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# The Kernel Theorem of Hilbert–Schmidt operators

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**Abstract**

All Hilbert–Schmidt operators acting on $L^2$-sections of a vector bundle with fiber a separable Hilbert space $H$ over a compact Riemannian manifold $M$, are characterized. This is achieved by defining the vector bundle of Hilbert–Schmidt operators on $H$, and then making use of a classical result known as the Kernel Theorem of Hilbert–Schmidt operators.


## 1. Introduction

In this paper we are concerned with the following problem: to characterize all the Hilbert–Schmidt operators acting on $L^2$-sections of a vector bundle with fiber a separable Hilbert space $H$ over a compact Riemannian manifold $M$. This is achieved by defining the vector bundle of Hilbert–Schmidt operators on $H$, and then making use of a classical result known as the Kernel Theorem of Hilbert–Schmidt operators [1, p. 306].

This characterization of Hilbert–Schmidt operators has broadly been used in the context of geometric functional analysis and mathematical physics (see, for instance, [2]), but a rigorous proof is missing in the literature.

Let $E$ and $F$ be separable Hilbert spaces. The operators called Hilbert–Schmidt are introduced in the following fashion: if $\{e_n\}$ and $\{f_m^*\}$ are orthonormal bases of $E$ and of $F^*$, and if for an operator $A : E \to F$ the series

$$\|A\|_2 = \left( \sum_{n,m=1}^{\infty} \left| \langle f_m^*, Ae_n \rangle \right|^2 \right)^{1/2}$$

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is convergent, then this series does not depend on the choice of the bases, where $\langle\,,\rangle$ denotes the duality pairing on $F^* \times F$. Thus we say that $A$ is a Hilbert–Schmidt operator if $\|A\|_2$ is finite. These operators are compact and can be represented by "infinite matrices" $\{a_{m,n}\} \in l^2(\mathbf{N} \times \mathbf{N})$ in the following fashion:

$$Au = \sum_{m,n=1}^{\infty} a_{m,n} \langle e_n^*, u \rangle f_m,$$

where $\{e_n^*\}$ and $\{f_m\}$ are orthonormal bases of $E^*$ and of $F$.

The most important example is that of integral operators $\widetilde{K}$ associated with the functions $K \in L^2(\Omega_1 \times \Omega_2)$, where $\Omega_i$ is an open set in $\mathbf{R}^n$, by the formula

$$(\widetilde{K}u)(\omega_2) = \int_{\Omega_1} K(\omega_1, \omega_2) u(\omega_1)\, d\omega_1.$$

These integral operators, therefore, form a class of compact operators. Moreover, we establish in Section 3 that every Hilbert–Schmidt operator $\widetilde{K}$ from $L^2(\Omega_1)$ to $L^2(\Omega_2)$ is defined by a *kernel* $K \in L^2(\Omega_1 \times \Omega_2)$. This is known as the Kernel Theorem.

When $\Omega_i$ is replaced by a Riemannian manifold $M$ and the function $u$ is allowed to take values on a Hilbert space $H$, the Kernel Theorem generalizes where now the kernel $K$ is a section of a bundle over $M$ of Hilbert–Schmidt operators acting on $H$.

The paper is organized as follows: In Section 2 we define the vector bundle of Hilbert–Schmidt operators over a manifold.

In Section 3 we prove the Kernel Theorem in this general context. As a corollary, we obtain that operators mapping $L^2$-sections into differentiable sections (in the distributional sense) have an integral kernel.

## 2. The vector bundle of Hilbert–Schmidt operators

In this section we define the vector bundle of Hilbert–Schmidt operators over a manifold. Let $M$ be a $C^\infty$ manifold. Consider a disjoint union $\mathcal{V}_M = \bigcup_{x\in M} F_x$ of a family of vector spaces parameterized by the set $M$. It is sometimes convenient to describe a point of $\mathcal{V}_M$ by $(x; w)$, where $x \in M$ and $w \in F_x$.

The map $\pi : \mathcal{V}_M \to M$, $\pi(x; w) = x$ is called the *projection*. $F_x$ is called the *fiber* of $\mathcal{V}_M$ at $x$.

**Definition 2.1.** $\mathcal{V}_M$ is called a $C^\infty$ *vector bundle* over $M$ with fiber $F$, if there is an open covering $\{U_\alpha;\ \alpha \in M\}$ of $M$ satisfying the following:

(1) For each $U_\alpha$, there is a map $\tau_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha)$ such that $\pi\tau_\alpha(x, u) = x$, and for any fixed $x$, $\tau_\alpha(x) : F \to F_x$ $(= \pi^{-1}(x))$ is a linear isomorphism, where $\tau_\alpha(x)u = \tau_\alpha(x, u)$.
(2) If $U_\alpha \cap U_\beta \neq \emptyset$, then $\Phi : U_\alpha \cap U_\beta \times F \to F$ defined by $\Phi(x)u = \tau_\alpha(x)^{-1}\tau_\beta(x)u$ is a $C^\infty$ map.
(3) $A$ is maximal among indexed families satisfying (1) and (2).

**Theorem 2.1.** *For $i = 1, 2$; let $M_i$ be a $C^\infty$ manifold. Let $\mathcal{V}_{M_i}$ be a vector bundle over $M_i$ with fiber a separable Hilbert space $F_i$. Consider the disjoint union*

$$\mathcal{V}_{M_1\times M_2} = \bigcup_{(x,y)\in M_1\times M_2} \mathcal{L}_2(F_{1x}, F_{2y})$$

*of a family of Hilbert–Schmidt operators from $F_{1x}$ into $F_{2y}$, parameterized by the set $M_1 \times M_2$. Then $\mathcal{V}_{M_1\times M_2}$ is a $C^\infty$ vector bundle over $M_1 \times M_2$ with fiber $\mathcal{L}_2(F_1, F_2)$, the space of Hilbert–Schmidt operators from $F_1$ into $F_2$.*

**Proof.** Let $\{U_{i,\alpha_i};\ \alpha_i \in A_i\}$ be an open covering of $M_i$ satisfying the conditions of Definition 2.1. Then $\{U_{1,\alpha_1} \times U_{2,\alpha_2};\ (\alpha_1, \alpha_2) \in A_1 \times A_2\}$ is an open covering of $M_1 \times M_2$ which satisfies the following: Let $\tau_{i,\alpha_i} : U_{i,\alpha_i} \times F_i$ be a local trivialization of $\mathcal{V}_{M_i}$. For $(x, y) \in U_{1,\alpha_1} \times U_{2,\alpha_2}$ and $L \in \mathcal{L}_2(F_1, F_2)$, set

$$\tau_{\alpha_1\alpha_2}(x, y; L) = \tau_{2,\alpha_2}(y) L \tau_{1,\alpha_1}^{-1}(x). \tag{1}$$

By Proposition 12.1.2 in [1], $\tau_{\alpha_1\alpha_2}(x, y; L) \in \mathcal{L}_2(F_{1x}, F_{2y})$. Define the map $\rho_{\alpha_1\alpha_2}(x, y) : \mathcal{L}_2(F_{1x}, F_{2y}) \to \mathcal{L}_2(F_1, F_2)$ by

$$\rho_{\alpha_1\alpha_2}(x, y) L = \tau_{2,\alpha_2}^{-1}(y) L \tau_{1,\alpha_1}(x). \tag{2}$$

It follows that

$$\tau_{\alpha_1\alpha_2}\big(x, y; \rho_{\alpha_1\alpha_2}(x, y) L\big) = L,$$

and

$$\rho_{\alpha_1\alpha_2}(x, y) \tau_{\alpha_1\alpha_2}(x, y; L) = L.$$

Then $\rho_{\alpha_1\alpha_2}(x, y)$ is an inverse of $\tau_{\alpha_1\alpha_2}(x, y; \cdot)$. Hence, to show that

$$\tau_{\alpha_1\alpha_2}(x, y; \cdot) : \mathcal{L}_2(F_1, F_2) \to \mathcal{L}_2(F_{1x}, F_{2y})$$

is a linear isomorphism it suffices to show that it is a continuous map. We then estimate the Hilbert–Schmidt norm of $\tau_{\alpha_1\alpha_2}(x, y; L)$ for all $L \in \mathcal{L}_2(F_1, F_2)$. By Proposition 12.1.2 in [1] and by (1),

$$\big\|\tau_{\alpha_1\alpha_2}(x, y; L)\big\|_2 \leqslant \big\|\tau_{2,\alpha_2}(y)\big\| \|L\|_2 \big\|\tau_{1,\alpha_1}^{-1}(x)\big\|,$$

where $\|\ \|$ denotes the norm in $F_i$. It follows that $\tau_{\alpha_1\alpha_2}(x, y; \cdot)$ is continuous. This verifies condition (1) of Definition 2.1.

To check condition (2) we proceed as follows. If $U_{i,\alpha_i} \cap U_{i,\beta_i} \neq \emptyset$, let $\Phi : (U_{1,\alpha_1} \cap U_{1,\beta_1} \times U_{2,\alpha_2} \cap U_{2,\beta_2}) \times \mathcal{L}_2(F_1, F_2) \to \mathcal{L}_2(F_1, F_2)$ be defined by $\Phi(x, y)L = \tau_{\alpha_1\alpha_2}(x, y)^{-1} \tau_{\beta_1\beta_2}(x, y)L$. By (1) and (2)

$$\Phi(x, y)L = \tau_{2,\alpha_2}^{-1}(y) \tau_{2,\beta_2}(y) L \tau_{1,\beta_1}^{-1}(x) \tau_{1,\alpha_1}(x).$$

Since $\Phi_i : U_{i,\alpha_i} \cap U_{i,\beta_i} \times F_i \to F_i$ defined by $\Phi_i(x)u = \tau_{\alpha_i}^{-1}(x) \tau_{\beta_i}(x) u$ is a $C^\infty$ map, and $\Phi$ is linear in $L$, it follows that $\Phi$ is a $C^\infty$ map. This gives condition (2) of Definition 2.1.

Let $A$ be the maximal family among indexed families satisfying conditions (1) and (2) above. This shows that $\mathcal{V}_{M_1\times M_2}$ is a $C^\infty$ vector bundle over $M_1 \times M_2$ with fiber $\mathcal{L}_2(F_1, F_2)$. □

## 3. The Kernel Theorem

Assume in the rest of this paper that $M$ is a compact $n$-dimensional $C^\infty$ Riemannian manifold such that if $\partial M \neq \emptyset$, then the boundary is a closed submanifold and near its boundary $M$ is a direct product $\partial M \times [0,\delta)$, $\delta > 0$. To avoid technical difficulties we use also a manifold $\widetilde{M} = ((-\infty, 0] \times \partial M) \cup M$.

For a vector bundle $\mathcal{V}_M$, let $\boldsymbol{\Gamma}(\mathcal{V}_M)$ be the space of all $C^\infty$ sections of $\mathcal{V}_M$. For $u \in \boldsymbol{\Gamma}(\mathcal{V}_M)$, we define the norm $\|u\|$ by

$$\|u\|^2 = \int_M \bigl|u(x)\bigr|_x^2\, dx,$$

where $dx$ is the volume element defined by the Riemannian metric. Let $L^2(\mathcal{V}_M)$ be the completion of $\boldsymbol{\Gamma}(\mathcal{V}_M)$ with respect to the norm $\|u\|$.

**Theorem 3.1.** *Let $M_i$ be a compact $C^\infty$ Riemannian manifold satisfying the above hypothesis. Let $\mathcal{V}_{M_i}$ be a $C^\infty$ vector bundle over $M_i$ with fiber $F_i$ a separable Hilbert space. Finally, let $\mathcal{V}_{M_1 \times M_2}$ be the $C^\infty$ vector bundle over $M_1 \times M_2$ with fiber $\mathcal{L}_2(F_1^*, F_2)$. An operator $\widetilde{K}$ from $L^2(\mathcal{V}_{M_1})$ to $L^2(\mathcal{V}_{M_2})$ is a Hilbert–Schmidt operator if and only if it is associated with a kernel $K \in L^2(\mathcal{V}_{M_1 \times M_2})$.*

We give an application of this theorem. Let $\mathcal{V}_M$ be a $C^\infty$ vector bundle over $M$ with fiber a finite-dimensional Hermitian vector space $E$. Define an inner product $\langle\, ,\rangle_k$ on the space $\boldsymbol{\Gamma}(\mathcal{V}_M)$ of $C^\infty$ sections, by

$$\langle u, v\rangle = \sum_{i=0}^{k} \int_M \bigl\langle \bigl(D^i u\bigr)(x), \bigl(D^i v\bigr)(x)\bigr\rangle\, dx,$$

where $Du$ denotes covariant differentiation with respect to the Riemannian connection. The completion of $\boldsymbol{\Gamma}(\mathcal{V}_M)$ with respect to the norm $\|u\|_k = \langle u, u\rangle_k^{1/2}$ is denoted by $\Gamma^k(\mathcal{V}_M)$.

**Theorem 3.2.** *Let $\mathcal{V}_{M \times M}$ be the $C^\infty$ vector bundle over $M \times M$ with fiber*

$$\mathcal{L}_2(E^*, E).$$

*Let $k \geqslant [n/2] + 1$. Then every continuous operator $\widetilde{K} : L^2(\mathcal{V}_M) \to \Gamma^k(\mathcal{V}_M)$ is associated with a kernel $K \in L^2(\mathcal{V}_{M \times M})$.*

**Proof.** The proof follows from Theorem 3.1. By Lemma 5.1 of Chapter VII in [3], the inclusion map $i : \Gamma^k(\mathcal{V}_M) \hookrightarrow L^2(\mathcal{V}_M)$ is a Hilbert–Schmidt operator. By Proposition 12.1.2 of [1], the composition $i\widetilde{K}$ is Hilbert–Schmidt. Then by Theorem 3.1, this is associated with a kernel $K \in L^2(\mathcal{V}_{M \times M})$. □

The proof of Theorem 3.1 is based on the classical version of the Kernel Theorem (Lemma 3.1). We say that the space $\mathcal{L}_2(E^*, F)$ is the *Hilbert tensor product* of the separable Hilbert spaces $E$ and $F$. We denote this space by

$$E \hat{\otimes} F = \mathcal{L}_2(E^*, F).$$

**Lemma 3.1.** *Let $E$ and $F$ be separable Hilbert spaces, and $\Omega_i$ an open set in $\mathbf{R}^{n_i}$. An operator $\widetilde{K}$ from $L^2(\Omega_1, E)$ to $L^2(\Omega_2, F)$ is a Hilbert–Schmidt operator if and only if it is associated with a kernel $K \in L^2(\Omega_1 \times \Omega_2, \mathcal{L}_2(E^*, F))$.*

**Proof.** Indeed, the space $\mathcal{L}_2(L^2(\Omega_1, E), L^2(\Omega_2, F))$ of Hilbert–Schmidt operators from $L^2(\Omega_1, E)$ to $L^2(\Omega_2, F)$ is equal to $L^2(\Omega_1, E) \hat{\otimes} L^2(\Omega_2, F)$ (since $L^2(\Omega_1, E)$ is identified with its dual space). According to Theorem 12.6.1 in [1], it is isometric to the space $(L^2(\Omega_1) \hat{\otimes} E) \hat{\otimes} (L^2(\Omega_2) \hat{\otimes} F)$. By Proposition 12.3.1 in the same reference, this space is isometric to

$$L^2(\Omega_1) \hat{\otimes} \big(L^2(\Omega_2) \hat{\otimes} (E \hat{\otimes} F)\big) = L^2(\Omega_1) \hat{\otimes} \big(L^2(\Omega_2) \hat{\otimes} \mathcal{L}_2(E^*, F)\big).$$

Again by Theorem 12.6.1, this space is isometric to

$$L^2\big(\Omega_1, L^2\big(\Omega_2, \mathcal{L}_2(E^*, F)\big)\big).$$

But by the Fubini theorem the space $L^2(\Omega_1, L^2(\Omega_2, \mathcal{L}_2(E^*, F)))$ is isomorphic to the space $L^2(\Omega_1 \times \Omega_2, \mathcal{L}^2(E^*, F))$ of square summable Hilbert–Schmidt operators on $\Omega = \Omega_1 \times \Omega_2$. □

**Proof of Theorem 3.1.** For $i = 1, 2$; consider a finite open covering

$$\{U_{i\alpha};\ \alpha \in A_i\}$$

of $M_i$ by trivializing charts. Consider a partition of unity $\{\psi_{i\alpha}\}_{\alpha \in A_i}$ subordinated to the covering $\{U_{i\alpha};\ \alpha \in A_i\}$. For all $\alpha \in A_i$, let $\Psi_{i\alpha} : L^2(\mathcal{V}_{M_i}) \to L^2(\mathcal{V}_{M_i})$ be the operator given by

$$\Psi_{i\alpha} u = \psi_{i\alpha} u. \tag{3}$$

Assume that $\widetilde{K}$ is a Hilbert–Schmidt operator. By Proposition 12.1.2 in [1], $\Psi_{2\beta} \widetilde{K} \Psi_{1\alpha}$ is a Hilbert–Schmidt operator. By Lemma 3.1, $\Psi_{2\beta} \widetilde{K} \Psi_{1\alpha}$ is associated with a kernel $K_{\alpha\beta} \in L^2(U_{1\alpha} \times U_{2\beta}, \mathcal{L}^2(F_1^*, F_2))$. We show that $\widetilde{K}$ has the integral kernel,

$$\widetilde{K} u(y) = \int_{M_1} \sum_{\alpha \in A_1, \beta \in A_2} \psi_{2\beta}(y) K_{\alpha\beta}(x, y) \psi_{1\alpha}(x) u(x)\, dx.$$

This is a computation.

$$\begin{aligned} &\int_{M_1} \sum_{\alpha \in A_1,\ \beta \in A_2} \psi_{2\beta}(y) K_{\alpha\beta}(x, y) \psi_{1\alpha}(x) u(x)\, dx \\ &\quad = \sum_{\alpha \in A_1,\ \beta \in A_2} \int_{U_{1\alpha}} \psi_{2\beta}(y) K_{\alpha\beta}(x, y) \psi_{1\alpha}(x) u(x)\, dx \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\alpha\in A_1,\ \beta\in A_2} \bigl(\Psi_{2\beta}\widetilde{K}\Psi_{1\alpha}u\bigr)(y)\\
&= \sum_{\alpha\in A_1,\ \beta\in A_2} \psi_{2\beta}(y)\bigl(\widetilde{K}\psi_{1\alpha}u\bigr)(y) \quad \text{(by 3)}\\
&= \sum_{\alpha\in A_1} \bigl(\widetilde{K}\psi_{1\alpha}u\bigr)(y)\\
&= \Bigl(\widetilde{K}\sum_{\alpha\in A_1}\psi_{1\alpha}u\Bigr)(y) \quad \text{(since } A_1 \text{ is a finite set)}\\
&= \bigl(\widetilde{K}u\bigr)(y).
\end{aligned}
$$

Moreover, there exists a positive constant $c$ such that for all $(x, y) \in M_1 \times M_2$,

$$
\begin{aligned}
&\Bigl\| \sum_{\alpha\in A_1,\ \beta\in A_2} \psi_{2\beta}(y)K_{\alpha\beta}(x,y)\psi_{1\alpha}(x)\Bigr\|^2_{\mathcal{L}_2(F^*_{1x},F_{2y})}\\
&\quad\leqslant c \sum_{\alpha\in A_1,\ \beta\in A_2} \bigl\|\psi_{2\beta}(y)K_{\alpha\beta}(x,y)\psi_{1\alpha}(x)\bigr\|^2_{\mathcal{L}_2(F^*_{1x},F_{2y})}\\
&\quad= c \sum_{\alpha\in A_1,\ \beta\in A_2} \bigl(\psi_{2\beta}(y)\psi_{1\alpha}(x)\bigr)^2\bigl\|K_{\alpha\beta}(x,y)\bigr\|^2_{\mathcal{L}_2(F^*_{1x},F_{2y})}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\int_{M_1\times M_2} \Bigl\| \sum_{\alpha\in A_1,\ \beta\in A_2} \psi_{2\beta}(y)K_{\alpha\beta}(x,y)\psi_{1\alpha}(x)\Bigr\|^2_{\mathcal{L}_2(F^*_{1x},F_{2y})} dx\,dy\\
&\quad\leqslant c \sum_{\alpha\in A_1,\ \beta\in A_2} \int_{U_{1\alpha}} \psi_{1\alpha}(x)^2 \int_{U_{2\beta}} \psi_{2\beta}(y)^2 \bigl\|K_{\alpha\beta}(x,y)\bigr\|^2_{\mathcal{L}_2(F^*_{1x},F_{2y})} dy\,dx\\
&\quad\leqslant c \sum_{\alpha\in A_1,\ \beta\in A_2} \int_{U_{1\alpha}}\int_{U_{2\beta}} \bigl\|K_{\alpha\beta}(x,y)\bigr\|^2_{\mathcal{L}_2(F^*_{1x},F_{2y})} dy\,dx\\
&\quad< \infty,
\end{aligned}
$$

since $K_{\alpha\beta} \in L^2(U_{1\alpha} \times U_{2\beta}, \mathcal{L}^2(F_1{}^*, F_2))$. This shows that $\widetilde{K}$ is associated with a kernel $K \in L^2(\mathcal{V}_{M_1\times M_2})$.

Conversely, assume that the operator $\widetilde{K}$ is associated with a kernel $K \in L^2(\mathcal{V}_{M_1\times M_2})$. Then, for all $(\alpha, \beta) \in A_1 \times A_2$, $\Psi_{2\beta}\widetilde{K}\Psi_{1\alpha}$ is associated with a kernel $\psi_{2\beta}(y)K(x, y)\psi_{1\alpha}(x) \in L^2(U_{1\alpha} \times U_{2\beta}, \mathcal{L}^2(F_1^*, F_2))$. By Lemma 3.1, the operator $\Psi_{2\beta}\widetilde{K}\Psi_{1\alpha}$ is a Hilbert–Schmidt operator and then

$$
\widetilde{K} = \sum_{\alpha\in A_1,\ \beta\in A_2} \Psi_{2\beta}\widetilde{K}\Psi_{1\alpha},
$$

is also a Hilbert–Schmidt operator. □

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