Available online at www.sciencedirect.com
SCIENCE @ DIRECT•
Computers and Mathematics with Applications 50 (2005) 145–155
An International Journal computers & mathematics with applications
www.elsevier com/locate/camwa

Oscillation for a Class of Neutral Parabolic Differential Equations

ZIGEN OUYANG
Department of Mathematics, Shanghai University
Shanghai 200436, P.R China
and
Department of Mathematics, Nanhua University
Hengyang 421001, P.R China
Ouyangzigen@hotmail.com

SHENGFAN ZHOU AND FUQI YIN
Department of Mathematics, Shanghai University
Shanghai 200436, P.R. China

(Received July 2004; revised and accepted December 2004)

Abstract—Some sufficient conditions are established for the oscillation of a class of neutral parabolic differential equations of the form,

$$\frac{\partial^N\left(u(x,t)-\sum_{k=1}^{r}\lambda_k u(x,t-\rho_k)\right)}{\partial t^N}-a(t)\Delta u+\sum_{i=1}^{n}p_i(x,t)\,u(x,t-\sigma_i)-\sum_{j=1}^{m}q_j(x,t)\,u(x,t-\tau_j)$$
$$+h(t)\,f(u(x,t-r_1),\ \ ,u(x,t-r_l))=0,\qquad (x,t)\in\Omega\times[t_0,+\infty)\equiv G,\quad t_0\in R^+,$$

where N is an odd number, Ω is a bounded domain in R^M with a smooth boundary $\partial\Omega$, and $\triangle$ is the Laplacian operation with three different boundary conditions. We obtained some new oscillatory conditions for the odd-order neutral parabolic differential equation. To some extent, our results are new oscillatory conditions, and extended some oscillatory results of some references

Keywords—Oscillation, Neutral, Parabolic, Differential equation, Eventually positive solution.

1. INTRODUCTION

Consider the following neutral parabolic differential equation,

$$\frac{\partial^N\left(u(x,t)-\sum_{k=1}^{r}\lambda_k u(x,t-\rho_k)\right)}{\partial t^N}-a(t)\Delta u+\sum_{i=1}^{n}p_i(x,t)\,u(x,t-\sigma_i)-\sum_{j=1}^{m}q_j(x,t)\,u(x,t-\tau_j)\tag{1.1}$$
$$+h(t)\,f(u(x,t-r_1),\dots,u(x,t-r_l))=0,\qquad (x,t)\in\Omega\times[t_0,+\infty)\equiv G,\quad t_0\in R^+,$$

Supported by the Natural Science Foundation of China under Grant Nos 10171072, 10471086
I would like to express my indebtedness to the referees, whose searching questions helped me elaborate and improve my paper

0898-1221/05/$ - see front matter Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX
doi 10.1016/j camwa 2004 12.013

where N is an odd number, $n \le m$, a, p_i, q_j are nonnegative coefficients representing the phenomena which underline the diffusion process, $i = 1, \ldots, n$, $j = 1, \ldots, m$. For example, in population dynamics the term $a\triangle u$ corresponds to diffusion due to local concentration, while $p_i u$ and $q_j u$, correspond to death and birth rates, and f is nonlinear item, respectively, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

We'll assume throughout this paper that the following hold.

(H_1) $a \in C([t_0, \infty), R^+)$, $\rho_k \in [0, \infty)$, p_i, $q_i \in C(R^M \times [t_0, \infty), R^+)$, σ_i, τ_j, $r_{k'} \in [0, \infty)$, $t_0 \in R$ for $k = 1, \ldots, r$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, and $k' = 1, \ldots, l$.

(H_2) There exists a partition of the set $\{1, \ldots, m\}$ into n disjoint subsets $J_1, J_2, \ldots, J_n$, such that $j \in J_i$ implies that $\tau_j \le \sigma_i$, $i = 1, \ldots, n$.

(H_3) $P_i(t) = \min_{x \in \Omega} p_i(x, t)$ and $Q_j(t) = \max_{x \in \Omega} q_j(x, t)$ satisfy the condition,

$$P_i(t) \ge \sum_{k \in J_i} Q_k(t + \tau_k - \sigma_i), \qquad \text{for } t \ge t_0 + \sigma_i - \tau_k \text{ and } i = 1, \ldots, n.$$

(H_4) $\sum_{k=1}^{r} \lambda_k + \overline{\lim}_{t\to\infty} \int_\mu^t \int_\mu^{t_{N-1}} \cdots \int_\mu^{t_2} \sum_{i=1}^{p} \sum_{k \in J_i} \int_{t_1 - \sigma_i}^{t_1 - \tau_k} Q_k(s)\, ds\, dt_1 \ldots, dt_{N-1} = c < 1$ for $t \ge t_0 + \sigma_i$, $\lambda_i \ge 0$, $i = 1, \ldots, r$.

(H_5) $h(t) \in C([t_0, \infty), R^+)$, $f \in C(C(R^M \times [t_0 - r, \infty), R) \times \cdots \times C(R^M \times [t_0 - r_l, \infty), R), R)$, f is increasing respect to $u_i > 0$, $u_i f \ge 0$ for $i = 1, \ldots, l$, f is convex, $-f \ge f(-u_1, \ldots, -u_l)$, there exists a $t > T$, such that $P_i(t) - \sum_{k \in J_i} Q_k(t + \tau_k - \sigma_i) \ne 0$, or $hf \ne 0$, for all $T > t_0$.

(H_6) There exists $0 < K_1 < K_2$ $i, j = 1, \ldots, l$ and such that $K_1 u_i \le f(u_1, \ldots, u_l) \le K_2 u_j$ for each $u_i > 0$, $i = 1, \ldots, l$.

We consider the following boundary conditions,

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [t_0, \infty), \tag{B_1}$$

$$\frac{\partial u(x,t)}{\partial \bar{N}} = 0, \qquad (x,t) \in \partial\Omega \times [t_0, \infty), \tag{B_2}$$

$$\frac{\partial u(x,t)}{\partial \bar{N}} + \nu u = 0, \qquad (x,t) \in \partial\Omega \times [t_0, \infty), \tag{B_3}$$

where $\bar{N}$ is a unit exterior normal vector to $\partial\Omega$ and $\nu(x,t)$ is a nonnegative continuous function on $\partial\Omega \times [t_0, \infty)$,

$$\sigma = \max_{\substack{1 \le k \le r \\ 1 \le i \le n \\ 1 \le j \le m \\ 1 \le k' \le l}} \{\rho_k, \sigma_i, \tau_i, r_{k'}\},$$

and the initial data of the form,

$$u(x,t) = \varphi(x,t), \qquad \text{for } -\sigma \le t \le t_0, \quad x \in \Omega. \tag{1.2}$$

Let $N = 1$, $f \equiv 0$, and $\lambda_k = 0$, $k = 1, \ldots, r$, then equation (1.1) have the following special form,

$$\frac{\partial u(x,t)}{\partial t} - a(t)\Delta u + \sum_{i=1}^{n} p_i(x,t) u(x, t - \sigma_i) \sum_{j=1}^{m} q_j(x,t) u(x, t - \tau_j) = 0. \tag{1.3}$$

The oscillatory behavior of solutions for parabolic equation with functional arguments has been dealt in a few recent studies, we can refer to [1–6].

Recently, Kubiaczyk and Saker [1] investigated equation (1.3) with boundary conditions (B_1)–(B_3), and they obtained some oscillatory results. But Theorems 2.3, 3.3, and 4.3 in [1] are not correct.

For example, we consider

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{2}{e} u(x, t-1) - \frac{1}{e^2} u(x, t-2) = 0, \qquad x \in (0,1), \quad t \ge 1, \tag{$*$}$$

with boundary condition, $u_x|_{x=0} = u_x|_{x=1} = 0$.

It is easy to see that $P(t) = 2/e$, $Q(t) = 1/e^2$, $R(s) = 2/e - 1/e^2$, and $\int_t^{t+1}(2/e - 1/e^2)\,ds = 2/e - 1/e^2 > 0$, $\int_{t_0}^{\infty} R(t)e^{(1-\int_t^{t+1} R(s)\,ds)}\,dt = \int_{t_0}^{\infty}(2/e - 1/e^2)e^{(1-2/e-1/e^2)}\,dt = \infty$. The conditions of Theorem 2.3 of [1] are satisfied, but one solution of $(*)$, $u(x,t) = e^{-t}$ is nonoscillatory.

More recently, Ouyang, Zhou, and Yin [11] have corrected Theorem 2.3, 3.3, and 4.3 in [1]. They have obtained some new sufficient conditions for the oscillation of (1.3), (B_1)–(B_3).

In the present paper, we investigate equation (1.1) with boundary conditions (B_1)–(B_3), we obtain some sufficient conditions for the oscillation of (1.1), (B_1)–(B_3). These results generalize [1] and some related results of other references. By the time, we obtain some real oscillatory conditions for a class of high order neutral parabolic differential equations.

The paper is organized as follows. In Section 2, we establish some oscillation conditions for the solutions of equation (1.1) with boundary condition (B_1)–(B_3), respectively, in end, we correct Theorem A, which is Theorem 2.1, 3.1, and 4.1, respectively, and give some examples to explain the oscillation of (1.3), (1.1) with the above boundary conditions.

THEOREM A. *Assume that (H_1)–(H_4) hold, setting*

$$\tilde{R}_i(t) = \tilde{P}_i(t) - \sum_{k \in J_i} \tilde{Q}_k(t + \tau_k - \sigma_i)\, z(t - \sigma_i),$$

such that

$$0 \le \sum_{i-1}^{n} \int_t^{t+\sigma_i} \tilde{R}_i(t)ds, \qquad t \ge t_0$$

and

$$\int_{t_0}^{\infty} \sum_{i=1}^{n} \left(\tilde{R}_i(t)\, e^{1-\sum_{i-1}^{n}\int_t^{t+\sigma_i}\tilde{R}_i(t)\,ds}\right) dt = \infty. \tag{1.4}$$

Then, every solution of problem (1.3), (B_i) ($i = 1, 2, 3$) *oscillates, where* $\tilde{P}_i = P_i$, $\tilde{Q}_k = Q_k$ *for boundary condition* (B_2), (B_3), $\tilde{P}_i = P_i \exp\{\alpha_1 \int_{t-\sigma_i}^{t} a(s)ds\}$, $\tilde{Q}_k = Q_k \exp\{\alpha_1 \int_{t-\tau_k}^{t} a(s)ds\}$ *for boundary condition* (B_1).

For the reader's convenience, we recall some definitions.

DEFINITION 1.1. *A function $u(x,t) \in C^N(G) \cap C^1(\bar{G})$ is said to be a solution of the problem (1.1)–(B_i) ($i = 1, 2, 3$) if it satisfies (1.1) in the domain G and satisfies the boundary conditions, (B_i) ($i = 1, 2, 3$).*

DEFINITION 1.2. *The solution $u(x,t)$ of problem (1 1) is said to be oscillatory in the domain $G = \Omega \times [t_0, \infty)$ if for any positive number μ, there exists a point $(x_1, t_1) \in \Omega \times [\mu, \infty)$, such that the equality $u(x_1, t_1) = 0$ holds.*

If every solution of equation (1.1) is oscillatory, then equation (1.1) is called oscillatory.

DEFINITION 1.3. *A function $u(x,t)$ is called eventually positive (negative) if there exits a number $t_1 \ge t_0$, such that $u(x,t) > 0$ (< 0) for every $(x,t) \in \Omega \times [\mu, \infty)$.*

REMARK. If a solution is nonoscillatory, then it is eventually positive or eventually negative.

2. OSCILLATION OF BOUNDARY VALUE PROBLEM (1.1), (B_1)–(B_3)

First, we consider the following Dirichlet boundary value problem in the domain Ω,

$$\Delta u + \alpha u = 0, \qquad \text{in } (x,t) \in \Omega \times [t_0, \infty), \tag{2.1}$$

$$u = 0, \qquad \text{on } (x,t) \in \partial\Omega \times [t_0, \infty), \tag{2.2}$$

in which α is a constant.

It is well known from [7] that the smallest eigenvalue α_1 of problem (2.1) is positive and the corresponding eigenfunction $\phi(x) \geq 0$ for $x \in \Omega$.

Let $u(x,t)$ be a solution of problem (1.1),(B_1), we define throughout this paper,

$$U(t) = \frac{\int_\Omega u(x,t)\,\phi(x)\,dx}{\int_\Omega \phi(x)\,dx}. \tag{2.3}$$

Let $u(x,t)$ be a solution of problem (1.1),(B_i), $i = 2, 3$, we define,

$$V(t) = \frac{\int_\Omega u(x,t)\,dx}{\int_\Omega dx}. \tag{2.4}$$

LEMMA 2.1. *Suppose that (H_1)–(H_3), (H_5) hold. If any solution of differential inequality,*

$$\begin{aligned} &\left(z(t) - \sum_{k=1}^{r} \lambda_k z(t-\rho_k)\right)^{(N)} + \alpha_1 a(t) z(t) + \sum_{i=1}^{n} P_i(t) z(t-\sigma_i) \\ &- \sum_{j=1}^{m} Q_j(t) z(t-\tau_j) + h(t) f(z(t-r_1), \ldots, z(t-r_l)) \leq 0, \end{aligned} \tag{2.5}$$

isn't eventually positive, then every solution of (1.1),(B_1) is oscillatory.

PROOF. Assume that there exists a nonoscillatory solution $u(x,t)$ of the problem (1.1),(B_1). Without loss of generality, let $u(x,t)$ be an eventually positive solution of problem (1.1), (B_1) (if it is eventually negative, we can let $v(x,t) = -u(x,t)$, and using (H_5), the proof is similar), then there exists a number $\mu > t_0 - \sigma$, such that $u(x,t) > 0$, $u(x, t-\sigma_i) > 0$, $u(x, t-\tau_j) > 0$, $u(x, t-\rho_k) > 0$, $u(x, t-r_{k'}) > 0$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, $k = 1, \ldots, r$, $k' = 1, \ldots, l$.

Multiplying both sides of equation (1.1) by $\phi(x)$, and integrating both sides over the domain Ω with respect to x, we have

$$\begin{aligned} &\frac{d^N}{dt^N} \int_\Omega \left(u\phi - \sum_{k=1}^{r} \lambda_k u(x, t-\rho_k)\,\phi\right) dx - a(t) \int_\Omega \triangle u \phi\,dx \\ &+ \sum_{i=1}^{n} \int_\Omega \phi p_i(x,t)\,u(x,t-\sigma_i)\,dx - \sum_{j=1}^{m} \int_\Omega \phi q_j(x,t)\,u(x,t-\tau_j)\,dx \\ &+ h(t) \int_\Omega \phi f(u(x,t-r_1), \ldots, u(x,t-r_l))\,dx = 0, \qquad t \geq \mu. \end{aligned} \tag{2.6}$$

From (H_3), (2.6) becomes

$$\begin{aligned} &\frac{d^N}{dt^N} \int_\Omega \left(u\phi - \sum_{k=1}^{r} \lambda_k u(x, t-\rho_k)\,\phi\right) dx - a(t) \int_\Omega \triangle u \phi\,dx \\ &+ \sum_{i=1}^{n} P_i(t) \int_\Omega \phi u(x,t-\sigma_i)\,dx - \sum_{j=1}^{m} Q_j(t) \int_\Omega \phi u(x,t-\tau_j)\,dx \\ &+ h(t) \int_\Omega \phi f(u(x,t-r_1), \ldots, u(x,t-r_l))\,dx \leq 0, \qquad t \geq \mu. \end{aligned} \tag{2.7}$$

Using Green's formula, we get

$$\int_\Omega a(t) \triangle u \phi\,dx = -\alpha_1 \int_\Omega u\phi\,dx, \qquad t \geq \mu. \tag{2.8}$$

Combining (2.7), (2.8), and using Jensen's inequality, we obtain

$$\begin{aligned}&\frac{d^N}{dt^N}\int_\Omega\left(u\phi-\sum_{k=1}^r\lambda_k u\left(x,t-\rho_k\right)\phi\right)dx+\alpha_1 a\left(t\right)\int_\Omega u\phi\,dx\\&+\sum_{i=1}^n P_i\left(t\right)\int_\Omega\phi u\left(x,t-\sigma_i\right)dx-\sum_{j=1}^m Q_j\left(t\right)\int_\Omega\phi u\left(x,t-\tau_j\right)dx\\&+h\left(t\right)f\left(\frac{\int_\Omega\phi u\left(x,t-r_1\right)dx}{\int_\Omega\phi\,dx},\dots,\frac{\int_\Omega\phi u\left(x,t-r_l\right)dx}{\int_\Omega\phi\,dx}\right)\int_\Omega\phi\,dx\le 0,\qquad t\ge\mu\end{aligned}$$

According to (2.3), it follows

$$\begin{aligned}&\frac{d^N}{dt^N}\left(U\left(t\right)-\sum_{k=1}^r\lambda_k U\left(t-\rho_k\right)\right)+\alpha a\left(t\right)U\left(t\right)+\sum_{i=1}^n P_i\left(t\right)U\left(t-\sigma_i\right)-\sum_{j=1}^m Q_j\left(t\right)U\left(t-\tau_j\right)\\&\qquad+h\left(t\right)f\left(U\left(t-r_1\right),\dots,U\left(t-r_l\right)\right)\le 0,\qquad t\ge\mu.\end{aligned}\tag{2.9}$$

Because $u(x,t)$ is positive, from (2.3) again, we have that $U(t)$ is positive eventually. This means that $U(t)$ is an eventually positive solution of (2.5), this contradicts the assumption. The proof is complete.

Using (2.3), (2.4), and the proof of Lemma 2.1, it is easy to show the following.

COROLLARY 2.1. *Suppose that (H_1)–(H_3), (H_5) hold. If any solution of differential inequality,*

$$\begin{aligned}&\left(z\left(t\right)-\sum_{k=1}^r\lambda_k z\left(t-\rho_k\right)\right)^{(N)}+\sum_{i=1}^n P_i\left(t\right)z\left(t-\sigma_i\right)-\sum_{j=1}^m Q_j\left(t\right)z\left(t-\tau_j\right)\\&\qquad+h\left(t\right)f\left(z\left(t-r_1\right),\dots,z\left(t-r_l\right)\right)\le 0,\end{aligned}\tag{2.10}$$

isn't eventually positive, then every solution of (1.1), (B_1)–(B_3) is oscillatory.

Now, we give another main oscillatory condition of the problem (1.1), (B_1)–(B_3).

THEOREM 2.1. *Suppose that (H_1)–(H_5) hold. If any solution of the following differential equation,*

$$\begin{aligned}&\left(z\left(t\right)-\sum_{k=1}^r\lambda_k z\left(t-\rho_k\right)\right)^{(N)}+\sum_{i=1}^n\left[P_i\left(t\right)-\sum_{k\in J_i}Q_k\left(t+\tau_k-\sigma_i\right)\right]z\left(t-\sigma_i\right)\\&\qquad+h\left(t\right)f\left(z\left(t-r_1\right),\dots,z\left(t-r_l\right)\right)=0,\qquad t\ge\mu.\end{aligned}\tag{2.11}$$

isn't eventually positive, then every solution of the problem (1.1), (B_1)–(B_3) is oscillatory.

PROOF. We only need to prove that every solution of problem (1.1),(B_1) is oscillatory, the proof of problem (1.1), (B_2), (B_3) is similarly, and thus, we omit it. Assume that problem (1.1),(B_1) has a nonoscillatory solution $u(x,t)$, without loss of generally, we let $u(x,t)$ be eventually positive (if it is eventually negative, we let $-u(x,t)=v(x,t)$, and using (H_5), the proof is similar). Then, there exists a $\mu>t_0-\sigma$, such that $u(x,t)>0$, $u(x,t-\sigma_i)>0$, $u(x,t-\tau_j)>0$, $u(x,t-\rho_k)>0$, and $u(x,t-r_{k'})>0$, $i=1,\dots,n$, $j=1,\dots,m$, $k=1,\dots,r$, $k'=1,\dots,l$.

From Corollary 2.1, we get that (2.10) has an eventually positive solution $z(t)$. Let

$$\begin{aligned}y\left(t\right)&=z\left(t\right)-\sum_{k=1}^r\lambda_k z\left(t-\rho_k\right)\\&-\int_\mu^t\int_\mu^{t_{N-1}}\cdots\int_\mu^{t_2}\sum_{i=1}^n\sum_{k\in J_i}\int_{t_1-\sigma_i}^{t_1-\tau_k}Q_k\left(s+\tau_k\right)z\left(s\right)ds\,dt_1\dots dt_{N-1},\qquad t\ge\mu.\end{aligned}\tag{2.12}$$

It is easy to know by (H_4) that $y(t) \le z(t)$ and $y(t) > 0$. In fact, it is obvious that $y(t) \le z(t)$. Now, we prove $y(t) > 0$. Combining (2.10) and (2.12),

$$y^{(N)}(t) + \sum_{i=1}^{n}\left(P_i(t) - \sum_{k\in J_i} Q_k(t+\tau_k-\sigma_i)\right) z(t-\sigma_i) + h(t) f(z(t-r_1),\dots,z(t-r_l)) \le 0. \tag{2.13}$$

Thus, $y^{(N)}(t) < 0$, so $y^{(i)}(t)$ is eventually monotone, $i = 1,\dots,N-1$. We consider the following two cases.

CASE 1. $y'(t) < 0$.

Assume that there exists a t_1^*, such that $y(t_1^*) = \alpha \le 0$, because $y(t)$ is monotone decrease strictly, then there exists a $t_2^* > t_1^*$, such that $y(t_2^*) = \beta < 0$, it means that $y(t) \le y(t_2^*) = \beta < 0$ for $t > t_2^*$, by (H_4), we obtain

$$\begin{aligned} z(t) &= y(t) + \sum_{k=1}^{r} \lambda_k z(t-\rho_k) \\ &+ \int_\mu^t \int_\mu^{t_{N-1}} \cdots \int_\mu^{t_2} \sum_{i=1}^{n}\sum_{k\in J_i} \int_{t_1-\sigma_i}^{t_1-\tau_k} Q_k(s+\tau_k) z(s)\, ds\, dt_1 \dots dt_{N-1}, \\ &< \beta + c \max_{t-\sigma\le s\le t} z(s) < \beta + \max_{t-\sigma\le s\le t} z(s). \end{aligned} \tag{2.14}$$

By [8, Lemma 1.5.2], it is easy to show that $z(t) \le 0$, this is a contradiction.

CASE 2. $y'(t) > 0$.

If there exists a t_3^* such that $y(t_3^*) \ge 0$, then there exists a $t_4^* > t_3^*$, such that $y(t_4^*) > 0$, so for all $t > t_4^*$, we have $y(t) > 0$. Now, we assume, for the converse, that $y(t) \le 0$, for all $t > t_3^*$, one argues that $y''(t) < 0$, because n is an odd number, repeats the argument to obtain that $y^{(n)}(t) > 0$, which contradicts to that $y^{(n)}(t) < 0$. Thus, $y(t) > 0$ for all $t > t_4^*$.

From (2.13), (H_5), and $y(t) \le z(t)$, $y(t) > 0$, we obtain

$$y^{(N)}(t) + \sum_{i=1}^{n}\left(P_i(t) - \sum_{k\in J_i} Q_k(t+\tau_k-\sigma_i)\right) y(t-\sigma_i) + h(t) f(y(t-r_1),\dots,y(t-r_l)) \le 0, \qquad t \ge \mu, \tag{2.15}$$

have an eventually positive solution $y(t)$.

Now we prove that (2.11) has an eventually positive solution.

Since $y^{(N)}(t) < 0$, $y(t) > 0$, and n is an odd number, then there exists an even number N^* ($0 \le N^* \le N-1$), such that $y^{(i)}(t) > 0$, $0 \le i \le N^*$, $(-1)^i y^{(i)}(t) > 0$, $N^* \le i \le N$. Integrating both sides of (2.15) respects to t $N - N^*$ times, we obtain

$$y^{(N^*)}(t) \ge \int_t^\infty \frac{(s-t)^{N-N^*-1}}{(N-N^*-1)!}\left(\sum_{i=1}^{n}\left(P_i(s) - \sum_{k\in J_i} Q_k(s+\tau_k-\sigma_i)\right) y(s-\sigma_i) + h(s) f(y(s-r_1),\dots,y(s-r_l))\right) ds, \qquad t \ge T. \tag{2.16}$$

Integrating both sides of (2.16) respects to t N^* times, it follows

$$y(t) \ge \int_T^t \frac{(t-s)^{N^*-1}}{(N^*-1)!} \int_s^\infty \frac{(u-s)^{N-N^*-1}}{(N-N^*-1)!}\left(\sum_{i=1}^{n}\left(P_i(u) - \sum_{k\in J_i} Q_k(u+\tau_k-\sigma_i)\right) \times y(u-\sigma_i) + h(u) f(y(u-r_1),\dots,y(u-r_l))\right) du\, ds, \qquad t \ge T, \tag{2.17}$$

where $T > \mu$, and such that (2.10) and (2.16) hold and $t - \sigma > T - \mu$, $y(T - \mu) > 0$ for $t \geq T$. Set

$$K = \{v \in C(T - \mu, \infty), R^+ : 0 \leq v(t) \leq 1 \text{ for } t > T - \mu\}.$$

According (2.17) and the definities of K, we can define an operator S on K as follows,

$$(Sv)(t) = \begin{cases} (Sv)(T), & T - \mu \leq t < T, \\ \frac{1}{y(t)} \int_T^t \frac{(t-s)^{N^*-1}}{(N^*-1)!} \int_t^\infty \frac{(u-s)^{N-N^*-1}}{(N-N^*-1)!} & \\ \left(\sum_{i=1}^n \left(P_i(u) - \sum_{k \in J_i} Q_k(u + \tau_k - \sigma_i)\right) v(u - \sigma_i) y(u - \sigma_i) \right. & \\ \left. + h(u) f(v(u - r_1) y(u - r_1), \ldots, v(u - r_l) y(u - r_l)) \right) du\, ds, & t \geq T. \end{cases}$$

Using the similar method of the proof of Theorem 1 in [10], we can derive

$$P_i(t) - \sum_{k \in J_i} Q_k(s + \tau_k - \sigma_i) \equiv 0, \qquad \text{for } s \geq t^*, \quad i = 1, \ldots, n,$$

and

$$h(s) f(w(s - r_1), \ldots, w(s - r_l)) \equiv 0, \qquad \text{for } s \geq t^*,$$

this contradicts (H_5). The proof is complete.

Let

(H_7) There exist $\bar{P}_i(t), \bar{Q}_j(t), i = 1, \ldots, n; j = 1, \ldots, m$, such that $\bar{P}_i(t) \leq P_i(t)$, $\bar{Q}_j(t) \geq Q_j(t)$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, and $\bar{P}_i(t) \geq \sum_{k \in J_i} Q_k(s + \tau_k - \sigma_i)$.

(H_8) $\sum_{k=1}^r \lambda_k + \overline{\lim}_{t \to \infty} \int_\mu^t \int_\mu^{t_{N-1}} \cdots \int_\mu^{t_2} \sum_{i=1}^n \sum_{k \in J_i} \int_{t-\sigma_i}^{t-\tau_k} \bar{Q}_k(s + \tau_k)\, ds\, dt_1 \ldots dt_{N-1} < 1$.

Then, Theorem 2.1 can be improved as follows.

COROLLARY 2.2. *Suppose that (H_1)–(H_3), (H_5), (H_7), (H_8) hold. If any solution of the following differential equation,*

$$z^{(N)}(t) + \sum_{i=1}^n \left(\bar{P}_i(t) - \sum_{k \in J_i} \bar{Q}_k(t + \tau_k - \sigma_i) \right) z(t - \sigma_i) + h(t) f(z(t - r_1), \ldots, z(t - r_l)) = 0, \tag{2.18}$$

isn't eventually positive, then every solution of problem (1.1), (B_1), (B_2), (B_3) is oscillatory.

REMARK 2.1. Corollary 2.2 improves the Theorem 4.3 in [1].

Let

$$\tilde{R}(t) = \min_{1 \leq i \leq n} R_i(t) = \min_{1 \leq i \leq n} \left(P_i(t) - \sum_{k \in J_i} Q_k(t + \tau_k - \sigma_i) \right). \tag{2.19}$$

Then, we have the following.

THEOREM 2.2. $N > 1$, *Suppose that H_1–H_6 hold. If*

$$\int_T^\infty \tilde{R}(t)\, dt = \infty, \qquad t_0 > 0, \tag{2.20}$$

or

$$\int_T^\infty h(t)\, dt = \infty, \qquad t_0 > 0, \tag{2.21}$$

hold. If $u(x,t)$ is a solution of (1.1), B_1–B_3, then there exists at least one of the following holds.

(1) *$u(x,t)$ is oscillatory.*

(2) *If $u(x,t)$ is a nonoscillatory, then $\lim_{t \to \infty} u(x,t) = 0$.*

PROOF. We only need to prove that the solution of problem (1.1), B_1 satisfies (1) or (2), the remaining is similar. Assume that the solution $u(x,t)$ of problem (1.1), B_1 don't satisfies (1) and (2), that is, $u(x,t)$ is nonoscillatory and $\overline{\lim}_{t\to\infty} u(x,t) \neq 0$. Without loss of generality, assume that $u(x,t)$ is eventually positive (if it has an eventually negative solution, the proof is similar). By the assumption, we have that $\overline{\lim}_{\to\infty} u(x,t) = a > 0$. Using Lemma 2.1, we obtain that $z(t) = U(t) = \frac{\int_\Omega u(x,t)\phi(x)\,dx}{\int_\Omega \phi(x)\,dx}$ is a positive solution of

$$\left(z(t) - \sum_{k=1}^{r} \lambda_k z(t-\rho_k)\right)^{(N)} + \alpha_1 a(t) z(t) + \sum_{i=1}^{n} P_i(t) z(t-\sigma_i) - \sum_{j=1}^{m} Q_k(t) z(t-\tau_j)$$
$$+ h(t) f(z(t-r_1), \dots, z(t-r_l)) \le 0,$$

and that $\overline{\lim}_{\to\infty} z(t) > a > 0$. Let

$$y(t) = z(t) - \sum_{k=1}^{r} \lambda_k z(t-\rho_k)$$
$$- \int_\mu^t \int_\mu^{t_{N-1}} \cdots \int_\mu^{t_2} \sum_{i=1}^{n} \sum_{k\in J_i} \int_{t_1-\sigma_i}^{t_1-\tau_k} Q_k(s+\tau_k) z(s)\, ds\, dt_1 \dots dt_{N-1}, \qquad t \ge \mu. \tag{2.22}$$

Using (H_4) and the proof of Theorem 2.1, it is easy to show that $0 < y(t) < z(t)$. From (2.22), (H_4) and $\overline{\lim}_{\to\infty} z(t) > a > 0$, it is obvious that $\overline{\lim}_{\to\infty} y(t) = b > 0$, it satisfies

$$(y(t))^{(N)} + \alpha_1 a(t) \left(y(t) + \sum_{k=1}^{r} \lambda_k z(t-\rho_k) \right.$$
$$\left. + \int_\mu^t \int_\mu^{t_{N-1}} \cdots \int_\mu^{t_2} \sum_{i=1}^{n} \sum_{k\in J_i} \int_{t_1-\sigma_i}^{t_1-\tau_k} Q_k(s+\tau_k) z(s)\, ds\, dt_1 \dots dt_{N-1} \right)$$
$$+ \sum_{i=1}^{n} \left(P_i(t) - \sum_{k\in J_i} Q(t+\tau_k-\sigma_i) \right) y(t-\sigma_i)$$
$$+ h(t) f(y(t-r_1), \dots, y(t-r_l)) \le 0.$$

It is obvious that $y(t)$ is a positive solution of

$$(y(t))^{(N)} \le -\left(\sum_{i=1}^{n} \left(P_i(t) - \sum_{k\in J_i} Q(t+\tau_k-\sigma_i) \right) y(t-\sigma_i) \right.$$
$$\left. + h(t) f(y(t-r_1), \dots, y(t-r_l)) \right) \le 0. \tag{2.23}$$

Because $y(t)$ is eventually positive, it follows that $y^{(k)}(t)$ is eventually monotone, $i = 1, \dots, N-1$. If $y'(t) > 0$, integrating both sides of (2.23) from t to T, it follows

$$y^{(N-1)}(t) - y^{(N-1)}(T)$$
$$\le - \int_T^t \left(\sum_{i=1}^{m} \left[P_i(s) - \sum_{k\in J_i} Q_k(s+\tau_k-\sigma_i) \right] + K_1 h(s) \right) y(s-\bar{\sigma})\, ds$$
$$\le -y(T-\bar{\sigma}) \int_T^t \left(\sum_{i=1}^{m} \left[P_i(s) - \sum_{k\in J_i} Q_k(s+\tau_k-\sigma_i) \right] + K_1 h(s) \right) ds, \qquad t > T, \tag{2.24}$$

that is,

$$\int_T^t \left(\sum_{i=1}^{m} \left[P_i(s) - \sum_{k\in J_i} Q_k(s+\tau_k-\sigma_i) \right] + K_1 h(s) \right) ds$$
$$\le \frac{1}{y(T-\bar{\sigma})} \left[-y^{(n-1)}(t) + y^{(n-1)}(T) \right], \qquad t > T. \tag{2.25}$$

Taking limitation of both sides of (3.8), using the fact that $y^{(N-1)}(t)$ is eventually positive, and according (2.20), (2.21), we derive a contradiction. If $y'(t) < 0$, then $y(t)$ is eventually decreasing, and we have $\lim_{t\to\infty} y(t) = \overline{\lim}_{t\to\infty} y(t) = b > 0$. Integrating both sides of (2.23) respect to t in T to t, we obtain

$$\begin{aligned} &\int_T^t \left(\sum_{i=1}^m \left[P_i(s) - \sum_{k\in J_i} Q_k(s+\tau_k-\sigma_i) \right] + K_1 h(s) \right) ds \\ &\quad \le \frac{1}{y(t)} \left[-y^{(n-1)}(t) + y^{(n-1)}(T) \right], \qquad t > T. \end{aligned} \tag{2.26}$$

Taking limitation both sides of (2.26), it follows

$$\lim_{t\to\infty} \int_T^\infty \left(\sum_{i=1}^m \left[P_i(s) - \sum_{k\in J_i} Q_k(s+\tau_k-\sigma_i) \right] + K_1 h(s) \right) ds < \frac{1}{b} y^{(n-1)}(T), \qquad t > T. \tag{2.27}$$

This contradicts to (2.20), (2.21). The proof is completed.

Theorem 2.2 can be rewritten as follows.

COROLLARY 2.3. $N > 1$, *Suppose that* H_1–H_6 *and (2.20) or (2.21) hold. If (1.1),* B_1–B_3 *has a nonoscillatory solution* $u(x,t)$, *then* $\lim_{t\to\infty} u(x,t) = 0$.

3. APPLICATION

Using Theorem 2.1, we consider the special case. $N = 1$, $f \equiv 0$, we can obtain some special results for the oscillation of problem (1.1) with boundary condition (B_1) or (B_2) or (B_3).

THEOREM 3.1. *Suppose that* (H_1)–(H_4), (H_5), (H_7), *hold and further assume that*

(H_9) $\underline{\lim}_{t\to\infty} \int_t^{t+\sigma_i} \bar{R}_i(s)\, ds > 0$;

(H_{10}) $\overline{\lim}_{t\to\infty} \sum_{i=1}^n \sum_{k\in J_i} \int_{t-\sigma_i}^{t-\tau_k} \bar{Q}_k(s)\, ds = c < 1$ *for* $t \ge t_0 + \sigma$;

(H_{11}) $\int_{t_0}^\infty (\sum_{i=1}^n \bar{R}_i(t) \ln(e \sum_{j=1}^n \int_t^{t+\sigma_j} \bar{R}_j(s)\, ds))\, dt = \infty$, *where* $\bar{R}_i(s) = \bar{P}_i(t) - \sum_{k\in J_i} \bar{Q}_k(t + \tau_k - \sigma_i))$, *and* $\bar{P}_i, \bar{Q}_k$ *is defined by Theorem A,* $i = 1, 2, \dots, n$, $k = 1, \dots, J_i$. *Then, every solution of equation (1.3) with boundary condition* (B_1) *or* (B_2) *or* (B_3) *oscillates.*

PROOF. We only prove that every solution of equation (1.3) with boundary condition (B_2) oscillates, the proof of the remaining is similar. It is obvious that $\bar{P}_i = P_i$, $\bar{Q}_k = Q_k$, $i = 1, 2, \dots, n$, $k = 1, \dots, J_i$. Assume $u(x,t)$ is a nonoscillatory solution of the problem (1.3)–(B_2), without loss of generally, assume $u(x,t)$ is eventually positive, according the proof of Theorem 2.1 and (H_{10}), we obtain

$$z'(t) + \sum_{i=1}^n R_i(t) z(t - \sigma_i) = 0, \qquad t \ge \mu.$$

has an eventually positive solution $z(t)$, let $\lambda(t) = -z'(t)/z(t)$. Then, $\lambda(t)$ is a nonnegative and continuous,and there exists $t_1 > t_0$ with $y(t_1) > 0$ and such that $y(t) = y(t_1)\exp(-\int_{t_1}^t \lambda(s)\, ds)$. Furthermore, $\lambda(t)$ satisfies

$$\lambda(t) = \sum_{i=1}^n R_i(t) \exp\left(-\int_{t-\sigma_i}^t \lambda(s)\, ds \right), \qquad t \ge \mu.$$

Let $B(t) = \sum_{i=1}^n \int_t^{t+\sigma_i} R_i(s)\, ds$, by using inequality $e^{rx} \ge x + \ln(er)/r$ for $r > 0$. By (H_9), we obtain that $B(t) > 0$, then

$$\begin{aligned} \lambda(t) &= \sum_{i=1}^n R_i(t) \exp\left(B(t) \frac{1}{B(t)} \int_{t-\sigma_i}^t \lambda(s)\, ds \right) \\ &\ge \sum_{i=1}^n R_i(t) \left[\frac{1}{B(t)} \int_{t-\sigma_i}^t \lambda(s)\, ds + \frac{\ln(eB(t))}{B(t)} \right], \qquad t \ge \mu. \end{aligned}$$

Using the sketch of interchanging the order of integration, and using the same method of the proof of Theorem 3.2 in [9] and (H_{11}), it will derive a contradiction.

If $u(x,t)$ is eventually negative, we can change (1.3) as

$$\frac{\partial u(x,t)}{\partial t} - a(t)\triangle(-u) + \sum_{i=1}^{n} p_i(x,t)(-u(x,t-\sigma_i)) - \sum_{j=1}^{m} q_j(x,t)(-u(x,t-\tau_j)) = 0.$$

Let $v(x,t) = -u(x,t)$, it follows that $v(x,t)$ is an eventually positive solution of

$$\frac{\partial v(x,t)}{\partial t} - a(t)\Delta v + \sum_{i=1}^{n} p_i(x,t)v(x,t-\sigma_i) - \sum_{j=1}^{m} q_j(x,t)v(x,t-\tau_j) = 0.$$

According Theorem 2.1, we obtain that (2.11) has an eventually positive solution, using the similar method of the above, we can derive a contradiction also. The proof is complete.

REMARK. Theorem 3.1 is a nearly sharp condition for the oscillation of (1.3).

EXAMPLE 3.1. Consider the parabolic equation,

$$u_t - \frac{1}{4}u_{xx} + e^{-3\pi/2}u\left(x, t - \frac{3\pi}{2}\right) - \frac{5}{4}e^{-2\pi}u(x, t-2\pi) = 0, \qquad (x,t) \in (0,\pi) \times R^+, \tag{3.1}$$

with the boundary condition,

$$\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(\pi,t)}{\partial x} = 0, \qquad t \geq 0. \tag{3.2}$$

where $a(t) = 1/4$, $R(t) = e^{-3\pi/2} - (5/4)e^{-2\pi}$, and $\sigma = 3\pi/2$.

Since $\int_0^\infty R(t)\ln(e\int_t^{t+3\pi/2} R(s)\,ds)\,dt = \infty$, so it satisfies condition (H_1)–(H_4) and every solution of problem (3.1),(3.2) is oscillatory. In fact, $u(x,t) = e^{-t}\sin t\cos x$ is such a solution.

EXAMPLE 3.2. Consider the parabolic equation,

$$\begin{aligned} &\frac{\partial^3\left(u(x,t) - (1/2)e^{-2\pi}u(x,t-2\pi)\right)}{\partial t^3} - u_{xx} + \frac{1}{2}e^{-3\pi/2}u\left(x, t-\frac{3\pi}{2}\right) \\ &\quad -\frac{1}{2}e^{-5\pi/2}u\left(x, t - \frac{5\pi}{2}\right), \qquad (x,t) \in (a,b) \times R^+, \end{aligned} \tag{3.3}$$

with the boundary condition,

$$\frac{\partial u(a,t)}{\partial x} = \frac{\partial u(b,t)}{\partial x} = 0, \qquad t \geq 0, \tag{3.4}$$

It is easy to show that $\int_T^\infty (P(t) - Q(t + 5\pi/2 - 3\pi/2))\,dt = \infty$, by Theorem 2.2, we obtain that if $u(x,t)$ is a solution of (3.3)–(3.4). Then, either it is oscillatory, or $\lim_{t\to\infty} u(x,t) = 0$. In fact, if we taking $a = 0$, $b = \pi$, $u(x,t) = e^{-t}\sin t\cos x$ is a oscillatory solution of it. If we taking $a = 0$, $b = \sqrt{2}\pi$, then $u(x,t) = e^{-t}\cos(\sqrt{2}/2)x$ is a asymptotic solution of it, such that $\lim_{t\to\infty} u(x,t) = 0$.

REFERENCES

1 I Kubiaczyk and S H Saker, Oscillation of delay parabolic differential equations with several coefficients, *J of Compl. and Appl. Math.* **147**, 263–275, (2002).

2. V. Bykov and T Ch Kultaev, Oscillation of solutions of parabolic equations, *Izv Akad, Nauk Kirgiz SSR* **6**, 3–9, (1983).

3. X L Fu and L Q Zhang, Forced oscillation of solutions of parabolic equations, *J Partial Differential Equations* **8**, 82–88, (1995).

4. T. Kusano and N. Yoshida, Oscillation of parabolic equations with oscillating coefficients, *Hiroshima Math. J* **24**, 123–133, (1994).
5 N Yoshida, Oscillation of nonlinear parabolic equations with functional arguments, *Hiroshima Math, J* **16**, 305-314, (1986).
6 N. Yoshida, Forced oscillation certain nonlinear delay parabolic equations, *Bull. Austral Math. Soc.* **36**, (1987).
7. V.S. Vladimirov, *Equations of Mathematical Physics*, Nauka, Moscow, (1981).
8 I. Gyori and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, (1991).
9 E M. Elabbsy, A.S. Hegazi and S.H. Saker, Oscillation of solutions to delay differential equations with positive and negative coefficients, *Electron. J. Differential Equations* **13**, 1–13, (2000).
10 Z Ouyang, Y Li and M Qing, Eventually positive solutions of odd-order neutral differential equations, *Appl. Math. Lett.* **17** (2), 159–166, (2004).
11 Z Ouyang, S. Zhou and F Yin, Oscillation for a class of odd delay parabolic differential equations, *J. Compl. and Appl. Math.* (to appear)