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The Banach–Stone theorem revisited

Z. Ercan [a,*], S. Önal [b,1]

[a] Abant Izzet Baysal University, Department of Mathematics, Gölköy Kampusu, 14280 Bolu, Turkey
[b] Middle East Technical University, Department of Mathematics, 06531 Ankara, Turkey

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ABSTRACT

Let X and Y be compact Hausdorff spaces, and E and F be locally solid Riesz spaces. If $\pi : C(X,E) \to C(Y,F)$ is a 1-biseparating Riesz isomorphism then X and Y are homeomorphic, and E and F are Riesz isomorphic. This generalizes the main results of [Z. Ercan, S. Önal, Banach–Stone theorem for Banach lattice valued continuous functions, Proc. Amer. Math. Soc. 135 (9) (2007) 2827–2829] and [X. Miao, C. Xinhe, H. Jiling, Banach–Stone theorems and Riesz algebras, J. Math. Anal. Appl. 313 (1) (2006) 177–183], and answers a conjecture in [Z. Ercan, S. Önal, Banach–Stone theorem for Banach lattice valued continuous functions, Proc. Amer. Math. Soc. 135 (9) (2007) 2827–2829].

1. Introduction and brief history

Let X be a topological space and E be a topological vector space. The vector space of all continuous functions from X into E is denoted by $C(X,E)$. Under the ordering

$$f \leqslant g: \quad \Longleftrightarrow \quad f(x) \leqslant g(x) \quad \text{for each } x \in X,$$

$C(X,E)$ is a Riesz space whenever E is Riesz space. If X is compact and E is a normed space then $C(X,E)$ is a normed space under the norm

$$\|f\| = \sup_{x \in X} \|f(x)\|$$

and it is complete if E is Banach space. In particular, if E is a Banach lattice then $C(X,E)$ is a Banach lattice. We write $C(X)$ instead of $C(X,\mathbb{R})$.

The Banach–Stone theorem tells us that every surjective isometry between $C(K)$ and $C(M)$ must be of the form $T(f)(k) = h(k)f(\sigma(k))$ where σ is a homeomorphism of the compact Hausdorff space K onto the compact Hausdorff space M. On the other hand, the isomorphic type of $C(K)$ is not sufficient to characterize K, for by Milutin's theorem, if K and M are uncountable metrizable compact spaces, then $C(K)$ and $C(M)$ are isomorphic. For compact metric spaces K and M the Banach–Stone theorem proved by Banach [4] and generalized for arbitrary compact Hausdorff spaces by Stone [13]. We refer to [9] for Riesz isomorphic and algebraic isomorphic versions of the Banach–Stone theorem.

* Corresponding author. Tel.: +90 (374) 2541307; fax: +90 (374) 253 46 42.
E-mail addresses: zercan@ibu.edu.tr (Z. Ercan), osul@metu.edu.tr (S. Önal).
1 Tel.: +90 (313) 2105374; fax: +90 (312) 210 29 72.

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Jerison [11] was the first to consider Banach–Stone type theorem on $C(X,E)$. We refer to [5] and [7] fore more details of the Banach–Stone theorem on $C(X,E)$.

Definition 1. Let X and Y be compact Hausdorff spaces, E and F be topological vector space and n be a natural number. A map $T : C(X,E) \to C(Y,F)$ is called n-separating if the following implication holds

$$\bigcap_{i=1}^{n} f_i^{-1}(0) = \emptyset \quad \Longrightarrow \quad \bigcap_{i=1}^{n} T(f_i)^{-1}(0) = \emptyset.$$

T is called n-biseparating if

$$\bigcap_{i=1}^{n} f_i^{-1}(0) = \emptyset \quad \Longleftrightarrow \quad \bigcap_{i=1}^{n} T(f_i)^{-1}(0) = \emptyset.$$

It is clear that if T is n-separating and $m \leqslant n$ then T is also m-separating. In the literature 2-separating linear map is called *separating* and 2-biseparating map is called *biseparating*, which was introduced in [3] and [10]. For realcompact spaces X and Y it is an open question whether a linear separating bijection $T : C(X) \to C(Y)$ is automatically biseparating (see [2]). In the last twenty years it has been shown that Banach–Stone theorem enjoys with separating property. Recently the following theorems are proved:

Theorem 2. *(See [8].) Let X and Y be compact Hausdorff spaces and, E and F be Banach spaces. Then every biseparating linear map $T : C(X,E) \to C(Y,F)$ is a weighted composition operator, $T(f)(y) = h(y)f(\alpha(y))$ for each $f \in C(X,E)$ and $y \in Y$, where α is a homeomorphism from Y into X and $h(y)$ is an invertible linear map from E into F.*

A Riesz space (vector lattice) E is an ordered vector space in which $\sup\{x,y\}$ exists for every $x, y \in E$. To each element $x \in E$ we associate the absolute value $|x| = \sup\{x,-x\}$, its positive part $x^+ = \{x,0\}$ and the negative part $x^- = \sup\{-x,0\}$. A subset A of a Riesz space is said to be *solid* if whenever $|x| \leqslant |y|$ imply that $y \in A$. A topological vector space on a Riesz space E is said to be *locally solid* if zero admits a fundamental system of solid neighborhoods. A norm $\|.\|$ on a Riesz space is said to be a *lattice norm* whenever $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$. A Riesz space equipped with a lattice norm is known as a *normed Riesz space*. If a normed Riesz space is also norm complete, then it is referred to as a *Banach lattice*. Normed Riesz spaces are locally solid. If X is compact Hausdorff space and E is a normed Riesz space then $C(X,E)$ is also a normed Riesz space. The Riesz spaces E and F are said to be *Riesz isomorphic* if there exists a one-to-one and onto linear map $T : E \to F$ such that $T(x) \geqslant 0$ if and only if $x \geqslant 0$. In this case T is called a *Riesz isomorphism*. For more details on Riesz spaces, locally solid Riesz spaces and Riesz algebras we refer to [1] and [14].

Theorem 3. *(See [12].) Let X, Y be compact Hausdorff spaces and let E, F be both Banach lattices and Riesz algebras. It is shown that if F has no zero-divisor and if there exists a 1-separating Riesz algebraic isomorphism $\Phi : C(X,E) \to C(Y,F)$, then X is homeomorphic to Y and E is Riesz algebraically isomorphic to F.*

Theorem 4. *(See [6].) For compact Hausdorff spaces X, Y and Z, and Banach lattice E, if $\pi : C(X,E) \to C(Y,C(Z))$ is a 1-separating Riesz isomorphism then X and Y are homeomorphic, and E and $C(Z)$ are Riesz isomorphic spaces.*

In [6] it was asked whether $C(Z)$ can be replaced by an arbitrary Banach lattice. The aim of this paper is to give a positive answer to this conjecture and generalize main result of [7].

2. The result

We are ready to state our main result as follows. This generalizes Theorems 4 and 5 and gives answer to a conjecture in [6].

Theorem 5. *Let X and Y be compact Hausdorff spaces and, E and F be locally solid Riesz spaces. If $\pi : C(X,E) \to C(Y,F)$ is a 1-biseparating Riesz isomorphism then X and Y are homeomorphic and, E and F are Riesz isomorphic spaces.*

If $e \in E$ and $f \in C(X)$ then $f \otimes e : X \to E$ is defined by $f \otimes e(x) = f(x)e$. Similar notation is used for $e \in F$ and $f \in C(Y)$.

Lemma 6. *For each $x \in X$ there exists a unique $\sigma(x) \in Y$ such that*

$$UZ(x) := \bigcap \{\pi(f)^{-1}(0)\colon\ f \in Z(x)\} = \{\sigma(x)\},$$

where

$$Z(x) = \{f \in C(X,E)_+\colon\ f(x) = 0\}.$$

Proof. Let $f_1, \dots, f_n \in Z(x)$ be given. Then $f_1 \vee f_2 \vee \cdots \vee f_n \in Z(x)$. From the following equation

$$\pi(f_1)^{-1}(0) \cap \pi(f_2)^{-1}(0) \cap \cdots \cap \pi(f_n)^{-1}(0) = \pi(f_1 \vee f_2 \vee \cdots \vee f_n)^{-1}(0)$$

$\{\pi(f)^{-1}(0)\colon f \in Z(x)\}$ has the finite intersection property. Since Y is compact $UZ(x)$ is nonempty. Let $a, b \in UZ(x)$ and $a \neq b$. Then there exists $g \in C(Y, F)_+$ such that $g(a) = 0$ and $g(b) \neq 0$. Then $\pi^{-1}(g)(x) > 0$. (Indeed, if $\pi^{-1}(g)(x) = 0$ then $\pi^{-1}(g) \in Z(x)$, so $g(b) = \pi(\pi^{-1}(g))(b) = 0$.) Let

$$e = \pi^{-1}(g)(x) \quad \text{and} \quad t = \left|\pi^{-1}(g) - 1 \otimes e\right|.$$

Then $t(x) = 0$, so $t \in Z(x)$. From the following

$$\pi(t)(a) = \left|\pi\left(\pi^{-1}(g)\right) - \pi(1 \otimes e)\right|(a) = 0$$

we have

$$0 = g(a) = \pi(1 \otimes e)(a) > 0.$$

This contradiction completes the proof. □

Lemma 7. *For each $f \in C(X, E)$ and $x \in X$ we have*

$$\pi(f)\big(\sigma(x)\big) = \pi\big(1 \otimes f(x)\big)\big(\sigma(x)\big).$$

Proof. Let $f_x = |f - 1 \otimes f(x)|$. Then $0 \leqslant f$ and $f(x) = 0$, so $f_x \in Z(x)$. Hence $\pi(f_x)(\sigma(x)) = 0$. As π is a Riesz homomorphism we have $\pi(f)(\sigma(x)) = \pi(1 \otimes f(x))(\sigma(x))$. □

Lemma 8. *Let $x \in X$. Then $\pi(f)(\sigma(x)) = 0$ if and only if $f(x) = 0$.*

Proof. Let $f \in C(X, E)_+$ be given. By the definition, if $f(x) = 0$ we have $\pi(f)(\sigma(x)) = 0$. If $\pi(f)(\sigma(x)) = 0$, then $\pi(1 \otimes f(x))(\sigma(x)) = 0$. This implies $0 \in 1 \otimes f(x)(X)$, so we must have $f(x) = 0$. □

Lemma 9. *σ is one-to-one.*

Proof. Let $a, b \in X$ with $\sigma(a) = \sigma(b)$. Suppose that $a \neq b$. Then $f(a) > 0$ and $f(b) = 0$ for some $f \in C(X, E)_+$. From the previous lemma we have

$$\pi(f)\big(\sigma(a)\big) \neq 0 \quad \text{and} \quad \pi(f)\big(\sigma(b)\big) = 0.$$

This is a contradiction. □

Lemma 10. *σ is continuous.*

Proof. Let $x_\alpha \to x$ in X. We show that $\sigma(x_\alpha) \to \sigma(x)$. $U \subset Y$ be an open set with $\sigma(x) \in U$. There exists $f \in C(X, E)_+$ with $\pi(f)(\sigma(x)) \neq 0$ and $\pi(f)(Y \setminus U) = \{0\}$. Then $f(x) > 0$. Since $f(x_\alpha) \to f(x)$ there exists α_0 such that $f(x_\alpha) > 0$ for each $\alpha > \alpha_0$. This implies that $\sigma(x_\alpha) \notin Y \setminus U$ for each $\alpha \geqslant \alpha_0$, so $\sigma(x_\alpha) \in U$ for each $\alpha \geqslant \alpha_0$. Hence $\sigma(x_\alpha) \to \sigma(x)$. □

Lemma 11. *σ is onto.*

Proof. Suppose that there exists $y \in Y$ with $y \notin \sigma(X)$. Since σ a continuous and X is compact, $\sigma(X)$ is compact. Then there exists $f \in C(X, E)_+$ with

$$\pi(f)(y) \neq 0 \quad \text{and} \quad \pi(f)\big(\sigma(X)\big) = \{0\}.$$

From Lemma 9 $f(X) = \{0\}$, so $\pi(f)(y) = 0$. This is a contradiction. Hence, σ is onto. □

Hence we have proved the following.

Corollary 12. *X and Y are homeomorphic.*

Lemma 13. *E and F are Riesz isomorphic spaces.*

Proof. Let $x \in X$ be given. Then it is clear that the map $T: E \to F$ defined by $T(e) = \pi(1 \otimes e)(\sigma(x))$ is a one to one Riesz homomorphism and $T(e) \geqslant 0$ if and only if $e \geqslant 0$. Let $a \in F$ be given. Then

$$\pi(1 \otimes e)\big(\sigma(x)\big) = a,$$

where $e = \pi^{-1}(1 \otimes a)(x)$. □

Hence combining the above lemmas we have proved Theorem 6.

3. Questions

Let X and Y be compact Hausdorff spaces and E, F be locally convex spaces with duals E' and F', respectively. We shall call a map $T: C(X, E) \to C(Y, F)$ is *n-weakly separating* if for each $v \in F'$ and $f_1, f_2, \ldots, f_n \in C(X, E)$,

$$\bigcap_{i=1}^{n} \big(v \circ T(f_i)\big)^{-1} = \emptyset$$

whenever $\bigcap_{i=1}^{n} (u \circ f_i)^{-1} = \emptyset$ for each $u \in E'$. It is clear that if E and F are normed spaces then an n-separating map T is weakly n-separating. But we do not know the converse of this. On the other hand, we believe that many Banach Stone type theorem related to n-biseparating can be obtained also for n-weakly biseparated linear maps.

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