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ORIGINAL ARTICLE

# The tanh–coth method combined with the Riccati equation for solving nonlinear coupled equation in mathematical physics

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**Abstract** In this work, we established abundant travelling wave solutions for nonlinear coupled evolution equation. This method was used to construct solitons and travelling wave solutions of nonlinear coupled evolution equation. The tanh–coth method combined with the Riccati equation presents a wider applicability for handling nonlinear wave equations.


## 1. Introduction

The investigation of the travelling wave solutions for nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena. Nonlinear wave phenomena appear in various scientific and engineering field, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. New exact solutions may help to find new phenomena. In recent years, a variety of powerful methods such as inverse scattering method (Ablowitz and Segur, 1981; Vakhnenko et al., 2003), the tanh–sech method (Malfliet, 1992; Malfliet and Hereman, 1996; Wazwaz, 2004a), extended tanh method (El-Wakil and Abdou, 2007; Fan, 2000), sine–cosine method (Wazwaz, 2004b; Bekir, 2008), homogeneous balance method (Fan and Zhang, 1998), Exp-function method (Bekir and Boz, 2008; He and Wu, 2006), and the $\left(\frac{G'}{G}\right)$-expansion method (Wang et al., 2008; Bekir, 2008) were used to develop nonlinear dispersive and dissipative problems.

The pioneer work of Malfliet (1992), Malfliet and Hereman (1996) introduced the powerful tanh method for a reliable treatment of the nonlinear wave equations. The useful tanh method is widely used by many such as in (Wazwaz, 2004a, 2005, 2006) and by the references therein. Later, the tanh–coth method, developed by Wazwaz (2007a), is a direct and effective algebraic method for handling nonlinear equations. Various extensions of the method were developed as well in

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Wazzan (2009a,b) and systematically studied in Gómez and Salas (2008a,b).

Our first interest in the present work is in implementing the tanh–coth method combined with Riccati equation method to stress its power in handling nonlinear equations, so that one can apply it to models of various types of nonlinearity. The next interest is in determining the exact travelling wave solutions for (2 + 1)-dimensional breaking soliton equations. Searching for exact solutions of nonlinear problems has attracted a considerable amount of research work where computer symbolic systems facilitate the computational work.

(2 + 1)-Dimensional breaking soliton equations (Hirota and Ohta, 1991):

$$\begin{aligned} &u_t + \alpha u_{xxy} + 4\alpha u v_x + 4\alpha u_x v = 0, \\ &u_y = v_x, \end{aligned} \tag{1}$$

where $\alpha$ is a known constant. Eq. (1) describes the (2 + 1)-dimensional interaction of a Riemann wave propagating along the $y$-axis with a long wave along the $x$-axis. In the past years, many authors have studied Eq. (1). For instance, Zhang has successfully extended the generalized auxiliary equation method to the (2 + 1)-dimensional breaking soliton equations in Zhang (2007)). Some soliton-like solutions were obtained by the generalized expansion method of Riccati equation in Cheng and Li (2003). Recently, a class of periodic wave solutions were obtained by the mapping method in Peng (2005). Two classes of new exact solutions were obtained by the singular manifold method in Peng and Krishna (2005). Very recently, Jacobi elliptic function solutions and their degenerate solutions are obtained by a generalized extended $F$-expansion method in Ren et al. (2006).

## 2. The tanh–coth method

Wazwaz has summarized for using tanh–coth method. A PDE

$$P(u, u_t, u_x, u_y, u_{xx}, u_{tt}, u_{yy}, u_{xxx}, \ldots) = 0. \tag{2}$$

can be converted to on ODE

$$Q(U, U', U'', U''', \ldots) = 0. \tag{3}$$

upon using a wave variable $\xi = x + y - \beta t$. Eq. (3) is integrated as long as all terms contain derivatives where integration constants are considered zeros. Introducing a new independent variable

$$Y = \tanh(\xi) \quad \text{or} \quad Y = \coth(\xi) \quad \xi = x + y - \beta t, \tag{4}$$

leads to change of derivatives:

$$\begin{aligned} &\frac{d}{d\xi} = (1 - Y^2)\frac{d}{dY} \\ &\frac{d^2}{d\xi^2} = (1 - Y^2)\left(-2Y\frac{d}{dY} + (1 - Y^2)\frac{d^2}{dY^2}\right) \\ &\frac{d^3}{d\xi^3} = (1 - Y^2)\left((6Y^2 - 2)\frac{d}{dY} - 6Y(1 - Y^2)\frac{d^2}{dY^2}\right. \\ &\qquad \left. + (1 - Y^2)^2\frac{d^3}{dY^3}\right) \end{aligned} \tag{5}$$

The tanh–coth method (Wazwaz, 2007a,b) admits the use of the finite expansion

$$U(\xi) = S(Y) = \sum_{k=0}^{m} a_k Y^k + \sum_{k=1}^{m} b_k Y^{-k}, \tag{6}$$

where $m$ is a positive integer, for this method, that will be determined. Expansion (6) reduces to the standard tanh method (Malfliet, 1992) for $b_k = 0, 1 \leqslant k \leqslant m$. The parameter $m$ is usually obtained, as stated before, by balancing the linear terms of the highest order in the resulting equation with the highest order nonlinear terms. If $m$ is not an integer, then a transformation formula should be used to overcome this difficulty. Substituting (6) into the ODE results is an algebraic system of equations in powers of $Y$ that will lead to the determination of the parameters $a_k$ $(k = 0, \ldots, m), b_k$ $(k = 1, \ldots, m)$ and $\beta$.

The function $Y$ satisfies the Riccati equation

$$Y' = A + BY + CY^2, \tag{7}$$

where $A, B$ and $C$ are constants (Wazwaz, 2007c), and

$$Y' = \frac{dY(\xi)}{d\xi}, \quad \xi = x + y - \beta t. \tag{8}$$

## 3. The Riccati equation and its special solutions

The Riccati equation

$$Y' = A + BY + CY^2, \tag{9}$$

has specific solutions for $B = 0$ given in Wang et al. (2006) by

$$\begin{aligned} &A = \frac{1}{2}, C = -\frac{1}{2}, \quad Y_1 = \tanh\frac{\xi}{2}, \coth\frac{\xi}{2}, \\ &A = \frac{1}{2}, C = \frac{1}{2}, \quad Y_2 = \tan\xi \pm \sec\xi, \tan\frac{\xi}{2}, -\cot\frac{\xi}{2}, \\ &A = 1, C = -1, \quad Y_4 = \tanh\xi, \coth\xi, \\ &A = 1, C = 1, \quad Y_5 = \tan\xi, -\cot\xi, \\ &A = 1, C = -4, \quad Y_7 = \frac{1}{2}\tanh 2\xi, \frac{1}{2}\coth 2\xi, \\ &A = 1, C = 4, \quad Y_8 = \frac{1}{2}\tan 2\xi, -\frac{1}{2}\cot 2\xi. \end{aligned} \tag{10}$$

Other values for $Y$ can be derived for other arbitrary values for $A$ and $C$. To show the efficiency of the method described in the previous part, we present some example.

## 4. Travelling wave solutions of the (2 + 1)-dimensional breaking soliton equations

We consider the (2 + 1)-dimensional breaking soliton Eqs. (1). Using the wave variable $\xi = x + y - \beta t$ and proceeding as before we find

$$\begin{aligned} &-\beta u' + \alpha u''' + 4\alpha u v' + 4\alpha u' v = 0, \\ &u' = v'. \end{aligned} \tag{11}$$

Integrating the second equation in the system and neglecting the constants of integration, we find

$$u = v. \tag{12}$$

Substituting (12) into the first equation of the system and integrating, we find

$$-cu + 4\alpha u^2 + \alpha u'' = 0. \tag{13}$$

Balancing $u''$ with $u^2$ in (13) gives

$$m + 2 = 2m, \tag{14}$$

so that

$$m = 2. \tag{15}$$

The tanh–coth method admits the use of the finite expansion

$$U(\xi) = S(Y) = a_0 + a_1 Y^2 + \frac{b_1}{Y^2}. \tag{16}$$

By substituting Eq. (16) in Eq. (13), collecting the coefficients of $Y^i$ $(i = 0, \ldots, 8)$ and setting it to zero, we obtain the system

$$\begin{aligned}
&6\alpha a_1 C^2 + 4\alpha a_1^2 = 0,\\
&10\alpha a_1 BC = 0,\\
&8\alpha a_0 a_1 + 8\alpha a_1 AC + 4\alpha a_1 B^2 - \beta a_1 = 0,\\
&6\alpha a_1 AB = 0,\\
&8\alpha a_1 b_1 + 4\alpha a_0^2 - \beta a_0 + 2\alpha b_1 C^2 + 2\alpha a_1 A^2 = 0,\\
&6\alpha b_1 BC = 0,\\
&4\alpha b_1 B^2 + 8\alpha a_0 b_1 + 8\alpha b_1 AC - \beta b_1 = 0,\\
&10\alpha b_1 AB = 0,\\
&4\alpha b_1^2 + 6\alpha b_1 A^2 = 0.
\end{aligned} \tag{17}$$

Solving this system by Maple gives $B = 0$ and the following six sets of solutions:

(i) *The first set:*

$$a_0 = -\frac{3}{2}AC, \quad a_1 = 0, \quad b_1 = -\frac{3}{2}A^2, \quad \beta = -4AC\alpha. \tag{18}$$

(ii) *The second set:*

$$a_0 = -\frac{1}{2}AC, \quad a_1 = 0, \quad b_1 = -\frac{3}{2}A^2, \quad \beta = 4AC\alpha. \tag{19}$$

(iii) *The third set:*

$$a_0 = -\frac{3}{2}AC, \quad a_1 = -\frac{3}{2}C^2, \quad b_1 = 0, \quad \beta = -4AC\alpha. \tag{20}$$

(iv) *The fourth set:*

$$a_0 = -\frac{1}{2}AC, \quad a_1 = -\frac{3}{2}C^2, \quad b_1 = 0, \quad \beta = 4AC\alpha. \tag{21}$$

(v) *The fifth set:*

$$a_0 = AC, \quad a_1 = -\frac{3}{2}C^2, \quad b_1 = -\frac{3}{2}A^2, \quad \beta = 16AC\alpha. \tag{22}$$

(vi) *The sixth set:*

$$a_0 = -3AC, \quad a_1 = -\frac{3}{2}C^2, \quad b_1 = -\frac{3}{2}A^2, \quad \beta = -16AC\alpha. \tag{23}$$

This in turn gives the following general set of solutions

$$u_I = -\frac{3}{2}AC - \frac{3}{2}A^2 Y^{-2}(\xi), \quad \beta = -4AC\alpha, \tag{24}$$

$$u_{II} = -\frac{1}{2}AC - \frac{3}{2}A^2 Y^{-2}(\xi), \quad \beta = 4AC\alpha, \tag{25}$$

$$u_{III} = -\frac{3}{2}AC - \frac{3}{2}C^2 Y^2(\xi), \quad \beta = -4AC\alpha, \tag{26}$$

$$u_{IV} = -\frac{1}{2}AC - \frac{3}{2}C^2 Y^2(\xi), \quad \beta = 4AC\alpha, \tag{27}$$

$$u_V = AC - \frac{3}{2}C^2 Y^2(\xi) - \frac{3}{2}A^2 Y^{-2}(\xi), \quad \beta = 16AC\alpha, \tag{28}$$

$$u_{VI} = -3AC - \frac{3}{2}C^2 Y^2(\xi) - \frac{3}{2}A^2 Y^{-2}(\xi), \quad \beta = -16AC\alpha, \tag{29}$$

where $A$ and $C$ are arbitrary constants and $Y$ takes many trigonometric and hyperbolic functions as shown in (10).

*Case I:* We first consider $u_I(x, t)$. We use the first result of (18). We then apply the related $Y$ functions for this choice of $A$ and $C$.

Using the first case in (10) where $A = \frac{1}{2}$ and $C = -\frac{1}{2}$ gives the solution

$$u_1 = -\frac{3}{8}\operatorname{csch}^2\left(\frac{\xi}{2}\right) \quad \text{and} \quad v_1 = -\frac{3}{8}\operatorname{csch}^2\left(\frac{\xi}{2}\right), \tag{30}$$

and soliton solution

$$u_2 = \frac{3}{8}\operatorname{sech}^2\left(\frac{\xi}{2}\right) \quad \text{and} \quad v_2 = \frac{3}{8}\operatorname{sech}^2\left(\frac{\xi}{2}\right), \tag{31}$$

where $\beta = \alpha$.

For $A = \frac{1}{2}$ and $C = \frac{1}{2}$ we find $\beta = -\alpha$, and we therefore obtain the solution

$$u_3 = -\frac{3}{8}\csc^2(\xi) \quad \text{and} \quad v_3 = -\frac{3}{8}\csc^2(\xi), \tag{32}$$

and the soliton solution

$$u_4 = -\frac{3}{8}\sec^2(\xi) \quad \text{and} \quad v_4 = -\frac{3}{8}\sec^2(\xi), \tag{33}$$

$$u_5 = -\frac{3}{8}\left[1 + \frac{1}{(\tan\xi \pm \sec\xi)^2})\right]. \tag{34}$$

For $A = 1$ and $C = -1$ we find $\beta = 4\alpha$, and we therefore obtain the solution

$$u_6 = -\frac{3}{2}\operatorname{csch}^2(\xi) \quad \text{and} \quad v_6 = -\frac{3}{2}\operatorname{csch}^2(\xi), \tag{35}$$

and the soliton solution

$$u_7 = \frac{3}{2}\operatorname{sech}^2(\xi) \quad \text{and} \quad v_7 = \frac{3}{2}\operatorname{sech}^2(\xi). \tag{36}$$

For $A = 1$ and $C = 1$ we find $\beta = -4\alpha$, and we therefore obtain the solutions

$$u_8 = -\frac{3}{2}\csc^2(\xi) \quad \text{and} \quad v_8 = -\frac{3}{2}\csc^2(\xi), \tag{37}$$

$$u_9 = -\frac{3}{2}\sec^2(\xi) \quad \text{and} \quad v_9 = -\frac{3}{2}\sec^2(\xi). \tag{38}$$

For $A = 1$ and $C = -4$ we find $\beta = 16\alpha$, and we therefore obtain the solution

$$u_{10} = \frac{3}{8}[16 - \coth^2(2\xi)] \quad \text{and} \quad v_{10} = \frac{3}{8}[16 - \coth^2(2\xi)], \tag{39}$$

and the soliton solution

$$u_{11} = \frac{3}{8}[16 - \tanh^2(2\xi)] \quad \text{and} \quad v_{11} = \frac{3}{8}[16 - \tanh^2(2\xi)]. \tag{40}$$

For $A = 1$ and $C = 4$ we find $\beta = -16\alpha$, and we therefore obtain the solutions

$$u_{12} = -\frac{3}{8}[16 + \cot^2(2\xi)] \quad \text{and} \quad v_{12} = -\frac{3}{8}[16 + \cot^2(2\xi)], \tag{41}$$

$$u_{13} = -\frac{3}{8}[16 + \tan^2(2\xi)] \quad \text{and} \quad v_{13} = -\frac{3}{8}[16 + \tan^2(2\xi)]. \tag{42}$$

*Case II:* We first consider $u_{II}(x,t)$. We use the second result of (19). Using the first case in (10) where $A=\frac{1}{2}$ and $C=-\frac{1}{2}$ gives the solution

$$u_{14}=\frac{1}{8}\left[1-3\coth^2\left(\frac{\xi}{2}\right)\right]\quad\text{and}\quad v_{14}=\frac{1}{8}\left[1-3\coth^2\left(\frac{\xi}{2}\right)\right], \tag{43}$$

and the soliton solution

$$u_{15}=\frac{1}{8}\left[1-3\tanh^2\left(\frac{\xi}{2}\right)\right]\quad\text{and}\quad v_{15}=\frac{1}{8}\left[1-3\tanh^2\left(\frac{\xi}{2}\right)\right], \tag{44}$$

where $\beta=-\alpha$.

For $A=\frac{1}{2}$ and $C=\frac{1}{2}$ we find $\beta=\alpha$, and we therefore obtain the solutions

$$u_{16}=-\frac{1}{8}[1+3\cot^2(\xi)]\quad\text{and}\quad v_{16}=-\frac{1}{8}[1+3\cot^2(\xi)], \tag{45}$$

$$u_{17}=-\frac{1}{8}[1+3\tan^2(\xi)]\quad\text{and}\quad v_{17}=-\frac{1}{8}[1+3\tan^2(\xi)], \tag{46}$$

$$u_{18}=-\frac{1}{8}\left[1+\frac{3}{(\tan\xi\pm\sec\xi)^2})\right]\quad\text{and}\quad v_{18}=-\frac{1}{8}\left[1+\frac{3}{(\tan\xi\pm\sec\xi)^2})\right]. \tag{47}$$

For $A=1$ and $C=-1$ we find $\beta=-4\alpha$, and we therefore obtain the solution

$$u_{19}=\frac{1}{2}[1-3\coth^2(\xi)]\quad\text{and}\quad v_{19}=\frac{1}{2}[1-3\coth^2(\xi)], \tag{48}$$

and the soliton solution

$$u_{20}=\frac{1}{2}[1-3\tanh^2(\xi)]\quad\text{and}\quad v_{20}=\frac{1}{2}[1-3\tanh^2(\xi)]. \tag{49}$$

For $A=1$ and $C=1$ we find $\beta=4\alpha$, and we therefore obtain the solutions

$$u_{21}=-\frac{1}{2}[1+3\cot^2(\xi)]\quad\text{and}\quad v_{21}=-\frac{1}{2}[1+3\cot^2(\xi)], \tag{50}$$

$$u_{22}=-\frac{1}{2}[1+3\tan^2(\xi)]\quad\text{and}\quad v_{22}=-\frac{1}{2}[1+3\tan^2(\xi)]. \tag{51}$$

For $A=1$ and $C=-4$ we find $\beta=-16\alpha$, and we therefore obtain the solution

$$u_{23}=2-\frac{3}{8}\coth^2(2\xi)\quad\text{and}\quad v_{23}=2-\frac{3}{8}\coth^2(2\xi), \tag{52}$$

and the soliton solution

$$u_{24}=2-\frac{3}{8}\tanh^2(2\xi)\quad\text{and}\quad v_{24}=2-\frac{3}{8}\tanh^2(2\xi). \tag{53}$$

For $A=1$ and $C=4$ we find $\beta=16\alpha$, and we therefore obtain the solutions

$$u_{25}=-2-\frac{3}{8}\cot^2(2\xi)\quad\text{and}\quad v_{25}=-2-\frac{3}{8}\cot^2(2\xi), \tag{54}$$

$$u_{26}=-2-\frac{3}{8}\tan^2(2\xi)\quad\text{and}\quad v_{26}=-2-\frac{3}{8}\tan^2(2\xi). \tag{55}$$

*Case III:* We next consider $u_{III}(x,t)$. We use the third result of (20). Using the first case in (10) where $A=\frac{1}{2}$ and $C=-\frac{1}{2}$ gives the solutions $u_1, u_2$ and $v_1, v_2$.

For $A=\frac{1}{2}$ and $C=\frac{1}{2}$ we find $\beta=-\alpha$, and we therefore obtain the solutions $u_3, u_4, u_5, v_3, v_4$ and $v_5$.

For $A=1$ and $C=-1$ we find $\beta=4\alpha$, and we therefore obtain the solutions $u_6, u_7, v_6$ and $v_7$.

For $A=1$ and $C=1$ we find $\beta=-4\alpha$, and we therefore obtain the solutions $u_8, u_9, v_8$ and $v_9$.

For $A=1$ and $C=-4$ we find $\beta=16\alpha$, and we therefore obtain the soliton solution

$$u_{27}=6\mathrm{sech}^2(2\xi)\quad\text{and}\quad v_{27}=6\mathrm{sech}^2(2\xi), \tag{56}$$

and the solution

$$u_{28}=-6\mathrm{csch}^2(2\xi)\quad\text{and}\quad v_{28}=-6\mathrm{csch}^2(2\xi). \tag{57}$$

For $A=1$ and $C=4$ we find $\beta=-16\alpha$, and we therefore obtain the solutions

$$u_{29}=-6\sec^2(2\xi)\quad\text{and}\quad v_{29}=-6\sec^2(2\xi), \tag{58}$$

$$u_{30}=-6\csc^2(2\xi)\quad\text{and}\quad v_{30}=-6\csc^2(2\xi). \tag{59}$$

*Case IV:* We next consider $u_{IV}(x,t)$. We use the fourth result of (21). Using the first case in (10) where $A=\frac{1}{2}$ and $C=-\frac{1}{2}$ gives the solutions $u_{14}, u_{15}, v_{14}$ and $v_{15}$.

For $A=\frac{1}{2}$ and $C=\frac{1}{2}$ we find $\beta=\alpha$, and we therefore obtain the solutions $u_{16}, u_{17}, u_{18}, v_{16}, v_{17}$ and $v_{18}$.

For $A=1$ and $C=-1$ we find $\beta=-4\alpha$, and we therefore obtain the solutions $u_{19}, u_{20}, v_{19}$ and $v_{20}$.

For $A=1$ and $C=1$ we find $\beta=4\alpha$, and we therefore obtain the solutions $u_{21}, u_{22}, v_{21}$ and $v_{22}$.

For $A=1$ and $C=-4$ we find $\beta=16\alpha$, and we therefore obtain the soliton solution

$$u_{31}=2[1-3\tanh^2(2\xi)]\quad\text{and}\quad v_{31}=2[1-3\tanh^2(2\xi)], \tag{60}$$

and the solution

$$u_{32}=2[1-3\coth^2(2\xi)]\quad\text{and}\quad v_{32}=2[1-3\coth^2(2\xi)]. \tag{61}$$

For $A=1$ and $C=4$ we find $\beta=-16\alpha$, and we therefore obtain the solutions

$$u_{33}=-2[1+3\tan^2(2\xi)]\quad\text{and}\quad v_{33}=-2[1+3\tan^2(2\xi)], \tag{62}$$

$$u_{34}=-2[1+3\cot^2(2\xi)]\quad\text{and}\quad v_{34}=-2[1+3\cot^2(2\xi)]. \tag{63}$$

*Case V:* We next consider $u_V(x,t)$. We use the fifth result of (22). Using the first case in (10) where $A=\frac{1}{2}$ and $C=-\frac{1}{2}$ gives the solution

$$u_{35}=v_{35}=-\frac{1}{8}\left[2+3\tanh^2\left(\frac{\xi}{2}\right)+3\coth^2\left(\frac{\xi}{2}\right)\right], \tag{64}$$

where $\beta=-4\alpha$.

For $A=\frac{1}{2}$ and $C=\frac{1}{2}$ we find $\beta=4\alpha$, and we therefore obtain the solutions

$$u_{36}=v_{36}=\frac{1}{8}\left[2-3\tan^2\left(\frac{\xi}{2}\right)-3\cot^2\left(\frac{\xi}{2}\right)\right], \tag{65}$$

$$u_{37}=v_{37}=\frac{1}{8}[2-3(\tan\xi\pm\sec\xi)^2-3(\tan\xi\pm\sec\xi)^{-2}], \tag{66}$$

For $A=1$ and $C=-1$ we find $\beta=-16\alpha$, and we therefore obtain the solution

$$u_{38}=v_{38}=-\frac{1}{2}[2+3\tanh^2(\xi)+3\coth^2(\xi)]. \tag{67}$$

For $A=1$ and $C=1$ we find $\beta=16\alpha$, and we therefore obtain the solution

$$u_{39}=v_{39}=\frac{1}{2}[2-3\tan^2(\xi)-3\cot^2(\xi)]. \tag{68}$$

For $A = 1$ and $C = -4$ we find $\beta = -64\alpha$, and we therefore obtain the solution

$$u_{40} = v_{40} = -4 - 6\tanh^2(2\xi) - \frac{3}{8}\coth^2(2\xi), \tag{69}$$

For $A = 1$ and $C = 4$ we find $\beta = 64\alpha$, and we therefore obtain the solution

$$u_{41} = v_{41} = 4 - 6\tan^2(2\xi) - \frac{3}{8}\cot^2(2\xi), \tag{70}$$

*Case VI:* We next consider $u_{VI}(x,t)$. We use the sixth result of (23). Using the first case in (10) where $A = \frac{1}{2}$ and $C = -\frac{1}{2}$ gives the solution

$$u_{42} = v_{42} = \frac{3}{8}\left[1 - \tanh^2\left(\frac{\xi}{2}\right) - \coth^2\left(\frac{\xi}{2}\right)\right], \tag{71}$$

where $\beta = 4\alpha$.

For $A = \frac{1}{2}$ and $C = \frac{1}{2}$ we find $\beta = -4\alpha$, and we therefore obtain the solutions

$$u_{43} = v_{43} = -\frac{3}{8}\left[1 + \tan^2\left(\frac{\xi}{2}\right) + \cot^2\left(\frac{\xi}{2}\right)\right], \tag{72}$$

$$u_{44} = v_{44} = -\frac{3}{8}[1 + (\tan\xi \pm \sec\xi)^2 + (\tan\xi \pm \sec\xi)^{-2}], \tag{73}$$

For $A = 1$ and $C = -1$ we find $\beta = 16\alpha$, and we therefore obtain the solution

$$u_{45} = v_{45} = \frac{3}{2}[2 - \tanh^2(\xi) - \coth^2(\xi)]. \tag{74}$$

For $A = 1$ and $C = 1$ we find $\beta = -16\alpha$, and we therefore obtain the solution

$$u_{46} = v_{46} = -\frac{3}{2}[2 + \tan^2(\xi) + \cot^2(\xi)]. \tag{75}$$

For $A = 1$ and $C = -4$ we find $\beta = 64\alpha$, and we therefore obtain the solution

$$u_{47} = v_{47} = 12 - 6\tanh^2(2\xi) - \frac{3}{8}\coth^2(2\xi)]. \tag{76}$$

For $A = 1$ and $C = 4$ we find $\beta = -64\alpha$, and we therefore obtain the solution

$$u_{48} = v_{48} = -12 - 6\tan^2(2\xi) - \frac{3}{8}\cot^2(2\xi)]. \tag{77}$$

Comparing some of our results with Zhang's (2007), Cheng and Li's (2003) and Peng's (2005) results, it can be seen that the results are the same. Some of these results are in agreement with the results reported by others in the literature, and new results are formally developed in this work.

## 5. Conclusion

The tanh–coth method combined with the Riccati equation was successfully used to establish solitary wave solutions. Many well known nonlinear wave equations were handled by this method. The performance of the this method is reliable and effective and gives more solutions. The applied method will be used in further works to establish more entirely new solutions for other kinds of nonlinear wave equations. The availability of computer systems like *Mathematica*or *Maple* facilitates the tedious algebraic calculations. The method which we have proposed in this letter is also a standard, direct and computerizable method, which allows us to solve complicated and tedious algebraic calculation.

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