this basis we have \( \mathfrak{H}_+ (B^{(1)}_j) = B^{(1)}_j \), \( \mathfrak{H}_- (B^{(2)}_j) = B^{(1)}_j \) and \( \mathfrak{H}_+ (B^{(1)}_j) = \mathfrak{H}_- (B^{(2)}_j) = 0 \). Here \( \mathfrak{H}_+ \) and \( \mathfrak{H}_- \) are the projections of Lemma 6 restricted to \( W \). The projections \( \mathfrak{H}_\pm \) acting on matrices \( \mathcal{V} \rightarrow \mathcal{V} \), reduce to the projections \( \mathfrak{H}_\pm \) acting on matrices \( \mathcal{W} \rightarrow \mathcal{W} \). After some computations we find another basis of \( \mathfrak{b}_W \) compatible with \( \mathfrak{u} = \mathfrak{gl}(2n) \cap \mathfrak{gl}(2n) \). This basis is shown in Table I, where \( r \) means \( R \)-reversible, \( e \) means \( R \)-equivariant, \( h \) means Hamiltonian and \( a \) means anti Hamiltonian. Next we extend the basis to \( V \). We obtain the following basis for \( \mathfrak{b} \cap (\mathfrak{gl}(2n) \cap \mathfrak{gl}(2n)) \)

\[
N^{2j+1} \mathfrak{H}_+ (B^{(1)}_j), \quad N^{2j} \mathfrak{H}_- (B^{(2)}_j), \quad N^{2j+1} B^{(1)}_i, \quad N^{2j} B^{(2)}_i, \quad i < j.
\]
We used the same notation $B_{ij}^{(k)}$ for matrices defined on $W$ and their extensions to $V$.

(b) Zero eigenvalues. There are two cases of zero eigenvalues, depending on the reduced symplectic form $\tau$ on $W$. We will only consider the case that $\tau$ is symmetric. Using the reduction lemma we restrict to $W$. Then $A = S = \text{diag}(0, \ldots, 0)$, $R = \text{diag}(p_1, \ldots, p_r)$ and $T = \text{diag}(e_1, \ldots, e_r)$. The $e_i$ are the reversible symplectic signs of the spaces $V_i$ and the $p_i$ are the reversible signs. First we find a basis of $b_{\mathfrak{w}_1}$, the matrices commuting with $S_1$. There is only one such matrix namely $B_1 = 1$. According to Proposition 2 the matrices $B_{ij}$ form a basis of $b_{\mathfrak{h}_i}$. For this basis we have

$$T(B_{ij}) = B_{ij}^T, \quad R(B_{ij}) = B_{ij}^R.$$ 

Since the projections $T_\tau$ and $R_\tau$ commute we can combine the previous results, see Table II. After extending the basis to $V$ we obtain the following basis for $b \cap (\mathfrak{gl}(2n) \cap \mathfrak{gl}_r(2n))$

$$N^{2j+1}f_+(B_{ij}), \quad N^{2j}f_-(B_{ij}), \quad N^{2j+1}B_{ii}, \quad i < j.$$ 

We used the same notation $B_{ij}^{(k)}$ for matrices defined on $W$ and their extensions to $V$.

### TABLE I

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$r_+(B_{ij}^{(1)})$</th>
<th>$r_-(B_{ij}^{(2)})$</th>
<th>$s_+(B_{ij}^{(1)})$</th>
<th>$s_-(B_{ij}^{(2)})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>$e, a$</td>
<td>$r, a$</td>
<td>$e, a$</td>
<td>$0$</td>
</tr>
<tr>
<td>-</td>
<td>$e, h$</td>
<td>$r, h$</td>
<td>$0$</td>
<td>$r, h$</td>
</tr>
</tbody>
</table>

### TABLE II

<table>
<thead>
<tr>
<th>$\rho_i\rho_j$</th>
<th>$f_+(B_{ij})$</th>
<th>$f_-(B_{ij})$</th>
<th>$B_{ii}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+1$</td>
<td>$e, a$</td>
<td>$e, h$</td>
<td>$e, a$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$r, a$</td>
<td>$r, h$</td>
<td>$r, a$</td>
</tr>
</tbody>
</table>
5. Illustration

The example given in this section not only illustrates the method of finding a normal form and its versal unfolding, but also shows the importance of the signs.

Let us consider the following nilpotent matrix

\[ A = N = \begin{pmatrix} N_1 & 0 \\ 0 & N_1 \end{pmatrix} \quad \text{with} \quad N_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

on \( V = \mathbb{R}^4 \). We take \( \langle e, Ne, f, Nf \rangle \) as a basis of \( V \), where \( e \) and \( f \) are linearly independent vectors in \( V \). \( A \) consists of two Jordan blocks with zero eigenvalues. Restricted to either block the nilpotent part has height two. As can be read off from Tables V, VIII, and IX this matrix is the simplest example for which the reversible, symplectic and reversible symplectic signs are all relevant. Namely we consider the “collision” of zero eigenvalues depending on the signs. Then \( \tau \) must be symmetric, so the simplest case is \( N^2 = 0 \).

We already know that the signs label inequivalent normal forms. Now we wish to show that the deformations of inequivalent normal forms are qualitatively different. In this example we study the collision of zero eigenvalues with equal or different signs. Compare example 1 of the 1:1 and the 1:–1 resonance in Section 1.1. With the involution

\[ R = \begin{pmatrix} \rho_1 R_1 & 0 \\ 0 & \rho_2 R_1 \end{pmatrix} \quad \text{with} \quad R_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

the matrix \( A \) is \( R \)-reversible. \( \rho_1 \) and \( \rho_2 \) are the reversible signs of the first and second block of \( A \). In other words \( e \in E_\rho_1 \) and \( f \in E_\rho_2 \). For the linear case the overall sign of \( R \) is unimportant. Namely, if \( A \) is \( R \)-reversible then \( A \) is also \(-R\)-reversible. Therefore we set \( \rho_1 = 1 \) and \( \rho_2 = \rho \). Note that for nonlinear vector fields we can not ignore the overall sign of \( R \). For if the equation \( x^* = f(x) \) is \( R \)-reversible, it will in general not be \((-R)\)-reversible.

The symplectic form \( \omega \) defined by the matrix \( J \)

\[ J = \begin{pmatrix} e_1 J_1 & 0 \\ 0 & e_2 J_1 \end{pmatrix} \quad \text{with} \quad J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]

satisfies the relation \( R^* \omega = -\omega \) of Definition 1. \( A \) is Hamiltonian with respect to \( \omega \). Now \( e_1 \) and \( e_2 \) are the symplectic signs of the first and the second block. That is \( \omega(e, Ne) = e_1 \) and \( \omega(f, Nf) = e_2 \). For Hamiltonian systems the overall sign of \( \omega \) is unimportant, therefore we set \( e_1 = 1 \) and
Thus $A$ is also reversible Hamiltonian matrix. From Tables V, VIII, and IX we deduce that $A$ is already in normal form.

The first part of the construction of a versal deformation of $A$ is equal for all possible structures. On the reduced space $W = NV$ with basis $\langle e, f \rangle$, we have $S = \text{diag}(S_1, S_1)$ with $S_1 = 0$. Using the first row of Table XI we take the following in $b_W$:

$$\langle B_1, B_2, B_3, B_4 \rangle = \langle B_{11} + B_{22}, B_{12} - B_{21}, B_{13} + B_{24}, B_{14} - B_{23} \rangle$$

$$= \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right).$$

Now we extend $B_1, B_2, B_3$ and $B_4$ to $V$. For example, we define $B_1 Ne$ as $N B_1 e$. According to the deformation lemma (4)

$$\langle B_1, B_2, B_3, B_4, NB_1, NB_2, NB_3, NB_4 \rangle$$

is a basis of $b$. In Table III we summarize the properties of the basis vectors. The final step is to find a basis of $b \cap u$, where $u$ is the space of reversible, Hamiltonian or reversible Hamiltonian matrices. We treat each case separately. Let $\mathcal{H}$ be the isotropy group of $b \cap u$, see Remark 1.

$A$ is $R$-reversible. The basis of $b_W$ is such that each basis vector is either $R$-reversible or $R$-equivariant. Using this property of the basis of $b_W$, a basis of $b \cap \mathfrak{gl}_d(4)$ is

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\langle NB_1, NB_2, NB_3, NB_4 \rangle$</td>
</tr>
<tr>
<td>-1</td>
<td>$\langle NB_1, NB_2, B_3, B_4 \rangle$</td>
</tr>
</tbody>
</table>

Thus we obtain two different versal deformations of $A$ depending on the reversible sign. Only if $\rho = 1$ we can effectively reduce the number of

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\varepsilon$</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
<th>$B_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$e, a$</td>
<td>$e, a$</td>
<td>$e, a$</td>
<td>$e, h$</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>$e, a$</td>
<td>$e, a$</td>
<td>$e, h$</td>
<td>$a, a$</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>$e, a$</td>
<td>$e, a$</td>
<td>$r, a$</td>
<td>$r, h$</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>$e, a$</td>
<td>$e, a$</td>
<td>$r, h$</td>
<td>$r, a$</td>
</tr>
</tbody>
</table>
parameters by one. In that case there is a transformation in $\mathcal{H}$ which maps $NB_3$ into a linear combination of $NB_1$ and $NB_2$.

*A is Hamiltonian.* Fortunately, the basis vectors of $b_w$ are also either Hamiltonian or anti Hamiltonian. Again we can quite easily find a basis of $b \cap \mathfrak{gl}_c(4)$.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\langle NB_1, NB_2, NB_4, B_3 \rangle$</td>
</tr>
<tr>
<td>-1</td>
<td>$\langle NB_1, NB_2, B_3, NB_4 \rangle$</td>
</tr>
</tbody>
</table>

If $\varepsilon = 1$ there is a transformation in $\mathcal{H}$ which maps $NB_3$ into a linear combination of $NB_1$ and $NB_2$, thereby reducing the number of parameters by one.

*A is reversible Hamiltonian.* In both the reversible case and the Hamiltonian case, the codimension of the orbit, that is the dimension of the transverse section, equals four. For eigenvalues zero, both the reversible sign $\rho$ and the reversible symplectic sign are necessary to label the normal forms. Since the latter coincides with the symplectic sign we use $\varepsilon$ to label the classes. Thus we are left with four different cases that illustrate the essential differences. Using Table III it is easy to select a basis of $b \cap (\mathfrak{gl}_c(4) \cap \mathfrak{gl}_w(4))$, namely

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\varepsilon$</th>
<th>Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$\langle NB_1, NB_3, NB_4 \rangle$</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>$\langle NB_1, NB_2, NB_4 \rangle$</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>$\langle NB_1, NB_2, B_3 \rangle$</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>$\langle NB_1, NB_2, B_4 \rangle$</td>
</tr>
</tbody>
</table>

Reduction of the number of parameters is possible only if $\rho = 1$. For example, the versal deformation of $A$ for $\rho = -1$ and $\varepsilon = +1$ is given by

$$A(\mu) = \begin{pmatrix} 0 & \mu_1 + \mu_2 & -\mu_4 & 0 \\ 1 & 0 & 0 & -\mu_4 \\ +\mu_4 & 0 & 0 & \mu_1 - \mu_2 \\ 0 & +\mu_4 & 1 & 0 \end{pmatrix}$$

Let us summarize the results of the example in this section in a table. Table IV schematically shows the splitting of the $\text{Gl}(4)$-orbit of $A$. We list the normal forms together with the codimension of the deformation, depending on the structure present in the system. Note that when the linear
TABLE IV

Normal Forms and Deformations of \( A \) Depending on the Structure. \( \rho \) Is the Reversible Sign of the Second Block of \( A \). \( \varepsilon \) Is the Symplectic Sign of the Second Block of \( A \). \( c \) Is the Codimension of the Orbit. The Number of Deformation Parameters Is \( c \), in General, but \( c - 1 \) for the Cases \( \rho = 1 \) and \( \varepsilon = 1 \).

<table>
<thead>
<tr>
<th>( \mathfrak{gl}(4) )</th>
<th>( \mathfrak{gl}_-(4) )</th>
<th>( \mathfrak{gl}<em>-(4) \cap \mathfrak{gl}</em>+(4) )</th>
<th>( \mathfrak{gl}_+(4) )</th>
<th>( \mathfrak{gl}(4) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho = +1, \varepsilon = +1, c = 4 )</td>
<td>( \rho = +1, \varepsilon = +1, c = 3 )</td>
<td>( \varepsilon = +1, c = 4 )</td>
<td>( c = 8 )</td>
<td></td>
</tr>
<tr>
<td>( c = 8 )</td>
<td>( \rho = -1, \varepsilon = +1, c = 3 )</td>
<td>( \varepsilon = -1, c = 4 )</td>
<td>( \rho = -1, \varepsilon = -1, c = 3 )</td>
<td></td>
</tr>
</tbody>
</table>

classification is a subproblem of a nonlinear study, one should take into account both reversible signs \( \rho_1 \) and \( \rho_2 \).

APPENDIX

To complete the description of the real normal forms we still have to do the nilpotent case. Fortunately we can easily reconstruct the matrix \( A \) on

TABLE V

Normal Forms for Semisimple Reversible Matrices. \( p(t) \) Is the Characteristic Polynomial of \( S \) Restricted to \( W \). \( \rho \) Is the Reversible Sign Associated to the Eigenvalues of \( S \). The Reversible Sign only Applies to Zero Eigenvalues.

<table>
<thead>
<tr>
<th>( p(t) )</th>
<th>Basis of ( W )</th>
<th>( \rho )</th>
<th>( S )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ( t )</td>
<td>( \langle a \rangle )</td>
<td>+1</td>
<td>0</td>
</tr>
<tr>
<td>2 ( t^2 - x^2 ), ( \alpha &gt; 0 )</td>
<td>( \langle a, b \rangle )</td>
<td>n.a.</td>
<td>( \begin{pmatrix} 0 &amp; \alpha \ \alpha &amp; 0 \end{pmatrix} )</td>
</tr>
<tr>
<td>3 ( t^2 + \beta^2 ), ( \beta &gt; 0 )</td>
<td>( \langle a, b \rangle )</td>
<td>n.a.</td>
<td>( \begin{pmatrix} 0 &amp; \beta \ -\beta &amp; 0 \end{pmatrix} )</td>
</tr>
<tr>
<td>4 ( \left( t - \alpha t^2 + \beta^2 \right) \times \left( t + \alpha t^2 + \beta^2 \right) ), ( \alpha &gt; 0, \beta &gt; 0 )</td>
<td>( \langle a_1, a_2, b_1, b_2 \rangle )</td>
<td>n.a.</td>
<td>( \begin{pmatrix} 0 &amp; \alpha &amp; \beta \ \alpha &amp; -\beta &amp; \alpha \ -\beta &amp; \alpha &amp; 0 \end{pmatrix} )</td>
</tr>
</tbody>
</table>
TABLE VI
Normal Forms for Reversible Matrices Restricted to an Indecomposable A, R-invariant Subspace V. Numbers Correspond to Those of Table V. For All Cases $A = (a_i, x_i) \text{ and } C_2 = (\xi_2, \tilde{z})$. The Example of Iooss [19] Mentioned in Section 2 Corresonds to Row 1a with $k = 1$.

<table>
<thead>
<tr>
<th>$p(t)$</th>
<th>dim. of $V$</th>
<th>Basis of $V$</th>
<th>$p$</th>
<th>$A_1$</th>
<th>$A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a</td>
<td>$t$</td>
<td>$2k$</td>
<td>$\langle a, N^2a, \ldots, N^{2k-2}a \rangle$</td>
<td>+1</td>
<td>$\begin{pmatrix} 0 \ 1 \ \vdots \ 1 \end{pmatrix}$</td>
</tr>
<tr>
<td>1b</td>
<td>$t$</td>
<td>$2k + 1$</td>
<td>$\langle a, N^2a, \ldots, N^{2k}a \rangle$</td>
<td>+1</td>
<td>$\begin{pmatrix} 0 \ 1 \ \vdots \ 1 \end{pmatrix}$</td>
</tr>
<tr>
<td>2</td>
<td>$\tilde{t}^2 - \sigma^2$, $\sigma &gt; 0$</td>
<td>irr.</td>
<td>$\langle a, Nh, N^2a, \ldots, b, Na, N^2b, \ldots \rangle$</td>
<td>n.a.</td>
<td>$\begin{pmatrix} \sigma \ 1 \ \vdots \ 1 \end{pmatrix}$</td>
</tr>
<tr>
<td>3</td>
<td>$\tilde{t}^2 + \beta^2$, $\beta &gt; 0$</td>
<td>irr.</td>
<td>$\langle a, Nh, -N^2a, N^2h, \ldots, b, -Na, -N^2b, N^2a, \ldots \rangle$</td>
<td>n.a.</td>
<td>$\begin{pmatrix} \beta \ 1 \ \vdots \ 1 \end{pmatrix}$</td>
</tr>
<tr>
<td>4</td>
<td>$[(t-x)^2 + \beta^2]$ $\times [(t-x)^2 + \beta^2]$, $x &gt; \beta &gt; 0$</td>
<td>irr.</td>
<td>$\langle a_1, a_2, Nh_1, Nh_2, N^2a_1, N^2a_2, \ldots, b_1, b_2, Na_1, Na_2, N^2b_1, N^2b_2, \ldots \rangle$</td>
<td>n.a.</td>
<td>$\begin{pmatrix} C_2 \ I_2 \ \vdots \ I_2 \end{pmatrix}$</td>
</tr>
</tbody>
</table>
from the matrix $S$ on $W$. We only have to permute the basis vectors in order to get a basis of the form $\langle a_1, \ldots, a_n, b_1, \ldots, b_n \rangle$. The equivalence classes already follow from the semisimple case by the reduction procedure which implies that the signs found in the semisimple case (Table $\text{V}$), immediately carry over to the nilpotent case.

As mentioned before there is no agreement on what "the" Jordan normal form of a matrix should be. However, the splitting $V = \bigoplus_{r=1}^{n} N^r W$ suggests the following choice. If $\langle e_1, \ldots, e_k \rangle$ is a basis of $W$ then we take

$$\langle e_1, \ldots, e_k, Ne_1, \ldots, N^{n-1} e_1, \ldots, N^{n-1} e_k \rangle$$

as a basis for $V$. Note that this basis need not be of the form $\langle a_1, \ldots, a_n, b_1, \ldots, b_n \rangle$, therefore some permutations are needed.

Using the results of Theorem 5 we readily obtain the normal forms for reversible matrices. We list the normal forms in Table VI.

From Theorem 12 we obtain the normal forms for reversible Hamiltonian matrices (Table VIII). Due to our choice of basis we do not get the familiar form of $J$. This can be remedied by reversing the order of the second half of the basis vectors, but then the normal form of $A$ becomes less "nice". The result is in Table X.

**TABLE VII**

Deformations of Normal Forms of Reversible Matrices. The Indices $i$ and $j$ Run from 1 to $r$, the Number of Jordan Blocks of $A$. On Each Indecomposable $A$, $R$-invariant Subspace $V_i$, the Height of $N$ Is $n_i$, $l$ Must Be Such That $1 \leq 2l < \min(n_1, n_j)$ or $1 \leq 2l + 1 < \min(n_1, n_j)$. $I_2$ Is a $2 \times 2$ Identity Matrix and $J_2$ Is the $2 \times 2$ Matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

<table>
<thead>
<tr>
<th>$\rho(t)$</th>
<th>Basis of $b_W$</th>
<th>Basis of $b \cap gl \ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 $t$</td>
<td>$B = 1$</td>
<td>$N^{2^n+1} B_i$, if $\rho_i \neq -1$</td>
</tr>
<tr>
<td>2 $t^2 - \alpha^2$, $\alpha &gt; 0$</td>
<td>$B^{(1)} = \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix}$, $B^{(2)} = \begin{pmatrix} 0 &amp; 1 \ 1 &amp; 0 \end{pmatrix}$</td>
<td>$N^{2^n+1} B^{(1)}_i$, $N^{2^n} B^{(2)}_i$</td>
</tr>
<tr>
<td>3 $t^2 + \beta^2$, $\beta &gt; 0$</td>
<td>$B^{(1)} = \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix}$, $B^{(2)} = \begin{pmatrix} 0 &amp; 1 \ -1 &amp; 0 \end{pmatrix}$</td>
<td>$N^{2^n+1} B^{(1)}_i$, $N^{2^n} B^{(2)}_i$</td>
</tr>
<tr>
<td>$[(t + \alpha)^2 + \beta^2]$, $\alpha &gt; 0$, $\beta &gt; 0$</td>
<td>$B^{(1)} = \begin{pmatrix} I_2 &amp; 0 \ 0 &amp; I_2 \end{pmatrix}$, $B^{(2)} = \begin{pmatrix} I_2 &amp; 0 \ 0 &amp; I_2 \end{pmatrix}$</td>
<td>$N^{2^n+1} B^{(1)}_i$, $N^{2^n+1} B^{(2)}_i$, $N^{2^n} B^{(3)}_i$, $N^{2^n} B^{(4)}_i$</td>
</tr>
</tbody>
</table>

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Codes: 2908 Signs: 1835 Length: 45 pic 190 pts
### Table VIII

Normal Forms for Semisimple Hamiltonian Matrices. \( \tau \) is the Reduced Bilinear Form and \( T \) is the Matrix Representation of \( \tau \). \( \sigma \) is the Symplectic Sign which Applies only to Purely Imaginary Eigenvalues and to Zero Eigenvalues, if \( \tau \) is Symmetric. In all cases \( \varepsilon \) is scaled so that \( \varepsilon^2 = 1 \).

<table>
<thead>
<tr>
<th>( p(t) )</th>
<th>Basis of ( W )</th>
<th>( \tau )</th>
<th>( T )</th>
<th>( \sigma )</th>
<th>( S )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \langle e \rangle )</td>
<td>Symm</td>
<td>( \varepsilon )</td>
<td>( \varepsilon )</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>( r^2 )</td>
<td>( \langle e, f \rangle )</td>
<td>( \begin{pmatrix} 0 &amp; 1 \ -1 &amp; 0 \end{pmatrix} )</td>
<td>n.a.</td>
<td>( \begin{pmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{pmatrix} )</td>
</tr>
<tr>
<td>3</td>
<td>( r^2 - x^2 ), ( x &gt; 0 )</td>
<td>( \langle e, f \rangle )</td>
<td>( \begin{pmatrix} 0 &amp; 1 \ 1 &amp; 0 \end{pmatrix} )</td>
<td>n.a.</td>
<td>( \begin{pmatrix} x &amp; 0 \ 0 &amp; -x \end{pmatrix} )</td>
</tr>
<tr>
<td>4</td>
<td>( r^2 + \beta^2 ), ( \beta &gt; 0 )</td>
<td>( \langle e, -\frac{1}{\beta} Se \rangle )</td>
<td>( \begin{pmatrix} \varepsilon &amp; 0 \ 0 &amp; \varepsilon \end{pmatrix} )</td>
<td>( \varepsilon )</td>
<td>( \begin{pmatrix} 0 &amp; \beta \ -\beta &amp; 0 \end{pmatrix} )</td>
</tr>
<tr>
<td>5</td>
<td>( [(t-x)^2 + \beta^2] \times [(t+x)^2 + \beta^2] ), ( x &gt; 0, \beta &gt; 0 )</td>
<td>( \langle f, -\frac{1}{\beta} (S + x) f \rangle )</td>
<td>( \begin{pmatrix} 0 &amp; I_x \ I_x &amp; 0 \end{pmatrix} )</td>
<td>n.a.</td>
<td>( \begin{pmatrix} x &amp; \beta &amp; 0 \ -\beta &amp; -x &amp; 0 \ 0 &amp; 0 &amp; \beta \end{pmatrix} )</td>
</tr>
</tbody>
</table>
**TABLE IX**

<table>
<thead>
<tr>
<th>$p(t)$</th>
<th>Basis of $W$</th>
<th>$\tau$</th>
<th>$T$</th>
<th>$\epsilon$</th>
<th>$\rho$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$t$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\langle a \rangle$</td>
<td>Symm</td>
<td>$\epsilon$</td>
<td>$\epsilon$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$\langle b \rangle$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$t^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\langle a, b \rangle$</td>
<td>Symp</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{pmatrix} 0 &amp; 1 \ -1 &amp; 0 \end{pmatrix}$</td>
<td>n.a.</td>
<td>n.a.</td>
<td>$\begin{pmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{pmatrix}$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$t^2 + \beta^2$, $\beta &gt; 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\langle a, b \rangle$</td>
<td>Symm</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{pmatrix} \epsilon &amp; 0 \ 0 &amp; -\epsilon \end{pmatrix}$</td>
<td>$\epsilon$</td>
<td>n.a.</td>
<td>$\begin{pmatrix} 0 &amp; \beta \ \epsilon &amp; 0 \end{pmatrix}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{pmatrix} 0 &amp; \epsilon \ -\epsilon &amp; 0 \end{pmatrix}$</td>
<td>$-\epsilon$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$t^2 + \beta^2$, $\beta &gt; 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\langle a, b \rangle$</td>
<td>Symm</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{pmatrix} \epsilon &amp; 0 \ 0 &amp; 0 \end{pmatrix}$</td>
<td>$\epsilon$</td>
<td>n.a.</td>
<td>$\begin{pmatrix} 0 &amp; \beta \ -\epsilon &amp; 0 \end{pmatrix}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{pmatrix} 0 &amp; \epsilon \ -\epsilon &amp; 0 \end{pmatrix}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$[(t-x)^2 + \beta^2] \times [(t+x)^2 + \beta^2]$, $\alpha &gt; 0$, $\beta &gt; 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\langle a_1, a_2, b_1, b_2 \rangle$</td>
<td>Symp</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{pmatrix} \alpha &amp; \beta \ -\beta &amp; \alpha \end{pmatrix}$</td>
<td>n.a.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{pmatrix} 0 &amp; \alpha \ -\beta &amp; 0 \end{pmatrix}$</td>
<td>$\epsilon$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{pmatrix} 0 &amp; \beta \ -\epsilon &amp; 0 \end{pmatrix}$</td>
<td>$\epsilon$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Normal Forms for Semisimple Reversible Hamiltonian Matrices. $p(t)$ is the Characteristic Polynomial of $S$ Restricted to $W$. $\tau$ is the Reduced Bilinear Form and $T$ is the Matrix Representation of $\tau$. $\rho$ is the Reversible Symplectic Sign. It Does Not Apply to Zero Eigenvalues, if $\tau$ is Symplectic. Only for Zero Eigenvalues and $\tau$ Symmetric, We Also Need the Reversible Sign $\rho$ to Label the Classes. For the Other Classes the Reversible Sign $\rho$ is Irrelevant. In All Cases $\epsilon$ Is Scaled so That $\epsilon^2 = 1$.  

$\langle \cdot, \cdot \rangle$ Basis of $W$: 

- $\langle a \rangle$: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ 
- $\langle b \rangle$: $\begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix}$ 
- $\langle a, b \rangle$: $\begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix}$ 
- $\langle a_1, a_2, b_1, b_2 \rangle$: $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ 

$\alpha$, $\beta$, and $\epsilon$ are real numbers.
### TABLE X
Normal Forms for Reversible Hamiltonian Matrices Restricted to an Indecomposable $A, R, J$-invariant Subspace $V$. Numbers Correspond to Those of Table IX. For All Cases $A = \begin{pmatrix} 0 & A_2 \\ A_1 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & P_1 \\ -P_1 & 0 \end{pmatrix}$. Furthermore, $P_1 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ and $C_2 = \begin{pmatrix} -1 & 0 \end{pmatrix}$.

<table>
<thead>
<tr>
<th>$p(t)$</th>
<th>Basis of $V$</th>
<th>$\tau$</th>
<th>$\chi$</th>
<th>$\rho$</th>
<th>$A_1$</th>
<th>$A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1a$</td>
<td>$t$</td>
<td>$\langle a, N^2 a, \ldots, N^{n-2} a \rangle$</td>
<td>Symm</td>
<td>$\varepsilon$</td>
<td>$+1$</td>
<td>\begin{pmatrix} 0 \ 1 \ 1 \end{pmatrix}</td>
</tr>
<tr>
<td>$1b$</td>
<td>$t$</td>
<td>$\langle b, N^2 b, \ldots, N^{n-2} b \rangle$</td>
<td>Symm</td>
<td>$\varepsilon$</td>
<td>$-1$</td>
<td>\begin{pmatrix} 1 \ \vdots \ 1 \end{pmatrix}</td>
</tr>
<tr>
<td>$2$</td>
<td>$t^2$</td>
<td>$\langle a, N b, N^2 a, \ldots, N^2 b, \ldots \rangle$</td>
<td>Symp</td>
<td>n.a.</td>
<td>n.a.</td>
<td>\begin{pmatrix} 0 \ 1 \ 1 \end{pmatrix}</td>
</tr>
</tbody>
</table>
### Reversible and Hamiltonian Linear Systems

<table>
<thead>
<tr>
<th>Symm</th>
<th>$\xi$</th>
<th>$\eta$</th>
<th>$\zeta$</th>
<th>$\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symm</td>
<td>$\zeta$</td>
<td>n.a.</td>
<td>n.a.</td>
<td>n.a.</td>
</tr>
<tr>
<td>Symm</td>
<td>$\eta$</td>
<td>n.a.</td>
<td>n.a.</td>
<td>n.a.</td>
</tr>
<tr>
<td>Symm</td>
<td>$\xi$</td>
<td>n.a.</td>
<td>n.a.</td>
<td>n.a.</td>
</tr>
</tbody>
</table>

#### Equations

3. \[ \frac{p^2 - \pi^2}{\pi > 0} \]

4. \[ \frac{p^2 + \beta^2}{\beta > 0} \]

5. \[ \frac{\pi^2 - \alpha^2 + \beta^2}{\pi > 0, \beta > 0} \]

\[ (\alpha, \beta, \gamma, \delta, \epsilon, \zeta) \]
### TABLE XI

Deformations of Normal Forms of Reversible Hamiltonian Matrices. The Indices $i$ and $j$ run from 1 to $r$, the number of Jordan Blocks of $A$. If the Indices $i$ and $j$ Both Appear then $i < j$. On Each Indecomposable $A, R, J$-invariant Subspace $V_i$ the Height of $N$ is $n_i$. $I$ Must Be Such That $1 \leq 2i < \min(n_1, n_j)$ or $1 \leq 2i + 1 < \min(n_1, n_j)$. $I_2$ is a $2 \times 2$ Identity Matrix and $J_2$ Is the $2 \times 2$ Matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

<table>
<thead>
<tr>
<th>$p(t)$</th>
<th>$\tau$</th>
<th>Basis of $b_{R_1}$</th>
<th>Basis of $b_{\cap g_{1-R}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$t$</td>
<td>Symm $B^{(1)} = 1$</td>
<td>$N^{2i+1} \cdot B^{(1)}<em>1$, $N^{2i+1} \cdot f</em>+ (B^{(1)}_1)$ if $p_1, p_2 = 1$</td>
</tr>
<tr>
<td></td>
<td>$r^2$</td>
<td>Symp $B^{(1)} = \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix}$, $B^{(2)} = \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix}$</td>
<td>$N^{2i+1} \cdot f_+ (B^{(1)}<em>1), N^{2i+1} \cdot f</em>+ (B^{(2)}<em>1), N^{2i+1} \cdot f</em>+ (B^{(3)}<em>1), N^{2i+1} \cdot f</em>+ (B^{(4)}_1)$</td>
</tr>
<tr>
<td>3</td>
<td>$r^2 - \pi^2$, $\pi &gt; 0$</td>
<td>n.a. $B^{(1)} = \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix}$, $B^{(2)} = \begin{pmatrix} 0 &amp; 1 \ 1 &amp; 0 \end{pmatrix}$</td>
<td>$N^{2i+1} \cdot f_+ (B^{(1)}<em>1), N^{2i+1} \cdot f</em>+ (B^{(2)}<em>1), N^{2i+1} \cdot f</em>+ (B^{(3)}<em>1), N^{2i+1} \cdot f</em>+ (B^{(4)}_1)$</td>
</tr>
<tr>
<td>4</td>
<td>$r^2 + \beta^2$, $\beta &gt; 0$</td>
<td>Symm $B^{(1)} = \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix}$, $B^{(2)} = \begin{pmatrix} 0 &amp; 1 \ 1 &amp; 0 \end{pmatrix}$</td>
<td>$N^{2i+1} \cdot f_+ (B^{(1)}<em>1), N^{2i+1} \cdot f</em>+ (B^{(2)}<em>1), N^{2i+1} \cdot f</em>+ (B^{(3)}<em>1), N^{2i+1} \cdot f</em>+ (B^{(4)}_1)$</td>
</tr>
<tr>
<td>5</td>
<td>$[(t-\pi)^2 + \beta^2]$</td>
<td>n.a. $B^{(1)} = \begin{pmatrix} I_2 &amp; 0 \ 0 &amp; I_2 \end{pmatrix}$, $B^{(2)} = \begin{pmatrix} J_2 &amp; 0 \ 0 &amp; J_2 \end{pmatrix}$</td>
<td>$N^{2i+1} \cdot f_+ (B^{(1)}<em>1), N^{2i+1} \cdot f</em>+ (B^{(2)}<em>1), N^{2i+1} \cdot f</em>+ (B^{(3)}<em>1), N^{2i+1} \cdot f</em>+ (B^{(4)}<em>1), N^{2i+1} \cdot f</em>+ (B^{(1)}<em>1), N^{2i+1} \cdot f</em>+ (B^{(2)}_1)$</td>
</tr>
<tr>
<td></td>
<td>$\times [(t+\pi)^2 + \beta^2]$, $\pi &gt; 0$, $\beta &gt; 0$</td>
<td>n.a. $B^{(1)} = \begin{pmatrix} I_2 &amp; 0 \ 0 &amp; I_2 \end{pmatrix}$, $B^{(2)} = \begin{pmatrix} J_2 &amp; 0 \ 0 &amp; J_2 \end{pmatrix}$</td>
<td>$N^{2i+1} \cdot f_+ (B^{(1)}<em>1), N^{2i+1} \cdot f</em>+ (B^{(2)}<em>1), N^{2i+1} \cdot f</em>+ (B^{(3)}<em>1), N^{2i+1} \cdot f</em>+ (B^{(4)}<em>1), N^{2i+1} \cdot f</em>+ (B^{(1)}<em>1), N^{2i+1} \cdot f</em>+ (B^{(2)}_1)$</td>
</tr>
</tbody>
</table>
ACKNOWLEDGMENTS

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REFERENCES


