Note

Cubicity of threshold graphs

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ABSTRACT

We show that the cubicity of a connected threshold graph is equal to \( \lceil \log_2 \alpha \rceil \), where \( \alpha \) is its independence number.

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1. Introduction

Let \( G(V, E) \) be a simple, undirected graph where \( V \) is the set of vertices and \( E \) is the set of edges. A \( b \)-dimensional box is a Cartesian product \( I_1 \times I_2 \times \cdots \times I_b \), where each \( I_i \) is a closed interval on the real line. When each interval has unit length, we will call such a box a \( b \)-dimensional cube. The cubicity (resp. boxicity) of \( G \), denoted by \( \text{cub}(G) \) (\( \text{box}(G) \)), is the minimum positive integer \( b \) such that the vertices in \( G \) can be mapped into axis-parallel \( b \)-dimensional cubes (boxes) in such a way that two vertices are adjacent in \( G \) if and only if their assigned cubes (boxes) intersect. Cubicity and boxicity were introduced by Roberts [11]. Yannakakis [12] proved that it is NP-complete to determine if the cubicity of a graph is at most 3.

Graphs with cubicity 1 are called indifference graphs or unit interval graphs. We can also define an indifference graph in the following way.

Definition 1. A graph \( G(V, E) \) is an indifference graph if and only if there exists a positive real number \( t \) and a function \( \Pi : V \to \mathbb{R} \) such that, for two distinct vertices \( u \) and \( v \), \( uv \in E \) if and only if \( |\Pi(u) - \Pi(v)| \leq t \).

For a graph \( G(V, E) \), if there is a set of \( k \) supergraphs \( G_i(V, E_i), i \in \{1, 2, \ldots, k\} \) such that \( E = E_1 \cap E_2 \cap \cdots \cap E_k \), then we say that \( G \) is the intersection of \( G_i \)'s. Roberts [11] gives a very useful characterization of cubicity in terms of intersection of indifference graphs. We state it below as a lemma.

Lemma 2. Given a graph \( G(V, E) \), \( \text{cub}(G) \) is the minimum positive integer \( b \) such that \( G \) is the intersection of \( b \) indifference graphs.

Cubicity of graphs with special properties has attracted considerable attention. Roberts [11] studied the cubicity of complete \( k \)-partite graph and showed that the cubicity of any graph cannot be greater than \( \left\lfloor \frac{2k-1}{3} \right\rfloor \). As part of this proof, it was shown that the cubicity of a star graph \( K_{1,n} \) is \( \lceil \log_2 n \rceil \). In [2], Chandran and Naveen studied the cubicity of hypercubes.

In this paper, we prove that the cubicity of a connected threshold graph is equal to \( \lceil \log_2 \alpha \rceil \), where \( \alpha \) is its independence number. Threshold graphs were introduced in [4,9] and, since then, have been extensively studied [10,8,17]. A threshold graph may be defined as follows:

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Definition 3 ([3]). A graph \( G(V, E) \) is a threshold graph if and only if there is a partition of the vertex set \( V = D_0 \uplus D_1 \uplus \cdots \uplus D_m \), such that vertices \( u \in D_i \) and \( v \in D_j \) are adjacent if and only if \( i + j > m \).

Since the family of star graphs forms a subclass of threshold graphs, our result can be considered as an extension of Robert’s result on the cubicity of star graphs.

2. Cubicity of a connected threshold graph

We observe that it is easy to determine a maximum independent set of a threshold graph and hence, its independence number. Let \( G(V, E) \) be a connected threshold graph with vertex partition as in Definition 3. When the threshold graph is connected, \( D_0 \) is an empty set. Since we will be dealing with only connected threshold graphs, from now on we will ignore \( D_0 \). Let \( v \) be a vertex in \( D_{(m/2)} \). We claim that the set \( I \) as defined below,

\[
I = \begin{cases} 
D_1 \cup D_2 \cup \cdots \cup D_{m/2}, & \text{if } m \text{ even}, \\
D_1 \cup D_2 \cup \cdots \cup D_{(m/2)} \cup \{v\}, & \text{if } m \text{ odd},
\end{cases}
\]

is a maximum independent set, that is \( |I| = \alpha \). We first note that \( I \) is a maximal independent set. Let \( u \in D_i \) be any vertex such that \( i > \lfloor m/2 \rfloor \). The maximal independent set containing \( u \) will be of the form \( I' = D_1 \cup D_2 \cup \cdots \cup D_{m-i-1} \cup \{u\} \). Since none of the \( D_i \)'s are empty, it immediately follows that \( |I'| \leq |I| \). Using similar arguments we can show that the set \( C \),

\[
C = \begin{cases} 
\{v\} \cup D_{m/2+1} \cup \cdots \cup D_m, & \text{if } m \text{ even}, \\
D_{(m/2)} \cup D_{(m/2)+1} \cup \cdots \cup D_m, & \text{if } m \text{ odd},
\end{cases}
\]

is a maximum clique. Finally, we note that \( C \cup I = V \) and \( C \cap I = \{v\} \). Now we state and prove our main result:

Theorem 4. The cubicity of a connected threshold graph \( G \) is equal to \( \lceil \log_2 \alpha \rceil \), where \( \alpha \) is the independence number of \( G \).

Proof. Let \( G(V, E) \) be a connected threshold graph with a vertex partition as in Definition 3 and independence number \( \alpha \). Since any vertex \( v_m \in D_m \) is a universal node in \( G \), \( \{v_m\} \cup I \) induces a subgraph isomorphic to \( K_{1,\alpha} \). Also, it is easy to see that the cubicity of any graph \( G \) is at least the cubicity of an induced subgraph of \( G \). Hence, it immediately follows that \( \text{cub}(G) \geq \text{cub}(K_{1,\alpha}) = \lceil \log_2 \alpha \rceil \).

We now show that this bound is sharp by constructing \( \lceil \log_2 \alpha \rceil \) indifference graphs whose intersection yields \( G \).

Suppose \( C \) is as defined in (2). Let \( I : V \longrightarrow \{0, \ldots, \alpha - 1\} \) be a labeling of vertices such that \( \forall u \in C, I(u) = 0 \), and for each \( u \in V \setminus C, I(u) \) is a distinct number from the set \( \{1, 2, \ldots, \alpha - 1\} \). Let \( b_i : V \longrightarrow \{0, 1\} \) be such that, for any \( u \in V \), \( b_i(u) = \left\lfloor \frac{I(u)}{2^{i-1}} \right\rfloor \mod 2 \). For each \( v \in V \), let \( g(v) \) be the index \( i \) such that \( v \in D_i \).

We now use Definition 1 to define indifference graphs \( G_i(V, E_i), i \in \{1, 2, \ldots, \lceil \log_2 \alpha \rceil\} \) whose intersection is \( G \). For each \( G_i \) let \( I_i : V \longrightarrow \{0, 1\} \) be defined such that,

\[
I_i(u) = (-1)^{b_i(u)}(m - g(u) + \delta), \quad \forall u \in V,
\]

where \( 0 < \delta < \frac{1}{2} \). For two distinct vertices \( u \) and \( v \), \( uv \) is an edge in \( G_i \), i.e. \( uv \in E_i \) if and only if \( |I_i(u) - I_i(v)| \leq m \). We mention two properties of \( G_i \)'s which help us gain more insight into their structure. Given graph \( G_i \) and two distinct vertices \( u \) and \( v \),

Property 1. Supposing \( b_i(u) = b_i(v) \), the distance

\[
|I_i(u) - I_i(v)| = |g(u) - g(v)| < m.
\]

Hence, if \( b_i(u) = b_i(v) \), then \( uv \) is an edge in \( G_i \).

Property 2. Suppose \( b_i(u) \neq b_i(v) \). We recall that \( \delta < \frac{1}{2} \). The distance

\[
|I_i(u) - I_i(v)| = |2(m + \delta) - (g(u) + g(v))| \leq m
\]

is \( \leq m \) if and only if \( g(u) + g(v) \leq m \). Hence, if \( b_i(u) \neq b_i(v) \), then \( uv \) is an edge in \( G_i \) if and only if \( g(u) + g(v) > m \).

Now we prove that \( G \) is the intersection of \( G_i \)'s. Suppose \( uv \) is an edge in \( G \), we note from Definition 3 that \( g(u) + g(v) > m \). Using this fact with Properties 1 and 2 we observe that irrespective of the values of \( b_i(u) \) and \( b_i(v) \), \( uv \) is an edge in every \( G_i \). Hence, we see that every \( G_i \) is a supergraph of \( G \).

Now, consider two vertices \( u \) and \( v \) such that \( uv \) is not an edge in \( G \). Since \( C \) is a clique, at most one of these vertices is in \( C \), which implies that \( I(u) \neq I(v) \) and hence \( b_i(u) \neq b_i(v) \) for some index \( i \). Also, according to Definition 3, we have \( g(u) + g(v) \leq m \). Together with Property 2, we conclude that \( uv \) is not an edge of \( G_i \). Thus we have proved that \( G \) is an intersection of \( \lceil \log_2 \alpha \rceil \) indifference graphs and hence \( \text{cub}(G) \leq \lceil \log_2 \alpha \rceil \).
A threshold cover of a graph $G(V, E)$ is a set of threshold graphs $G_i(V, E_i), i = 1, 2, \ldots, k$ such that $E = E_1 \cup E_2 \cup \ldots \cup E_k$. The threshold dimension $\theta(G)$ is the least integer $k$ such that a threshold cover of size $k$ exists. In [3], Chvátal and Hammer show that $\theta(G) \leq |V| - \alpha(G)$, where $\alpha(G)$ is the independence number of $G$. Cozzens and Halsey [5] proved that the boxicity of any graph $G(V, E)$ is not more than the threshold dimension of its complement $\overline{G}$, i.e. $\text{box}(G) \leq \theta(\overline{G})$. We have a similar result for the cubicity of any graph which follows as a corollary of Theorem 4. But first, we need to state two lemmas which will be used for this purpose.

**Lemma 5** ([6]). Let $G$ be a graph. $\theta(G)$ is the least integer $k$ such that $\overline{G}$ is the intersection of $k$ threshold graphs.

**Lemma 6** ([11]). Suppose $G$ is the intersection of graphs $G_1, G_2, \ldots, G_j$, then $\text{cub}(G) \leq \sum_{i=1}^{j} \text{cub}(G_i)$. The corollary follows.

**Corollary 7.** For a connected graph $G(V, E)$ with independence number $\alpha$, $\text{cub}(G) \leq \theta(\overline{G}) \lceil \log_2 \alpha \rceil$, where, $\overline{G}$ is the complement of $G$.

**Proof.** Applying Lemma 5 to $\overline{G}$, we immediately see that $G$ can be expressed as the intersection of $\theta(\overline{G})$ threshold graphs, say $G_i, i \in \{1, 2, \ldots, \theta(\overline{G})\}$. For each $i$, let $\alpha_i$ be the independence number of $G_i$. Since each $G_i$ is a supergraph of $G$, $\alpha \geq \alpha_i$. We use this fact and Lemma 6 to obtain the result.

$$\text{cub}(G) \leq \sum_{i=1}^{\theta(\overline{G})} \text{cub}(G_i) = \sum_{i=1}^{\theta(\overline{G})} \lceil \log_2 \alpha_i \rceil \leq \theta(\overline{G}) \lceil \log_2 \alpha \rceil. \quad \Box$$

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**References**