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Physical parameters reconstruction of a fixed–fixed mass-spring system from its characteristic data ☆

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Abstract

In this paper, an inverse problem of constructing a linear n degree of freedom mass-spring system from part of its physical parameters and part of modality of its maximal or minimal natural frequencies is considered. The solvability and the expression of the solution is derived. The numerical algorithms and some numerical examples are given.

MSC: 15A24; 65F18

Keywords: Jacobi matrix; Inverse eigenpairs problem; Inverse vibration problem; Mass-spring system

1. Introduction

A linear n degree of freedom system of n point masses simply interconnected by springs is characterized by the generalized eigenvalue equation (see [5]),

$$(K_n - \lambda M_n)u = 0, \tag{1.1}$$

where $\lambda = \omega^2$, ω is the natural frequency, u is the modality, K_n is the stiffness matrix,

$$K_n = \begin{bmatrix} k_1+k_2 & -k_2 & & & & \\ -k_2 & k_2+k_3 & -k_3 & & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -k_{n-1} & k_{n-1}+k_n & -k_n \\ & & & & -k_n & k_n+k_{n+1} \end{bmatrix}, \tag{1.2}$$

and $M_n = \mathrm{diag}(m_1, \ldots, m_n)$ is the mass matrix. Of particular interest are matrices K_n satisfying certain row sum conditions, namely (see [13]):

- K_n is a Jacobi matrix with all row sums equal to zero.We shall refer to such a Jacobi matrix as free–free.
- K_n is a Jacobi matrix with all row sums, save the first, equal to zero. We shall call this type fixed–free.

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- K_n is a Jacobi matrix with first and last row sums positive, while all others are zero. K_n with this property will be called fixed–fixed.

In this paper, we will consider reconstructing a fixed–fixed mass-spring system, with $k_1 > 0$, $k_{n+1} > 0$, $m_n > 0$, m_i, $k_{i+1} > 0$ for $i = 1, 2, \dots, n-1$.

Inverse vibration problem, roughly speaking, is how to reconstruct a stiffness matrix and a mass matrix from prescribed natural frequencies or modalities or some physical parameters. Inverse vibration problems for mass-spring system have been of interest for many applications, because of n degree-of-freedom mass-spring system may be thought of as finite-difference or finite-element approximations of continuous systems. Gantmakher and Krein [4], Boley and Golub [1], Gladwell [5], Zhou and Dai [17], Chu [3] had showed that inverse eigen-problems arise in classical vibration theory which the reconstruction of mass-spring system may be thought of as basic inverse vibration problem.

There are many important results on reconstructing a mass-spring system with different given data, for example, Hochstadt [7], Hald [6], de Boor and Golub [2], Ram [15,16] and Jiménez et al. [11] have given us the algorithms for computing physical parameter m_i and k_i with all or partial natural frequencies of system by constructing a Jacobi matrix; Peter Nylen and Frank Uhlig [12,13] focus on the reconstruction of mass-spring system with nature frequencies from the modified system; Hochstadt [8] and Hu etc. [9] has reconstructed a Jacobi matrix with its sub-matrix and all eigenvalue; Hu etc. [10] and Peng [14] has reconstructed a Jacobi matrix with its eigenpairs and subsystem; In this paper, we will also consider reconstructing a fixed–fixed mass-spring systems from some physical parameters of subsystem and two natural frequencies and parts of its modalities, where two natural frequencies are the maximal and the minimal one which have been of important interest for engineering applications. The problems are as follows.

Problem FDDMK I. Given $u \in R^n$ and $w_i \in R_+$, for $i = 1, 2, \dots, n$, find (M_n, K_n), such that, w_i is the maximal (or the minimal) natural frequency of (M_i, K_i), u is a modality corresponding to w_n.

Problem FDDMK II. Given $w_i, w_i^* \in R_+$ for $i = 1, 2, \dots, n$, find (M_n, K_n), such that w_i and w_i^* is the maximal and the minimal natural frequencies of (M_i, K_i), for $i = 1, 2, \dots, n$.

Problem FDDMK III. Given $w_n, w_n^* \in R_+$, $X_2, Y_2 \in R^{n-p}$, and $m_i, k_i \in R_+$ for $i = 1, 2, \dots, p$, find (M_n, K_n) and $X_1, Y_1 \in R^p$, such that, w_n and w_n^* is the maximal and the minimal natural frequencies of (M_n, K_n) respectively, x and y is the modality corresponding to w_n and w_n^* respectively.

Where, R_+ is the set of all positive number, R^n is the set of all n-dimension vector. M_i and K_i be i by i leading principal sub-matrix of M_n and K_n, respectively. (M_i, K_i) denote the i degree of-freedom mass-spring system including physical parameters $m_1, m_2, \dots, m_i$ and $k_1, k_2, \dots, k_i, k_{i+1}$. (M_i, K_i) is a subsystem of (M_n, K_n), $\sigma(M_i, K_i)$ is the set of all roots from $\det(K_i - \lambda M_i) = 0$. $x = (X_1^T, X_2^T)^T$ and $y = (Y_1^T, Y_2^T)^T$, $X_1 = (x_1, \dots, x_p)^T$, $Y_1 = (y_1, \dots, y_p)^T$, $X_2 = (x_{p+1}, \dots, x_n)^T$, $Y_2 = (y_{p+1}, \dots, y_n)^T$. $\mathrm{sign}(\alpha) = 1, 0$ or -1, when $\alpha > 0$, $\alpha = 0$ or $\alpha < 0$, respectively.

This paper is organized as follows. In Section 2, we discuss the properties of (1.1), and the uniqueness of the solutions for Problem FDDMK I and II. In Section 3, we show the existence of the solution for Problem FDDMK III, and derive an expression of the solution. In Section 4, we give an algorithm to compute the solution of problem FDDMK III, and some numerical examples to illustrate the results obtained in this paper are given.

2. Preliminary results, solving problem FDDMK I and II

Lemma 1. *If $m_i > 0$ for $i = 1, 2, \dots, n$, $a_i = (k_i + k_{i+1})m_i^{-1}$, $b_i = -k_{i+1}m_i^{-1}$, $c_i = -k_{i+1}m_{i+1}^{-1}$, then $(K_n - \lambda M_n)u = 0$ is equivalent to $Au = \lambda u$, where*

$$A = M_n^{-1}K_n = \begin{bmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & c_{n-2} & a_{n-1} & b_{n-1} \\ & & & c_{n-1} & a_n \end{bmatrix}. \tag{2.1}$$

Remark. $\lambda^* \in R_+$ is an eigenvalue of A and $\mu^* \in R^n$ is an eigenvector corresponding to λ^* if and only if λ^* is a general eigenvalue of matrix-pairs K_n, M_n, and $(K_n - \lambda^* M_n)u^* = 0$.

Note that $b_i c_i > 0$ for $i = 1, 2, \ldots, n-1$, matrix A is a Jacobian matrix which has some important properties as follows (see e.g., [5,17]).

Property 1. Suppose A_i be a $i \times i$ leading principal sub-matrix of Jacobi matrix A, $D_i(\lambda) = \det(A_i - \lambda I)$, I is the identity matrix of size i, then

$$D_i(\lambda) = (a_i - \lambda)D_{i-1}(\lambda) - b_{i-1}c_{i-1}D_{i-2}(\lambda), \quad i = 2, 3, \ldots, n, \tag{2.2}$$

where $D_0(\lambda) = 1$ and $D_1(\lambda) = a_1 - \lambda$.

Property 2. Suppose that $\lambda^{(i)}$ is an eigenvalue of A_i, $\lambda_1^{(i)}$ and $\lambda_i^{(i)}$ is the minimal and maximal zero point of $D_i(\lambda)$, then

(1) $D_i(\lambda^{(i)}) = 0$, $D_{i-1}(\lambda^{(i)}) \neq 0$, for $i = 2, \ldots, n$;
(2) $\lambda_1^{(n)} < \lambda_1^{(n-1)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_n^{(n)}$.

Property 3. Suppose that $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and $y = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$ is an eigenvector corresponding to maximal eigenvalue λ and minimal eigenvalue μ, respectively, then (1) $x_i x_{i+1} < 0$, $y_i y_{i+1} > 0$ for $i = 1, 2, \ldots, n-1$; or (2) $\operatorname{sign}(x_i) = (-1)^{j-i}\operatorname{sign}(x_j)$, $\operatorname{sign}(y_i) = \operatorname{sign}(y_j)$ for $i, j = 1, 2, \ldots, n$.

Let $\varphi_i(\lambda) = \det(K_i - \lambda M_i)$, it has the similar properties with $D_i(\lambda) = \det(A_i - \lambda I)$. The following lemmas are a direct result from Property 1–3. Their proofs are similar to the proof of Property 1–3, and are omitted (see [5,17]).

Lemma 2. *Suppose that* $\varphi_0(\lambda) = 1$, $\varphi_1(\lambda) = k_1 + k_2 - \lambda m_1$, $\varphi_i(\lambda) = \det(K_i - \lambda M_i)$ *for* $i = 2, 3, \ldots, n-1$, *then*

$$\varphi_i(\lambda) = (k_i + k_{i+1} - \lambda m_i)\varphi_{i-1}(\lambda) - k_i^2 \varphi_{i-2}(\lambda), \tag{2.3}$$

Note that $\varphi_i(\lambda)$ *is made of physical parameters* $m_1, m_2, \ldots, m_i$ *and* $k_1, k_2, \ldots, k_i, k_{i+1}$.

Lemma 3. *Suppose that* $\lambda^{(i)}$ *is a zero point of* $\varphi_i(\lambda)$, $\omega_1^{(i)}$ *and* $\omega_i^{(i)}$ *is the minimal and the maximal natural frequency of mass-spring system* (M_i, K_i) *for* $i = 1, 2, \ldots, n$ *then*

(1) $\varphi_{i-1}(\lambda^{(i)}) \neq 0$ *for* $i = 2, 3, \ldots, n$;
(2) $0 < \omega_1^{(n)} < \omega_1^{(n-1)} < \cdots < \omega_1^{(2)} < \omega_1^{(1)} < \omega_2^{(2)} < \cdots < \omega_n^{(n)}$.

Lemma 4. *Suppose that* $u = (u_1, u_2, \ldots, u_n)^{\mathrm{T}}$ *and* $v = (v_1, v_2, \ldots, v_n)^{\mathrm{T}}$ *is a modality corresponding to the maximal and the minimal natural frequency of mass-spring system* (M_n, K_n), *respectively, then it is true that* (1) $u_i u_{i+1} < 0$, $v_i v_{i+1} > 0$ *for* $i = 1, 2, \ldots, n-1$; *or* (2) $\operatorname{sign}(u_i) = (-1)^{j-i}\operatorname{sign}(u_j)$, $\operatorname{sign}(v_i) = \operatorname{sign}(v_j)$, *for* $i, j = 1, 2, \ldots, n$.

In the following, we find the solution of Problem FDDMK I and II. Obviously, solving Problem FDDMK I is equivalent to solving the inverse general eigen-pairs problem as follows: given $\lambda_i = \omega_i^2$ and real vector u, find $M_n, K_n \in R^{n\times n}$, such that

$$(K_n - \lambda_n M_n)u = 0, \quad \varphi_i(\lambda_i) = 0, \text{ for } i = 1, 2, \ldots, n. \tag{2.4}$$

Let $E_i = k_i(u_{i-1} - u_i)$, $F_i = (k_i^2 \varphi_{i-2}(\lambda_i)\varphi_{i-1}^{-1}(\lambda_i) - k_i)$, for $i = 2, \ldots, n$; $\triangle_i = (\lambda_n - \lambda_i)u_i + \lambda_i u_{i+1}$, $\triangle_{m_i} = (u_i - u_{i+1})F_i - E_i$, $\triangle_{k_{i+1}} = \lambda_n u_i F_i - \lambda_i E_i$, for $i = 1, 2, \ldots, n-1$.

Theorem 1. *Without loss of generality, we assume that w_i is the maximal natural frequency of (M_i, K_i). Suppose that $k_1 > 0$ and $\sum_{i=1}^{n} m_i = \Omega > 0$, Problem* FDDMK I *has unique solution, if and only if, the following conditions are satisfied*

(1) $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$, $\varphi_{i-1}(\lambda_i) \neq 0$, $i = 2, \ldots, n-1, n$, *where* $\lambda_i = w_i^2$;
(2) $\mathrm{sign}(u_i) = (-1)^{i-1}\mathrm{sign}(u_1)$, $i = 2, \ldots, n-1, n$;
(3) $\triangle_i \triangle_{m_i} \triangle_{k_{i+1}} \neq 0$, $\mathrm{sign}(\triangle_i) = \mathrm{sign}(\triangle_{m_i}) = \mathrm{sign}(\triangle_{k_{i+1}})$, *for* $i = 2, 3, \ldots, n-1$;
(4) $\mathrm{sign}(\lambda_1 \triangle_{m_1} \triangle_1^{-1} - k_1) = 1$;
(5) $u_{n-1}\varphi_{n-1}(\lambda_n)\ \triangle_{n-1} - u_n \varphi_{n-2}(\lambda_n)\triangle_{k_n} = 0$;
(6) $\mathrm{sign}(\Omega - \sum_{i=1}^{n-1} \triangle_{m_i} \triangle_i^{-1}) = 1$;
(7) $\mathrm{sign}(\lambda_n \Omega - \lambda_n \sum_{i=1}^{n-1} \triangle_{m_i} \triangle_i^{-1} + (u_{n-1}u_n^{-1} - 1)\triangle_{k_n}\triangle_{n-1}^{-1}) = 1$.

Proof. *Necessity*. Assume that Problem FDDMK I has unique solution, that is, Eq. (2.4) have unique solution. Hence, conditions (1) and (2) can be derived from Lemmas 3, 4. Expanding (2.4), we obtain for $i = 1, 2, \ldots, n$

$$(u_i - u_{i+1})k_{i+1} - \lambda_n u_i m_i = k_i(u_{i-1} - u_i), \tag{2.5}$$

$$\varphi_{i-1}(\lambda_i)k_{i+1} - \lambda_i \varphi_{i-1}(\lambda_i)m_i = k_i^2 \varphi_{i-2}(\lambda_i) - k_i \varphi_{i-1}(\lambda_i), \tag{2.6}$$

where $u_0 = u_{n+1} = 0$, $\varphi_{-1}(\lambda) = 0$.

Since $m_i > 0$ and $k_{i+1} > 0$ for $i = 2, \ldots, n-1$, we have condition (3) and

$$m_i = \triangle m_i / \triangle_i, \quad k_{i+1} = \triangle_{k_{i+1}} / \triangle_i \ \text{ for } i = 2, \ldots, n-1. \tag{2.7}$$

By Eqs. (2.5) and (2.6) for $i = 1$, we have

$$(u_1 - u_2)k_2 - \lambda_n u_1 m_1 = -k_1 u_1, \quad k_2 - \lambda_1 m_1 = -k_1. \tag{(2.8),(2.9)}$$

Since $m_1 > 0$ and $k_2 > 0$, we have conditions (3) and (4), and

$$m_1 = \triangle_{m_1} / \triangle_1, \quad k_2 = \triangle_{k_2} / \triangle_1 = m_1 \lambda_1 - k_1. \tag{2.10}$$

Similarly by Eqs. (2.5) and (2.6) for $i = n$, $u_n \neq 0$ and $\varphi_{n-1}(\lambda_n) \neq 0$, $k_n + k_{n+1} - \lambda_n m_n$ can be expressed as

$$k_n + k_{n+1} - \lambda_n m_n = \frac{k_n u_{n-1}}{u_n} \quad \text{and} \quad k_n + k_{n+1} - \lambda_n m_n = \frac{k_n^2 \varphi_{n-2}(\lambda_n)}{\varphi_{n-1}(\lambda_n)}. \tag{2.11}$$

Hence, condition (5) can be derived when substituting $k_n = \triangle_{k_n} / \triangle_{n-1}$ into (2.11).

Finally, since $\sum_{i=1}^{n} m_i = \Omega$ and (2.11) we obtain

$$m_n = \Omega - \sum_{i=1}^{n-1} m_i = \Omega - \sum_{i=1}^{n-1} (\triangle_{m_i} / \triangle_i), \quad k_{n+1} = \lambda_n m_n + k_n(u_{n-1}u_n^{-1} - 1), \tag{2.12}$$

where $m_n > 0$ and $k_{n+1} > 0$, so the conditions (6) and (7) can be derived.

Sufficiency: Since conditions (3) and (4) hold and k_1 is given, m_1 and k_2 can be computed according to (2.10). $\varphi_1(\lambda)$ can be reconstructed with m_1, k_1, k_2 and we can compute $\varphi_1(\lambda_2)$ and $\varphi_0(\lambda_2)$. When $\varphi_1(\lambda_2) \neq 0$ and condition (3) holds we can compute m_2 and k_3 by (2.7). Generally, if $m_1, \ldots, m_{i-1}$ and $k_1, \ldots, k_{i-1}, k_i$ are given, we can reconstruct $\varphi_{i-1}(\lambda)$ and we can compute $\varphi_{i-1}(\lambda_i)$ and $\varphi_{i-2}(\lambda_i)$. When $\varphi_{i-1}(\lambda_i) \neq 0$ and condition (3) holds we can compute m_i and k_{i+1} for $i = 2, 3, \ldots, n-1$. Finally m_n and k_{n+1} can be computed by (2.12) when the conditions (5)–(7) holds. So we get all physical parameters of a fixed–fixed mass-spring system. Such a system satisfy (2.4). This completes the proof. □

Next, we discuss Problem FDDMK II.

Obviously, it is an inverse general eigenvalues problem as follows: given λ_1 and $\lambda_i, \lambda_i^* \in R^+$, for $i = 2, 3, \ldots, n$, find $M_n, K_n \in R^{n \times n}$, such that

$$\varphi_1(\lambda_1) = 0, \quad \varphi_i(\lambda_i) = 0, \quad \varphi_i(\lambda_i^*) = 0 \ \text{ for } i = 2, 3, \ldots, n. \tag{2.13}$$

Expanding (2.13), if $\varphi_{i-1}(\lambda_i) \neq 0$, $\varphi_{i-1}(\lambda_i^*) \neq 0$, we obtain

$$k_1 + k_2 = \lambda_1 m_1, \quad k_{i+1} - \lambda_i m_i = W_i, \quad k_{i+1} - \lambda_i^* m_i = W_i^*, \text{ for } i = 2, 3, \ldots, n, \tag{2.14}$$

where $W_i = k_i(k_i\varphi_{i-2}(\lambda_i)\varphi_{i-1}^{-1}(\lambda_i) - 1)$, $W_i^* = k_i(k_i\varphi_{i-2}(\lambda_i^*)\varphi_{i-1}^{-1}(\lambda_i^*) - 1)$.

It is easy to prove Theorem 2 by the same method which we use to prove Theorem 1.

Theorem 2. *Suppose $m_1 > 0$ and $k_1 > 0$. Problem* FDDMK II *has unique solution, if and only if*

(1) $0 < \lambda_n^* < \lambda_{n-1}^* < \cdots < \lambda_2^* < \lambda_1 < \lambda_2 < \cdots < \lambda_n$;
(2) $\varphi_{i-1}(\lambda_i) \neq 0$, $\varphi_{i-1}(\lambda_i^*) \neq 0$, *for* $i = 2, 3, \ldots, n$;
(3) $\operatorname{sign}(m_1\lambda_1 - k_1) = 1$;
(4) $\delta_i\delta_{m_i}\delta_{k_{i+1}} \neq 0$, $\operatorname{sign}(\delta_i) = \operatorname{sign}(\delta_{m_i}) = \operatorname{sign}(\delta_{k_{i+1}})$, $i = 2, 3, \ldots, n$.

If the above conditions are satisfied, the physical parameters of mass-spring system have the expression as follows:

$$m_i = \delta_{m_i}/\delta_i, \quad k_{i+1} = \delta_{k_{i+1}}/\delta_i \text{ for } i = 2, \ldots, n,$$

where $\delta_i = \lambda_i - \lambda_i^*$, $\delta_{m_i} = W_i^* - W_i$, $\delta_{k_{i+1}} = W_i^* - \lambda_i^* W_i$.

3. The solvability conditions of problem FDDMK III

In fact, Problem FDDMK III is a general inverse eigen-problem as follows: given $\lambda, \mu \in R_+$, $X_2, Y_2 \in R^{n-p}$ and $K_{p-1} \in R^{(p-1)\times(p-1)}$, $M_p \in R^{p\times p}$, find $k_{p+1} \in R_+$, $M_n, K_n \in R^{n\times n}$ and $X_1, Y_1 \in R^p$, such that

$$(K_n - \lambda M_n)x = 0, \quad (K_n - \mu M_n)y = 0, \tag{3.1}$$

$$x^{\mathrm{T}} M_n y = 0. \tag{3.2}$$

Expanding (3.1) and (3.2), we obtain four questions as follows:

Problem FDDMK III.1. Given k_{p+1}, $x_{p+1} \neq 0$, $y_{p+1} \neq 0$, $M_p, K_p \in R^{p\times p}$, find $X_1, Y_1 \in R^p$ such that

$$(K_p - \lambda M_p)X_1 = k_{p+1}x_{p+1}E_{pp}, \quad (K_p - \mu M_p)Y_1 = k_{p+1}y_{p+1}E_{pp}, \tag{3.3}$$

where E_{pp} be the pth column of $p \times p$ identity matrix.

Problem FDDMK III.2. Given $k_{p+1} > 0$, x_i, y_i, for $i = p+1, p+2$, x_p and y_p is the solution for Problem FDDMK III.1, find $m_{p+1} > 0$ and $k_{p+2} > 0$, such that

$$\begin{cases} \lambda x_{p+1}m_{p+1} + (x_{p+2} - x_{p+1})k_{p+2} = k_{p+1}(x_{p+1} - x_p), \\ \mu y_{p+1}m_{p+1} + (y_{p+2} - y_{p+1})k_{p+2} = k_{p+1}(y_{p+1} - y_p). \end{cases} \tag{3.4}$$

Problem FDDMK III.3. Given $k_{p+1} > 0$, m_{p+1} and k_{p+2} is the solution for Problem FDDMK III.2, find $m_i > 0$ and $k_{i+1} > 0$, for $i = p+2, \ldots, n$, such that

$$\begin{cases} \lambda x_i m_i + (x_{i+1} - x_i)k_{i+1} = k_i(x_i - x_{i-1}), \\ \mu y_i m_i + (y_{i+1} - y_i)k_{i+1} = k_i(y_i - y_{i-1}), \end{cases} \tag{3.5}$$

where $i = p+2, \ldots, n$, $x_{n+1} = y_{n+1} = 0$.

Problem FDDMK III.4. Given m_i, k_i for $i = 1, 2, \ldots, p$ and x_i, y_i for $i = p+1, \ldots, n$, X_1, Y_1 is the solution for Problem FDDMK III.1, m_{p+1}, k_{p+2} is the solution for Problem FDDMK III.2, find $k_{p+1} > 0$, such that

$$X_1^{\mathrm{T}} M_p Y_1 + m_{p+1}x_{p+1}y_{p+1} + \sum_{i=p+2}^{n} m_i x_i y_i = 0. \tag{3.6}$$

Lemma 5. *Suppose $\alpha \notin \sigma(M_p, K_p)$, then there exists unique solution u in the equation*

$$(K_p - \alpha M_p)u = \beta E_{pp}, \tag{3.7}$$

and u can be expressed as follows:

$$u = \beta(K_p - \alpha M_p)^{-1} E_{pp} = \frac{\beta u^*}{\varphi_p(\alpha)}, \tag{3.8}$$

where $u^* = (u_1^*, u_2^*, \ldots, u_p^*)^{\mathrm{T}}$, $u_i^* = \varphi_{i-1}(\alpha)\prod_{j=i+1}^{p} k_j$, $u_p^* = \varphi_{p-1}(\alpha)$.

Proof. $\varphi_p(\alpha) = \det(K_p - \alpha M_p) \neq 0$ because of $\alpha \notin \sigma(M_p\ K_p)$, According to the definition of inverse matrix, we obtain

$$(K_p - \alpha M_p)^{-1} = \frac{1}{\varphi_p(\alpha)}(K_p - \alpha M_p)^*,$$

where $(K_p - \alpha M_p)^*$ is adjoint matrix of $K_p - \alpha M_p$. So there exists unique solution u in Eq. (3.7) as follows:

$$u = \frac{\beta}{\varphi(\alpha)}(K_p - \alpha M_p)^* E_{pp}.$$

We have (3.8) by straightforward computing the pth column of $(K_p - \alpha M_p)^*$.

Lemma 6. *Suppose that $\lambda, \mu \notin \sigma(M_p, K_p), \lambda \neq \mu$, and $X_1 = (x_1, x_2, \ldots, x_p)^{\mathrm{T}}, Y_1 = (y_1, y_2, \ldots, y_p)^{\mathrm{T}}$ are the solutions of equations $(K_p - \lambda M_p)X_1 = cE_{pp}$, $(K_p - \mu M_p)Y_1 = \mathrm{d}E_{pp}$, respectively. It is true that*

$$X_1^{\mathrm{T}} M_p Y_1 = \frac{cd\Phi_p(\lambda, \mu)}{(\lambda - \mu)\varphi_p(\lambda)\varphi_p(\mu)}, \tag{3.9}$$

where $\Phi_p(\lambda, \mu) = \varphi_{p-1}(\lambda)\varphi_p(\mu) - \varphi_{p-1}(\mu)\varphi_p(\lambda)$.

Proof. According to two equations $(K_p - \lambda M_p)X_1 = cE_{pp}$, $(K_p - \mu M_p)Y_1 = \mathrm{d}E_{pp}$, we obtain

$$Y_1^{\mathrm{T}}(K_p - \lambda M_p)X_1 = cY_1^{\mathrm{T}} E_{pp}, \quad X_1^{\mathrm{T}}(K_p - \lambda M_p)Y_1 = \mathrm{d}X_1^{\mathrm{T}} E_{pp}. \tag{3.10}$$

Noting that $X_1^{\mathrm{T}} M_p Y_1 = Y_1^{\mathrm{T}} M_p X_1$ and $X_1^{\mathrm{T}} K_p Y_1 = Y_1^{\mathrm{T}} K_p X_1$, it is true that

$$(\lambda - \mu)X_1^{\mathrm{T}} M_p Y_1 = \mathrm{d}X_1^{\mathrm{T}} E_{pp} - cY_1^{\mathrm{T}} E_{pp}. \tag{3.11}$$

By Lemma 5 we obtain

$$X_1^{\mathrm{T}} E_{pp} = \frac{c\varphi_{p-1(\lambda)}}{\varphi_p(\lambda)}, \quad Y_1^{\mathrm{T}} E_{pp} = \frac{\mathrm{d}\varphi_{p-1(\mu)}}{\varphi_p(\mu)}. \tag{3.12}$$

When $\lambda \neq \mu$, substituting (3.12) into (3.11) yields (3.9). This completes the proof. □

Using Lemmas 2–6, it is easy to have the following results.

Theorem 3. *Problem* FDDMK III.1 *has unique solution*, *if and only if*

(1) $\varphi_p(\lambda) \neq 0$, $\varphi_p(\mu) \neq 0$;
(2) $\mathrm{sign}(x_i x_{p+1}) = \mathrm{sign}(\varphi_{i-1}(\lambda)\varphi_p(\lambda))$, $\mathrm{sign}(y_i y_{p+1}) = \mathrm{sign}(\varphi_{i-1}(\mu)\varphi_p(\mu))$ *for* $i = 1, 2, \ldots, p$;
(3) $\mathrm{sign}(x_i x_{i+1}) = \mathrm{sign}(\varphi_{i-1}(\lambda)\varphi_i(\lambda)) = -1$, *for* $i = 1, 2, \ldots, p-1$;
(4) $\mathrm{sign}(y_i y_{i+1}) = \mathrm{sign}(\varphi_{i-1}(\mu)\varphi_i(\mu)) = 1$, *for* $i = 1, 2, \ldots, p-1$.

If the above conditions are satisfied, the solution has the expression as follows:

$$x_p = \frac{\varphi_{p-1}(\lambda)}{\varphi_p(\lambda)} k_{p+1} x_{p+1}, \quad x_i = \frac{\varphi_{i-1}(\lambda)}{\varphi_p(\lambda)} k_{p+1} x_{p+1} \prod_{j=i+1}^{p} k_j \quad \text{for } i = 1, 2, \ldots, p-1,$$

$$y_p = \frac{\varphi_{p-1}(\mu)}{\varphi_p(\mu)} k_{p+1} y_{p+1}, \quad y_i = \frac{\varphi_{i-1}(\mu)}{\varphi_p(\mu)} k_{p+1} y_{p+1} \prod_{j=i+1}^{p} k_j \quad \text{for } i = 1, 2, \ldots, p-1. \tag{3.13}$$

Now, by using the theory of linear systems we can get the solution of Problem FDDMK III.2 and III.3. At first, let $H_i = \lambda x_i(y_{i+1} - y_i) - \mu y_i(x_{i+1} - x_i)$, $G_i = (x_i - x_{i-1})(y_{i+1} - y_i) - (y_i - y_{i-1})(x_{i+1} - x_i)$, $Q_{i+1} = \lambda x_i(y_i - y_{i-1}) - \mu y_i(x_i - x_{i-1})$, for $i = p+1, \ldots, n$.

Theorem 4. *Problem* FDDMK III.2 *has unique solution, if and only if*

$$H_{p+1} G_{p+1} Q_{p+1} \neq 0, \quad \operatorname{sign}(H_{p+1}) = \operatorname{sign}(G_{p+1}) = \operatorname{sign}(Q_{p+2}),$$

and the solution has the expressions as follows:

$$m_{p+1} = k_{p+1} G_{p+1}/H_{p+1}, \quad k_{p+2} = k_{p+1} Q_{p+2}/H_{p+1}. \tag{3.14}$$

Theorem 5. *Problem* FDDMK III.3 *has unique solution, if and only if*

$$H_i G_i Q_{i+1} \neq 0, \quad \operatorname{sign}(H_i) = \operatorname{sign}(G_i) = \operatorname{sign}(Q_{i+1}) \ \textit{for } i = p+2, \ldots, n.$$

If the above conditions are satisfied, the solution has the expression as follows:

$$m_i = k_i G_i/H_i, \quad k_{i+1} = k_i Q_{i+1}/H_i. \tag{3.15}$$

Corollary. *Suppose that m_i and k_i have the expression of* (3.14) *and* (3.15), *respectively, there exists a relation formula*

$$k_i = k_{p+1} Q_{p+2} k_i^* \quad \textit{for } i = p+2, \ldots, n+1, \tag{3.16}$$

$$m_i = k_{p+1} Q_{p+2} m_i^* \quad \textit{for } i = p+2, \ldots, n, \tag{3.17}$$

where $k_{p+2}^* = H_{p+1}^{-1}$; $k_i^* = H_{p+1}^{-1} \prod_{j=p+2}^{i-1} Q_{j+1}/H_j$, $i = p+3, \ldots, n+1$; $m_i^* = k_i^* G_i/H_i$, *for* $i = p+2, \ldots, n$.

Proof. Using (3.15) to recur from i to $p+2$, we get

$$k_i = k_{p+2} \prod_{j=p+2}^{i-1} \frac{Q_{j+1}}{H_j}, \quad i = p+3, \ldots, n+1. \tag{3.18}$$

Substituting $k_{p+2} = k_{p+1} Q_{p+2}/H_{p+1}$ into (3.18) yields (3.16) and substituting (3.16) into (3.15) yields (3.17). This completes the proof. □

Remark. $\varphi_p(\lambda)$, $\varphi_p(\mu)$, $\Phi_p(\lambda, \mu)$, x_p, y_p, G_{p+1}, Q_{p+2}, $m_i (i = p+1, \ldots, n)$ are all function of k_{p+1}. Let

$$R_p(\lambda) = (k_p - \lambda m_p)\varphi_{p-1}(\lambda) - k_p^2 \varphi_{p-2}(\lambda), \quad R_p(\mu) = (k_p - \mu m_p)\varphi_{p-1}(\mu) - k_p^2 \varphi_{p-2}(\mu).$$
$$\delta_2^{(0)} = \varphi_{p-1}(\lambda)\varphi_{p-1}(\mu), \quad \delta_1^{(0)} = \varphi_{p-1}(\lambda) R_p(\mu) + \varphi_{p-1}(\mu) R_p(\lambda), \quad \delta_0^{(0)} = R_p(\lambda) R_p(\mu).$$
$$\delta_1^{(1)} = \varphi_{p-1}(\lambda)\varphi_{p-1}(\mu) - \varphi_{p-1}(\mu)\varphi_{p-1}(\lambda) = 0, \quad \delta_0^{(1)} = \varphi_{p-1}(\lambda) R_p(\mu) - \varphi_{p-1}(\mu) R_p(\lambda).$$
$$\delta_0^{(2)} = x_{p+1}(y_{p+2} - y_{p+1}) - y_{p+1}(x_{p+2} - x_{p+1}), \quad \delta_1^{(2)} = (x_{p+2} - x_{p+1}) y_{p+1} \varphi_{p-1}(\mu),$$
$$\delta_2^{(2)} = -(y_{p+2} - y_{p+1}) x_{p+1} \varphi_{p-1}(\lambda),$$
$$\delta_0^{(3)} = (\lambda - \mu) x_{p+1} y_{p+1}, \quad \delta_1^{(3)} = -\lambda x_{p+1} y_{p+1} \varphi_{p-1}(\mu), \quad \delta_2^{(3)} = \mu x_{p+1} y_{p+1} \varphi_{p-1}(\lambda).$$

$$\delta_i^{(4)} = \delta_i^{(2)}/H_{p+1} = \delta_i^{(2)} m_{p+1}^* \quad \text{for } i = 0, 1, 2.$$
$$\delta_{ij}^{(5)} = \delta_j^{(3)} * m_i^* \quad \text{for } i = p+2, \ldots, n, \quad j = 0, 1, 2.$$

The following notations can be introduced to solve Problem FDDMK III.4.

(i) $\varphi_p(\lambda) = (k_p + k_{p+1} - \lambda m_p)\varphi_{p-1}(\lambda) - k_p^2\varphi_{p-2}(\lambda) \triangleq k_{p+1}\varphi_{p-1}(\lambda) + R_p(\lambda)$.

(ii) $\varphi_p(\mu) = (k_p + k_{p+1} - \mu m_p)\varphi_{p-1}(\mu) - k_p^2\varphi_{p-2}(\mu) \triangleq k_{p+1}\varphi_{p-1}(\mu) + R_p(\mu)$.

(iii) $\varphi_p(\lambda)\varphi_p(\mu) = \delta_2^{(0)} k_{p+1}^2 + \delta_1^{(0)} k_{p+1} + \delta_0^{(0)}$.

(iv) $\Phi_p(\lambda, \mu) = \varphi_{p-1}\varphi_p(\mu) - \varphi_{p-1}(\mu)\varphi_p(\lambda) = \delta_1^{(1)} k_{p+1} + \delta_0^{(1)} = \delta_0^{(1)}$.

(v) $x_p = k_{p+1}x_{p+1}\varphi_{p-1}(\lambda)/[k_{p+1}\varphi_{p-1}(\lambda) + R_p(\lambda)]$,

(vi) $y_p = k_{p+1}y_{p+1}\varphi_{p-1}(\mu)/[k_{p+1}\varphi_{p-1}(\mu) + R_p(\mu)]$,

(vii) $G_{p+1} = [\delta_0^{(2)}\varphi_p(\lambda)\varphi_p(\mu) + \delta_1^{(2)} k_{p+1}\varphi_p(\lambda) + \delta_2^{(2)} k_{p+1}\varphi_p(\mu)]/[\varphi_p(\lambda)\varphi_p(\mu)]$.

(viii) $Q_{p+2} = [\delta_0^{(3)}\varphi_p(\lambda)\varphi_p(\mu) + \delta_1^{(3)}\varphi_p(\lambda)k_{p+1} + \delta_2^{(3)}\varphi_p(\mu)k_{p+1}]/[\varphi_p(\lambda)\varphi_p(\mu)]$.

(ix) $m_{p+1} = k_{p+1}(\delta_0^{(4)}\varphi_p(\lambda)\varphi_p(\mu) + \delta_1^{(4)}\varphi_p(\lambda)k_{p+1} + \delta_2^{(4)}\varphi_p(\mu)k_{p+1})/[\varphi_p(\lambda)\varphi_p(\mu)]$.

(x) $m_i = k_{p+1}(\delta_{i0}^{(5)}\varphi_p(\lambda)\varphi_p(\mu) + \delta_{i1}^{(5)}\varphi_p(\lambda)k_{p+1} + \delta_{i2}^{(5)}\varphi_p(\mu)k_{p+1})/[\varphi_p(\lambda)\varphi_p(\mu)]$.

Theorem 6. *Problem* FDDMK III.4 *has a solution*, *if and only if*,

(1) $\varphi_p(\lambda) \neq 0$, $\varphi_p(\mu) \neq 0$;

(2) $\gamma_2 t^2 + \gamma_1 t + \gamma_0 = 0$ *has a positive root* t^*, *where* $\gamma_2^2 + \gamma_1^2 \neq 0$, $\gamma_0^2 \neq 0$;

(3) $\gamma_2 t^2 + \gamma_1 t + \gamma_0 = 0$ *has infinite roots* $t^* > 0$, *when* $\gamma_2^2 + \gamma_1^2 + \gamma_0^2 = 0$.

If the above conditions are satisfied, the solution has the expression as follows:

$$k_{p+1} = t^* \neq -R_p(\lambda)/\varphi_{p-1}(\lambda) \quad \text{and} \quad -R_p(\mu)/\varphi_{p-1}(\mu),$$

where $\gamma_2 = \delta_0^{(6)}\delta_2^{(0)} + \delta_1^{(6)}\varphi_{p-1}(\lambda) + \delta_2^{(6)}\varphi_{p-1}(\mu)$, $\gamma_1 = \delta_0^{(6)}\delta_1^{(0)} + \delta_1^{(6)} R_p(\lambda) + \delta_2^{(6)} R_p(\mu) + \delta_3^{(6)}$, $\gamma_0 = \delta_0^{(6)}\delta_0^{(0)}$, $\delta_j^{(6)} = x_{p+1}y_{p+1}\delta_j^{(4)} + \sum_{i=p+2}^{n} x_i y_i \delta_{ij}^{(5)}$, $\delta_3^{(6)} = x_{p+1}y_{p+1}\delta_0^{(1)}/(\lambda - \mu)$.

Proof. Using Lemma 6 and X_1, Y_1 is the solution of Problem FDDMK III.1, when $\varphi_p(\lambda) \neq 0$, $\varphi_p(\mu) \neq 0$, it is true that

$$X_1^{\mathrm{T}} M_p Y_1 = \frac{k_{p+1}^2 x_{p+1} y_{p+1} \Phi_p(\lambda, \mu)}{(\lambda - \mu)\varphi_p(\lambda)\varphi_p(\mu)}.$$

Because of Problem FDDMK III.2 has a solution m_{p+1} which has the expression (ix), we can compute $m_{p+1}x_{p+1}y_{p+1}$. Similarly, we can compute $m_i x_i y_i$ for $i = p+2, \ldots, n$ by the expression (x). It holds that

$$\begin{aligned} X^{\mathrm{T}} MY &= k_{p+1}^2 x_{p+1} y_{p+1} \Phi_p(\lambda, \mu)/[(\lambda - \mu)\varphi_p(\lambda)\varphi_p(\mu)] \\ &\quad + \frac{k_{p+1}x_{p+1}y_{p+1}(\delta_0^{(4)}\varphi_p(\lambda)\varphi_p(\mu) + \delta_1^{(4)}\varphi_p(\lambda)k_{p+1} + \delta_2^{(4)}\varphi_p(\mu)k_{p+1})}{\varphi_p(\lambda)\varphi_p(\mu)} \\ &\quad + \sum_{i=p+2}^{n} \frac{k_{p+1}x_i y_i(\delta_{i0}^{(5)}\varphi_p(\lambda)\varphi_p(\mu) + \delta_{i1}^{(5)}\varphi_p(\lambda)k_{p+1} + \delta_{i2}^{(5)}\varphi_p(\mu)k_{p+1})}{\varphi_p(\lambda)\varphi_p(\mu)}. \end{aligned} \tag{3.19}$$

Reduce (3.19), we get by (3.6)

$$\delta_0^{(6)}\varphi_p(\lambda)\varphi_p(\mu) + \delta_1^{(6)}\varphi_p(\lambda)k_{p+1} + \delta_2^{(6)}\varphi_p(\mu)k_{p+1} + \delta_3^{(6)}k_{p+1} = 0. \tag{3.20}$$

Substituting (i)–(iii) into (3.20), we get

$$\gamma_2 k_{p+1}^2 + \gamma_1 k_{p+1} + \gamma_0 = 0. \tag{3.21}$$

Hence, Problem FDDMK III.4 has a solution if and only if $\gamma_2 t^2 + \gamma_1 t + \gamma_0 = 0$ has a positive root t^* which is not equal to $-R_p(\lambda)/\varphi_{p-1}(\lambda)$ and $-R_p(\mu)/\varphi_{p-1}(\mu)$. We can get $k_{p+1} = t^*$.

Theorem 7. *Problem* FDDMK III *has a solution, if and only if, the following conditions are satisfied*

(1) $\operatorname{sign}(x_i) = (-1)^{n-i}\operatorname{sign}(x_n)$, $\operatorname{sign}(y_i) = \operatorname{sign}(y_n)$, $i = p+1, \dots, n$;
(2) $\operatorname{sign}(\varphi_{i-1}(\lambda)\varphi_i(\lambda)) = -1$, $\operatorname{sign}(\varphi_{i-1}(\mu)\varphi_i(\mu)) = 1$, $i = 1, 2, \dots, p-1$;
(3) $\operatorname{sign}(\varphi_{i-1}(\lambda)\varphi_p(\lambda)) = (-1)^{p+1-i}$, $\operatorname{sign}(\varphi_{i-1}(\mu)\varphi_p(\mu)) = 1$, $i = 1, 2, \dots, p-1$;
(4) $\varphi_p(\lambda) \neq 0$, $\varphi_p(\mu) \neq 0$;
(5) $H_i G_i Q_{i+1} \neq 0$, $\operatorname{sign}(H_i) = \operatorname{sign}(G_i) = \operatorname{sign}(Q_{i+1})$, $i = p+1, \dots, n$;
(6) $\gamma_2 t^2 + \gamma_1 t + \gamma_0 = 0$ *has a positive root* t^* *which is not equal to* $-R_p(\lambda)/\varphi_{p-1}(\lambda)$ *and* $-R_p(\mu)/\varphi_{p-1}(\mu)$.

Proof. *Necessity*. It is clearly given by Theorems 3–6.

Sufficient: We can get physical parameters k_{p+1}, m_i, k_{i+1} for $i = p+1, \dots, n$ and x_i, y_i for $i = 1, 2, \dots, p$ by a construction method. Using m_i, k_i for $i = 1, 2, \dots, p$,we can get polynomial function to compute $\varphi_i(\lambda)$ and $\varphi_i(\mu)$ for $i = 0, 1, \dots, p-1$; Using x_i, y_i for $i = p+1, \dots, n$, we can calculate H_{p+1} and G_i, Q_{i+1} for $i = p+2, \dots, n$; Finally we compute γ_2, γ_1 and γ_0. If $\gamma_2 t^2 + \gamma_1 t + \gamma_0 = 0$ has a positive root t^* which is not equal to $-R_p(\lambda)/\varphi_{p-1}(\lambda)$ and $-R_p(\mu)/\varphi_{p-1}(\mu)$, we get $k_{p+1} = t^*$.

When $m_1, \dots, m_p$ and $k_1, \dots, k_p, k_{p+1}$ are given, we can calculate $\varphi_p(\lambda)$ and $\varphi_p(\mu)$. Then computing x_i, y_i for $i = 1, 2, \dots, p$ by (3.13), computing m_{p+1} and k_{p+1} by (3.14), computing m_i, k_{i+1} for $i = p+2, \dots, n$ by (3.18). So we get all physical parameters with $m_i > 0$ and $k_{i+1} > 0$ for $i = p+1, \dots, n$ and $\operatorname{sign}(x_i) = (-1)^{n-i}\operatorname{sign}(x_n)$, $\operatorname{sign}(y_i) = \operatorname{sign}(y_n)$ for $i = 1, 2, \dots, p$. This completes the proof. □

4. Algorithm and example

According to above discussion an algorithm to solve problem FDDMK-III is presented as follows.

Algorithm FDDMK. Given m_i, k_i for $i = 1, 2, \dots, p$ and x_i, y_i for $i = p+1, \dots, n$, $\omega_1, \omega_2 \in R_+$, then the Algorithm constructs k_{p+1}, m_i, k_{i+1} for $i = p+1, \dots, n$ and $X_1^{\mathrm{T}}, Y_1^{\mathrm{T}}$.

(1) If there is $i_1, i_2 \in \{1, 2, \dots, p\}$ such that $m_{i_1} < 0$ or $k_{i_2} < 0$, goto step (8). If there is $i_3, i_4 \in \{p+1, \dots, n\}$ such that $\operatorname{sign}(x_{i_3}) \neq (-1)^{n-i_3}\operatorname{sign}(x_n)$ or $\operatorname{sign}(y_{i_4}) \neq \operatorname{sign}(y_n)$, goto step (8).

(2) Compute $\lambda = \omega_1^2$, compute $\varphi_i(\lambda)$ for $i = 1, 2, \dots, p-1$ by (2.3). If there is $i_5 \in \{1, 2, \dots, p-1\}$ such that $\operatorname{sign}(\varphi_{i_5}(\lambda)\varphi_{i_5+1}(\lambda)) \neq -1$, goto step (8).

(3) Compute $\mu = \omega_2^2$,compute $\varphi_i(\mu)$ for $i = 1, 2, \dots, p-1$ by (2.3). If there is $i_6 \in \{1, 2, \dots, p-1\}$ such that $\operatorname{sign}(\varphi_{i_6}(\mu)) \neq \operatorname{sign}(\varphi_{i_6+1}(\mu))$, goto step (8).

(4) Compute H_{p+1} and H_i, G_i, Q_{i+1} for $i = p+2, \dots, n$. If $H_{p+1} = 0$ or $H_i G_i Q_i = 0$ then goto step (8); If there is $i_7 \in \{p+2, \dots, n\}$ such that fail to $\operatorname{sign}(H_{i_7}) = \operatorname{sign}(G_{i_7}) = \operatorname{sign}(Q_{i_7})$, goto step (8).

(5) Compute $R_p(\lambda)$, $R_p(\mu)$ and $\delta_i^{(j)}$ for $i = 0, 1, 2$ and $j = 0, 1, 2, 3, 4$, $\delta_{ij}^{(5)}$ for $j = 0, 1, 2$ and $i = p+2, \dots, n$, $\delta_j^{(6)}$ for $i = 0, 1, 2, 3$. Then calculate γ_i for $i = 0, 1, 2$, if $\gamma_2 t^2 + \gamma_1 t + \gamma_0 = 0$ has a positive root t^* which is not equal to $-R_p(\lambda)/\varphi_{p-1}(\lambda)$ and $-R_p(\mu)/\varphi_{p-1}(\mu)$, then let $k_{p+1} = t^*$, else goto step (8).

(6) Compute $\varphi_p(\lambda)$ and $\varphi_p(\mu)$, if $\varphi_p(\lambda) = 0$ or $\varphi_p(\mu) = 0$, goto step (8). If there is $i_8, i_9 \in \{1, 2, \dots, p-1\}$ such that $\operatorname{sign}(\varphi_{i_8-1}(\lambda)\varphi_p(\lambda)) \neq (-1)^{p+1-i_8}$ or $\operatorname{sign}(\varphi_{i_9-1}(\mu)\varphi_p(\mu)) \neq 1$, goto step (8).

(7) Compute x_p, y_p and G_{p+1}, Q_{p+2}, if $\operatorname{sign}(G_{p+1}) = \operatorname{sign}(Q_{p+2}) = \operatorname{sign}(H_{p+1})$, then calculate m_{p+1} and k_{p+2}, else goto step (8).

(8) Stop compute and exit program. Problem FDDMK III has no solution.

(9) Compute x_i and y_i for $i = 1, 2, \dots, p-1$ by (3.13). Compute m_i and k_i for $i = p+2, \dots, n$ by (3.17).

Example. Given $\lambda = 3.6926$, $\mu = 0.0965$, $p = 4$, $n = 9$, $\varepsilon = 1.0e - 012$, $m_1 = m_2 = m_3 = m_4 = 4$, $k_1 = k_2 = k_3 = k_4 = 4$, $X_2 = (x_5, x_6, x_7, x_8, x_9)^T = (0.0972, -0.2322, 0.2958, -0.2685, 0.1586)^T$, $Y_2 = (y_5, y_6, y_7, y_8, y_9)^T = (0.2365, 0.2243, 0.1904, 0.1382, 0.0726)^T$, Employing algorithm FDDMK, we can know that $v_2^2 + v_1^2 + v_0^2 < \varepsilon$ which is computed by MATLAB. Applying Algorithm FDDMK by MATLAB, we choose $k_{p+1} = 1.0$ in case 1 and let k_{p+1} has errors in case 2 where $k_{p+1} = 0.9999$, two solution are as follows:

Case 1: $k_{p+1} = 1.0$, a specially structured Jacobi matrix can be constructed as follows:

$$K_9 = \begin{pmatrix} 8.0000 & -4.0000 & & & & & & & \\ -4.0000 & 8.0000 & -4.0000 & & & & & & \\ & -4.0000 & 8.0000 & -4.0000 & & & & & \\ & & -4.0000 & 5.0000 & -1.0000 & & & & \\ & & & -1.0000 & 4.9842 & -3.9842 & & & \\ & & & & -3.9842 & 7.9576 & -3.9734 & & \\ & & & & & -3.9734 & 7.9537 & -3.9803 & \\ & & & & & & -3.9803 & 7.9563 & -3.9760 \\ & & & & & & & -3.9760 & 7.9524 \end{pmatrix},$$

$$M_9 = \mathrm{diag}(4, 4, 4, 4, 3.9852, 3.9775, 3.9771, 3.9782, 3.9765),$$

where $X_1 = (x_1, x_2, x_3, x_4)^T = (0.0142, -0.0240, 0.0265, -0.0208)^T$, $Y_1 = (y_1, y_2, y_3, y_4)^T = (0.0628, 0.1196, 0.1648, 0.1942)^T$, $(k_6, k_7, k_8, k_9, k_{10}) = (3.9842, 3.9734, 3.9803, 3.9760, 3.9764)$. $(m_5, m_6, m_7, m_8, m_9) = (3.9852, 3.9775, 3.9771, 3.9782, 3.9765)$.

From the above 9×9 Jacobi matrix K_9 and diagonal matrix M_9, we recompute the general spectrum of matrix-pair K_9 and M_9 by MATLAB 6.1, and get

$$\sigma(M_9, k_9) = (0.0965, 0.2506, 0.7559, 1.1083, 1.7792, 2.4036, 2.8667, 3.5477, 3.6926)^T,$$

and the eigenvector X to the maximal eigenvalue 3.6926 is

$$X = (0.0142, -0.0241, 0.0266, -0.0209, 0.0975, -0.2328, 0.2966, -0.2692, 0.1590)^T,$$

and the eigenvector X to the minimal eigenvalue 0.0965 is

$$Y = (-0.0629, -0.1198, -0.1651, -0.1945, -0.2369, -0.2247, -0.1907, -0.1384, -0.0727)^T.$$

Case 2: $k_{p+1} = 0.9999$, a specially structured Jacobi matrix can be constructed as follows:

$$K_9 = \begin{pmatrix} 8.0000 & -4.0000 & & & & & & & \\ -4.0000 & 8.0000 & -4.0000 & & & & & & \\ & -4.0000 & 8.0000 & -4.0000 & & & & & \\ & & -4.0000 & 4.9999 & \mathbf{-0.9999} & & & & \\ & & & \mathbf{-0.9999} & 4.9841 & -3.9842 & & & \\ & & & & -3.9842 & 7.9576 & -3.9734 & & \\ & & & & & -3.9734 & 7.9537 & -3.9803 & \\ & & & & & & -3.9803 & 7.9563 & -3.9760 \\ & & & & & & & -3.9760 & 7.9524 \end{pmatrix},$$

$$M_9 = \mathrm{diag}(4, 4, 4, 4, 3.9852, 3.9775, 3.9771, 3.9782, 3.9765),$$

where $X_1 = (x_1, x_2, x_3, x_4)^T = (0.0142, -0.0240, 0.0265, -0.0208)^T$, $Y_1 = (y_1, y_2, y_3, y_4)^T = (0.0628, 0.1196, 0.1648, 0.1942)^T$, $(k_6, k_7, k_8, k_9, k_{10}) = (3.9842, 3.9734, 3.9803, 3.9760, 3.9764)$. $(m_5, m_6, m_7, m_8, m_9) = (3.9852, 3.9775, 3.9771, 3.9782, 3.9765)$.

From the above 9×9 Jacobi matrix K_9 and diagonal matrix M_9, we recompute the general spectrum of matrix-pair K_9 and M_9 by MATLAB 6.1, and get

$$\sigma(M_9, K_9) = (0.0965, 0.2506, 0.7559, 1.1083, 1.7792, 2.4036, 2.8667, 3.5477, 3.6926)^T,$$

and the eigenvector X to the maximal eigenvalue 3.6926 is

$$X = (0.0142, -0.0241, 0.0266, \mathbf{-0.0208}, 0.0975, -0.2328, 0.2966, -0.2692, 0.1590)^{\mathrm{T}},$$

and the eigenvector X to the minimal eigenvalue 0.0965 is

$$Y = (-0.0629, -0.1198, -0.1651, -0.1945, -0.2369, -0.2247, -0.1907, -0.1384, -0.0727)^{\mathrm{T}}.$$

These obtained data show that Algorithm FDDMK is quite efficient.

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