Nonnegative Rank-Preserving Operators

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ABSTRACT

Analogues of characterizations of rank-preserving operators on field-valued matrices are determined for matrices with entries in certain structures $\mathbb{S}$ contained in the nonnegative reals. For example, if $\mathbb{S}$ is the set of nonnegative members of a real unique factorization domain (e.g. the nonnegative reals or the nonnegative integers), M is the set of $m\times n$ matrices with entries in $\mathbb{S}$, and $\min(m,n)\geqslant 4$, then a "linear" operator on M preserves the "rank" of each matrix in M if and only if it preserves the ranks of those matrices in M of ranks 1, 2, and 4. Notions of rank and linearity are defined analogously to the field-valued concepts. Other characterizations of rank-preserving operators for matrices over these and other structures $\mathbb{S}$ are also given.

1. INTRODUCTION AND SUMMARY

If $\mathbb{F}$ is an algebraically closed field, which linear operators T on the space of $m\times n$ matrices over $\mathbb{F}$ preserve the rank of each matrix? Evidently if U and V are invertible, then $X\to UXV$ is a rank-preserving, linear operator. When $m=n$, $X\to UX^tV$ is also. Marcus and Moyls [7] found that such "(U,V)-operators" were the only rank preservers. Later Marcus and Moyls [8] found that T preserves all ranks if and only if T "preserves rank 1." That is, the rank of $T(X)$ is 1 whenever the rank of X is 1. For further background, see Marcus's survey paper [6], Lautemann [5], Minc [9], and Westwick [11].

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In this paper we consider operators on the $m \times n$ matrices over various subsets of the nonnegative reals, $\mathbb{R}_+$. Our main result (Theorem 4.2) applies to subsets $\mathbb{U}_+$ consisting of the nonnegative members of a unique factorization domain $\mathbb{U}$—for example, the nonnegative reals themselves, the nonnegative rationals, the nonnegative integers, $\mathbb{R}_+ \cap \mathbb{Z}[\sqrt{2}]$, etc. Let M denote the $m \times n$ matrices with entries in $\mathbb{U}_+$. Theorem 4.2 asserts that whenever $\min(m, n) \geqslant 4$, a "linear" operator T on M preserves the "rank" of each member of M if and only if T is a "(U, V)-operator" on M if and only if T preserves "ranks" 1, 2, and 4. The concepts of "rank," "linearity," and "(U, V)-operator" are defined analogously to their field counterparts.

A weaker form of this theorem is obtained (Theorem 4.1) characterizing "linear" operators on M that preserve "ranks" 1 and 2.

Previously, similar results were obtained in [1] characterizing the rank-preserving operators on the $m \times n$ matrices over the Boolean algebra of two elements.

2. DEFINITIONS AND OTHER PRELIMINARIES

2.1. *Nonnegative Semidomains*

Let $\mathbb{S}$ be any subset of $\mathbb{R}_+$ (the nonnegative reals). We'll call it a *nonnegative semidomain* if it contains 0, 1 and is closed under multiplication and addition (the usual real operations). If $\mathbb{D}$ is a subring of $\mathbb{R}$ containing 1 (so $\mathbb{D}$ is an integral domain), let $\mathbb{D}_+$ denote the set of its nonnegative elements. Then $\mathbb{D}_+$ is a nonnegative semidomain. Examples are $\mathbb{R}_+$, $\mathbb{Q}_+$, $\mathbb{Z}_+$, $(\mathbb{Z}[\sqrt{2}\,])_+$, etc., where $\mathbb{Q}$ denotes the rationals and $\mathbb{Z}$ the integers. Note that $(\mathbb{Z}[\sqrt{2}\,])_+$ contains $\mathbb{Z}_+[\sqrt{2}\,]$ properly, since e.g. $\sqrt{2}-1$ is in the left member but not the right. There are other nonnegative semidomains: e.g. $\mathbb{H} = \{0,1,2,3\} \cup \{q \in \mathbb{Q} : q \geqslant 4\}$ is not of the form $\mathbb{D}_+$, for any integral domain $\mathbb{D}$ in $\mathbb{R}$.

Hereafter we will use the following notation unless otherwise specified:

$\mathbb{S}$ is an arbitrary nonnegative semidomain,
$\mathbb{D}$ is an arbitrary integral domain in $\mathbb{R}$,
$\mathbb{U}$ is an arbitrary unique factorization domain in $\mathbb{R}$, and
$\mathbb{F}$ is an arbitrary subfield of $\mathbb{R}$.

2.2. *Rank*

Let $\mathbb{S}$ be a subring of $\mathbb{R}$ containing 1, or a nonnegative semidomain of $\mathbb{R}$. Suppose X is an $m \times n$ matrix with all entries in $\mathbb{S}$, i.e., X is in $M_{m,n}(\mathbb{S})$. If $X \neq 0$, we define its $\mathbb{S}$-*rank*, $r_{\mathbb{S}}(X)$, as the least integer k such that there exist

$m \times k$ and $k \times n$ matrices Y and Z with entries in $\mathbb{S}$ such that $X = YZ$. The zero matrix is assigned the $\mathbb{S}$-rank 0.

(2.2.0) If $\mathbb{S}_1 \subseteq \mathbb{S}_2$, then $r_{\mathbb{S}_1}(X) \geqslant r_{\mathbb{S}_2}(X)$ for all X with entries in $\mathbb{S}_1$.

Here are some other properties of $\mathbb{S}$-rank that follow directly from the definitions. Letting $r(X) = r_{\mathbb{S}}(X)$ and A, B be matrices over $\mathbb{S}$:

(2.2.1) $r(AB) \leqslant \min(r(A), r(B))$,
(2.2.2) $r(A^t) = r(A)$,
(2.2.3) $r(C) \leqslant r(A)$ for all submatrices C of A,
(2.2.4) $r(A) \leqslant \min(m, n)$ if A is $m \times n$,
(2.2.5) If U, U^{-1} have all entries in $\mathbb{S}$, then $r(UA) = r(A)$.

2.3. *The Two-Element Boolean Algebra*

Let $\mathbb{B} = \{0, 1\}$. Define $x0 = 0x = 0$ and $x + 1 = 1 + x = 1$ for both x in $\mathbb{B}$. Then $\mathbb{B}$ is called the *2-element Boolean algebra*. It corresponds to the algebra of subsets of a singleton $\{a\}$ with 0 for $\varnothing$, 1 for $\{a\}$, $x + y$ for $x \cup y$, and xy for $x \cap y$. Note that $\mathbb{B}$ can't be embedded in a ring *under these operations* because in any ring $x + x \neq x$ unless $x = 0$. The $m \times n$ matrices over $\mathbb{B}$ have been studied extensively. (See Kim [4] for a compendium of results.)

If X is an $m \times n$ matrix over the nonnegative semidomain $\mathbb{S}$, define a Boolean $m \times n$ matrix $X^* = [x_{ij}^*]$ by $x_{ij}^* = 0$ if $x_{ij} = 0$ and $x_{ij}^* = 1$ if $x_{ij} > 0$. Then $*$ maps $M_{m,n}(\mathbb{S})$ onto $M_{m,n}(\mathbb{B})$, and preserves matrix addition, multiplication, and multiplication by scalars. That is, $*$ is a homomorphism.

It's well known (see e.g. Kim [4]) that the only invertible matrices in $M_{n,n}(B)$ are permutation matrices (matrices obtained by permuting the rows of I_n, the $n \times n$ identity matrix). Therefore if U is invertible over $\mathbb{S}$ (i.e. U, U^{-1} are $n \times n$ matrices over $\mathbb{S}$), then U^* is invertible over $\mathbb{B}$ and hence PU is a diagonal matrix over $\mathbb{S}$ for some permutation matrix P. [In fact, $P = (U^*)^t$, abusing the notation a bit.] Therefore a square matrix U over $\mathbb{S}$ is invertible if and only if some permutation of its rows is a diagonal matrix all of whose diagonal entries are units in $\mathbb{S}$.

The rank of matrices over $\mathbb{B}$ is defined just as in Section 2.1. We'll call it *Boolean* rank and denote it by $r_{\mathbb{B}}$. (Kim calls it Schein rank, [4].) If $r_{\mathbb{S}}(X) = k$, then $X = YZ$ for some $m \times k$, $k \times n$ matrices Y, Z over $\mathbb{S}$. Then $X^* = Y^* Z^*$, so $r_{\mathbb{B}}(X^*) \leqslant k$. In general

(2.3.1) $r_{\mathbb{B}}(X^*) \leqslant r_{\mathbb{S}}(X)$ for all X in $M_{m,n}(\mathbb{S})$.

The following will be used frequently.

EXAMPLE 2.3.1. Let

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix};$$

then $r_{\mathbb{B}}(\Lambda) = 4$ (see e.g. Berman and Plemmons [2, p. 84]). Therefore by inequalities (2.3.1), (2.2.4), $r_{\mathbb{S}}(L) = 4$ for every L in $M_{4,4}(\mathbb{S})$ for which $L^* = \Lambda$. On the other hand, $r_{\mathbb{R}}(\Lambda) = 3$.

2.4. When $r_{\mathbb{R}}(X) = r_{\mathbb{S}}(X)$

In this and all subsequent sections, when we refer to the *rank* of a matrix without specifying the kind of rank, we will always mean the $\mathbb{S}$-rank and we'll write $r(X)$ instead of $r_{\mathbb{S}}(X)$. The real rank of X will be denoted $\rho(X)$.

We remind the reader that $\mathbb{U}$ denotes an arbitrary unique factorization domain in $\mathbb{R}$, and $\mathbb{S}$ denotes an arbitrary nonnegative semidomain unless otherwise specified.

Let Γ be a nonempty subset of $\mathbb{S}^k$. We'll say that $\mathbf{g}$ is a *common factor* of Γ if $\Gamma \subseteq \{\sigma\mathbf{g} : \sigma \in \mathbb{S}\}$.

LEMMA 2.4.1. *Let Γ be any nonempty subset of $\mathbb{U}_+^k$. Each pair of nonzero vectors in Γ has a common nonzero scalar multiple in $\mathbb{U}_+^k$ if and only if Γ has a common factor in $\mathbb{U}_+^k$.*

Proof. We may suppose $\Gamma \neq \{\mathbf{0}\}$. Let $\mathbf{a}$ be a nonzero member of Γ, and α be a greatest common divisor (gcd) of the entries of $\mathbf{a}$. Then $\mathbf{a} = \alpha\mathbf{f}$ for an $\mathbf{f}$ in $\mathbb{U}_+^k$ which has 1 for a gcd of its entries. Let $\mathbf{x}$ be an arbitrary nonzero member of Γ. Then by our hypothesis, $\mathbf{a}$ and $\mathbf{x}$, and hence $\mathbf{f}$ and $\mathbf{x}$, have a nonzero common scalar multiple $\mathbf{c}$. Next, we show that the set $\{\mathbf{f},\mathbf{x}\}$ has a common factor $\mathbf{g}$ in $\mathbb{U}_+^k$. Suppose $\delta\mathbf{f} = \mathbf{c} = \varepsilon\mathbf{x}$. Let $\gamma = \gcd(\delta, \varepsilon)$, $\beta = \delta/\gamma$, and $\tau = \varepsilon/\gamma$. Then τ and β are in $\mathbb{U}_+$, and $\tau\mathbf{f} = \beta\mathbf{x}$. Therefore for every index i, τ divides βx_i. But τ is relatively prime to β, so τ divides every entry in $\mathbf{x}$, because $\mathbb{U}$ is a unique factorization domain. Therefore for some $\mathbf{g}$ in $\mathbb{U}_+^k$, $\mathbf{x} = \tau\mathbf{g}$. By cancellation, $\mathbf{f} = \beta\mathbf{g}$. Then β is a unit in $\mathbb{U}_+$, because β divides every entry in $\mathbf{f}$. Therefore $\mathbf{x} = \beta^{-1}\tau\mathbf{f}$. But $\mathbf{x}$ was arbitrary, so $\mathbf{f}$ is a common factor of Γ. The converse is immediate. ■

LEMMA 2.4.2. *If U is an $m \times n$ matrix over $\mathbb{S}$ and $\rho(U) = 1$, then $r(\alpha U) = 1$ for some α in $\mathbb{S}$.*

Proof. We may assume that $2 \leqslant m \leqslant n$. There exist vectors **b** and **c** in $\mathbb{R}_+^m$ and $\mathbb{R}_+^n$ such that $U = \mathbf{bc}^t$, because $\rho(U) = 1$. We may assume without loss of generality that for some $l \leqslant m$ and $k \leqslant n$, $b_i \neq 0$ and $c_j \neq 0$ if and only if $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant k$. We may also assume that $b_1 = 1$. Let $\alpha = c_1$; then α is in $\mathbb{S}$, because **c** is in $\mathbb{S}^n$. For each $i \leqslant m$, $b_i = u_{i1}/\alpha$; then $\alpha u_{ij} = u_{i1}c_j$ for all i, j. Therefore $r(\alpha U) = 1$. ∎

The following example demonstrates the existence of a nonnegative semidomain $\mathbb{S}$, a matrix U over $\mathbb{S}$, and α in $\mathbb{S}$ such that $\rho(U) = r(\alpha U) = 1$ but $r(U) = 2$.

EXAMPLE 2.4.1. Let $\mathbb{S} = \{a + b\sqrt{5} \geqslant 0 : a, b \in \mathbb{Z}\} = \mathbb{R}_+ \cap \mathbb{Z}[\sqrt{5}]$, $\alpha = 3 + \sqrt{5}$, $\beta = 1 + \sqrt{5}$, and

$$U = \begin{bmatrix} \beta & \alpha \\ 2 & \beta \end{bmatrix}.$$

Then

$$\alpha U = \begin{bmatrix} \alpha + \beta \\ \alpha \end{bmatrix} [2, \beta].$$

Therefore $r(\alpha U) = 1$ and hence $\rho(U) = 1$. Clearly $r(U) = 1$ or 2. If $r(U) = 1$, then

$$U = \begin{bmatrix} x \\ y \end{bmatrix} [u, v],$$

so $xv = \alpha$. But α is irreducible over $\mathbb{Z}[\sqrt{5}]$ (see, e.g. [3, Chapter IV, Exercise 14f]). Therefore x or v is a unit. If v is a unit, then $v^{-1}u\beta = 2$, since $yu = 2$ and $yv = \beta$. Multiplying by $\sqrt{5} - 1$ we have $4v^{-1}u = (2\sqrt{5}) - 2$, so $v^{-1}u = (-\frac{1}{2}) + \frac{1}{2}\sqrt{5}$, which is not in $\mathbb{S}$, a contradiction. If x is a unit we arrive at the same contradiction. Therefore $r(U) = 2$ even though $\rho(U) = 1$.

LEMMA 2.4.3. *If W is any $m \times m$ matrix over $\mathbb{U}_+$, then $\rho(W) = 1$ if and only if $r(W) = 1$.*

Proof. According to inequality (2.2.0), $\rho(W) = 1$ if $r(W) = 1$, because $\mathbb{U}_+ \subseteq \mathbb{R}$. Conversely, if $\rho(W) = 1$ then $r(\alpha W) = 1$ for some nonzero α in $\mathbb{U}_+$ by Lemma 2.4.2. Let Γ be the set of columns of αW. The members of Γ are all multiples of some nonzero vector **w**, because $r(\alpha W) = 1$. Therefore, by

Lemma 2.4.1, each pair $(\alpha \mathrm{w}_i, \alpha \mathrm{w}_j)$ of nonzero members of Γ have a common nonzero scalar multiple. Therefore each pair $(\mathrm{w}_i, \mathrm{w}_j)$ of nonzero columns of W have a nonzero scalar multiple. So by Lemma 2.4.1, the columns of W have a common factor in $\mathbb{U}_+^m$. Therefore $r(W) = 1$. ■

COROLLARY. *If X is any $m \times m$ matrix over $\mathbb{U}_+$ and the $\mathbb{U}_+$-rank of X is 2, then the real rank of X is 2.* ■

EXAMPLE 2.4.2. If $k > 1$, let

$$A(k) = \begin{bmatrix} 1 & 1 & k-1 \\ 1 & k & 0 \\ 1 & 0 & k \end{bmatrix}.$$

If $0 < k < 1$, let $p = \lceil 1/k \rceil$, $q = p - 1$, and

$$A(k) = \begin{bmatrix} 1 & 1-kq & kp-1 \\ 1 & k & 0 \\ 1 & 0 & k \end{bmatrix}.$$

LEMMA 2.4.4. *If k is a nonzero nonunit in $\mathbb{D}_+$, then $r(A(k)) = 3$.*

Proof. Let $A = A(k)$. Each entry in A is in $\mathbb{D}_+$. Also, $r(A) = 2$ or 3, because $\rho(A) = 2$. Suppose $r(A) = 2$. Then

$$A = [\mathrm{u}, \mathrm{v}] \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix},$$

where both factors have all entries in $\mathbb{D}_+$. Either y_1 or y_2 is not zero. Without loss of generality, assume $y_1 \neq 0$. Now $0 = a_{32} = y_1 u_3 + y_2 v_3$, so $y_2 v_3 = 0 = u_3$. Also, $1 = a_{31} = z_1 u_3 + z_2 v_3$, so $z_2 v_3 = 1$. Therefore $v_3 \neq 0$, so $y_2 = 0$. But $k = a_{22} = y_1 u_2 + y_2 v_2$, so $y_1 u_2 = k$. We have $1 = a_{21} = x_1 u_2 + x_2 v_2$. But $v_2 = 0$ (because $z_2 \neq 0$ and $0 = a_{23} = z_1 u_2 + z_2 v_2$). Therefore $x_1 u_2 = 1$, so u_2 is a unit. If $k > 1$, then $1 = a_{12} = y_1 u_2 + y_2 u_2$ but $y_2 = 0$, so y_1 is a unit. Thus $k = y_1 u_2$ as a product of two units, must be a unit, contrary to hypothesis. If $0 < k < 1$, then $1 - kq = a_{12} = y_1 u_1 + y_2 v_1$, but we've seen that $y_2 = 0$, $y_1 u_2 = k$, and $x_1 u_2 = 1$. Therefore $k(u_1 + q u_2) x_1 = 1$, so k is a unit, contrary to hypothesis. ■

The converse of the Corollary to Lemma 2.4.3 is false, because the $\mathbb{Z}_+$-rank of $A(2)$ is 3, while its real rank is 2.

Just as multiplication by a scalar can lower the rank of a rank-2 matrix over $\mathbb{S}$ (unless $\mathbb{S} = \mathbb{U}_+$), so multiplication by a scalar can lower the rank of a rank-3 matrix over $\mathbb{U}_+$. For example, the $\mathbb{Z}_+$-rank of $2A(2)$ is 2.

LEMMA 2.4.5. *If X is any $m \times m$ matrix over $\mathbb{F}_+$, then for each $k \leqslant 2$ we have $r(X) = k$ if and only if $\rho(X) = k$.*

Proof. By Lemma 2.4.3 and its corollary, we need only show that $r(X) = 2$ when $\rho(X) = 2$. We may also assume that $2 \leqslant m \leqslant n$. If $\rho(X) = 2$, then some $m \times 2$ submatrix of X has real rank 2. We may assume, without loss of generality, that no column of X is $\mathbf{0}$ and that $X = [\mathbf{a}, \mathbf{b}, \dots]$, where $\rho([\mathbf{a}, \mathbf{b}]) = 2$. We proceed by induction on n. If $n = 2$, the result is obvious. Suppose the result is true for all $m \times n'$ matrices with $2 \leqslant m \leqslant n' < n$. We have $X = [\mathbf{a}, \mathbf{b}, \mathbf{c}_3, \mathbf{c}_4, \dots, \mathbf{c}_{n-1}, \mathbf{c}_n]$. Let $Y = [\mathbf{a}, \mathbf{b}, \mathbf{c}_3, \dots, \mathbf{c}_{n-1}]$; then $\rho([\mathbf{a}, \mathbf{b}]) \leqslant \rho(Y) \leqslant \rho(X) = 2$, so $\rho(Y) = 2$ and hence $r(Y) = 2$ by the induction hypothesis. Let $\mathbb{S} = \mathbb{F}_+$. Therefore for some $\mathbf{u}, \mathbf{v}$ in $\mathbb{S}^m$, each column of Y is an $\mathbb{S}$-linear combination of $\mathbf{u}$ and $\mathbf{v}$. Let $\mathbf{c} = \mathbf{c}_n$. There exist real scalars x, y, z such that $x\mathbf{u} + y\mathbf{v} + z\mathbf{c} = \mathbf{0}$. We may assume that exactly one of x, y, z is negative. If $z < 0$, then $\mathbf{c} = \alpha\mathbf{u} + \beta\mathbf{v}$, where $\alpha = -x/z \geqslant 0$ and $\beta = -y/z \geqslant 0$. Some 2×2 submatrix of $[\mathbf{u}, \mathbf{v}]$ has real rank 2; call that matrix W. Then

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = W^{-1} \begin{bmatrix} c_i \\ c_j \end{bmatrix}$$

and hence α, β are in $\mathbb{F} \cap \mathbb{R}_+ = \mathbb{S}$. So every column of X is a linear combination of $\mathbf{u}$ and $\mathbf{v}$. If $x < 0$, then $\mathbf{u} = \alpha\mathbf{v} + \beta\mathbf{c}$, where $\alpha = -y/x \geqslant 0$ and $\beta = -z/x \geqslant 0$. By an argument similar to the case $z < 0$, we can show that α, β are in $\mathbb{S}$. Therefore every column of X is an $\mathbb{S}$-linear combination of $\mathbf{y}$ and $\mathbf{c}$. If $y < 0$, every column of X is an $\mathbb{S}$-linear combination of $\mathbf{u}$ and $\mathbf{c}$ similarly. Since $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{c}$ are in $\mathbb{S}^m$, in each of the cases above, it follows that $r(X) = 2$. ∎

Example 2.3.1 shows that Lemma 2.4.5 cannot be extended for $k > 2$.

LEMMA 2.4.6. *Suppose k is a nonzero nonunit in $\mathbb{D}_+$ and $\mu = \min(m, n) \geqslant 3$. Then for each $3 \leqslant r \leqslant \mu$, there exists an $m \times n$ matrix X over $\mathbb{D}_+$ of $\mathbb{D}_+$-rank r, such that the matrix obtained by multiplying the jth column of X by k has rank $r - 1$.*

Proof. Let A be $A(k)$ as in Example 2.4.2. Then multiplying the first column of A by k reduces its rank to 2. Obtain P from I_n by interchanging

I_n's first and jth columns, and let B be any $(m-3)\times(n-3)$ matrix over $\mathbb{D}_+$ of rank $r-3$. Then $X=(A\oplus B)P$ is the required matrix. ■

3. FACTOR SPACES AND RANK-1 SPACES

If X is a matrix over $\mathbb{S}$ and $X=\mathbf{a}\mathbf{x}^t$, then the vectors $\mathbf{a},\mathbf{x}$ are called *left* and *right factors* of X respectively. Both $\mathbf{a}$ and $\mathbf{x}$ are referred to as *factors* of X.

LEMMA 3.1. *Suppose A, B are $m\times n$ rank-1 matrices over $\mathbb{S}$ and* $\min(m,n)\geqslant 2$. *Then*

(a) $r(A+B)=1$ *only if $\alpha A, \alpha B$ have a common factor for some nonzero α in $\mathbb{S}$.*

(b) *If $\mathbb{S}=\mathbb{U}_+$, then $r(A+B)=1$ if and only if A, B have a common factor.*

Proof. Suppose $A=\mathbf{a}\mathbf{x}^t$ and $B=\mathbf{b}\mathbf{y}^t$. Let $U=[\mathbf{a},\mathbf{b}]$ and $V=[\mathbf{x},\mathbf{y}]$. Then $A+B=UV^t$. If $r(A+B)=1$ then $\rho(UV^t)=1$. But U is $m\times 2$ and V is $n\times 2$. Therefore $\rho(U)=1$ or $\rho(V)=1$. If $\rho(U)=1$, then $r(\alpha U)=1$ for some α in $\mathbb{S}$ by Lemma 2.4.2. Therefore $\alpha U=\mathbf{f}[\sigma,\tau]$ for some σ,τ in $\mathbb{S}$ and $\mathbf{f}$ in $\mathbb{S}^m$. Hence αA and αB have a common left factor, $\mathbf{f}$. If $\rho(V)=1$, we can show that they have a common right factor similarly.

If $\mathbb{S}=\mathbb{U}_+$, then by Lemma 2.4.3, we can take α to be 1 in the previous paragraph. Then A, B have a common factor when $r(A+B)=1$. The converse is immediate. ■

Any subset $\mathbf{V}$ of $\mathbb{S}^k$ closed under addition and under multiplication by scalars in $\mathbb{S}$ is called a (*vector*) *space* over $\mathbb{S}$. Identifying $\mathbb{S}^{mn}$ with $M_{m,n}(\mathbb{S})$, we transfer the definition to $M_{m,n}(\mathbb{S})$. If $\mathbf{V}\neq\{0\}$ is a space in $M_{m,n}(\mathbb{S})$ whose members have rank at most 1, then $\mathbf{V}$ is a *rank*-1 space. If $\mathbf{V}$ is a space all of whose members have the same left factor $\mathbf{a}$, then $\mathbf{V}$ is called a *left factor space*. Notice that in that case $\mathbf{W}=\{\mathbf{x}\in\mathbb{S}^n:\mathbf{a}\mathbf{x}^t\in\mathbf{V}\}$ is a space in $\mathbb{S}^n$. Conversely, if $\mathbf{W}$ is a space in $\mathbb{S}^n$ then $\{\mathbf{a}\mathbf{x}^t:\mathbf{x}\in\mathbf{W}\}$ is a left factor space. *Right factor spaces* are defined symmetrically. We call $\mathbf{V}$ a *factor space* if it is either a left or a right factor space.

Evidently factor spaces are rank-1 spaces. If $\mathbb{S}=\mathbb{U}_+$, then the converse is true, as we will see in Theorem 3.1 below.

Define a relation λ on the $m\times n$ rank-1 matrices over $\mathbb{S}$ by: $A\lambda B$ if A, B have a common left factor.

LEMMA 3.2.

(a) λ *is an equivalence relation on the* $m \times n$ *rank*-1 *matrices over* $\mathbb{U}_+$.

(b) *For any nonempty set* E *of* $m \times n$ *rank*-1 *matrices over* $\mathbb{U}_+$, *the members of* E *have a common left factor if and only if* $X \lambda Y$ *for all* X, Y *in* E.

Proof. Part (a): Evidently λ is a reflexive and symmetric. Suppose A, B, C are rank-1 $m \times n$ matrices over $\mathbb{U}_+$, $A \lambda B$, and $B \lambda C$. Then A, B, and C can be factored as $A = \mathbf{a}\mathbf{x}^t$, $\mathbf{a}\mathbf{y}^t = B = \mathbf{b}\mathbf{z}^t$, and $C = \mathbf{b}\mathbf{w}^t$. Now $\mathbf{a}, \mathbf{b}$ have a common nonzero scalar multiple because the factors of B are nonzero. Therefore $\mathbf{a}, \mathbf{b}$ have a common factor by Lemma 2.4.1, and hence $A \lambda C$. Consequently λ is also transitive.

Part (b): For each X in E select a left factor $\mathbf{g}_X$ and put $\Gamma = \{\mathbf{g}_X : X \in E\}$. By the proof of part (a), if A, C are in Γ, then $\mathbf{g}_A$ and $\mathbf{g}_C$ have a common nonzero scalar multiple. Therefore Γ has a common factor $\mathbf{f}$, by Lemma 2.4.1. Thus $\mathbf{f}$ is a common left factor of all X in E. The converse is immediate. ∎

Thus the λ-equivalence classes are the maximal left factor spaces in $M_{m,n}(\mathbb{U}_+)$. These in turn are of the form $\mathbf{V}(\mathbf{a}) = \{\mathbf{a}\mathbf{x}^t : \mathbf{x} \in \mathbb{U}_+^n\}$, where the gcd of the entries of $\mathbf{a}$ is a unit.

THEOREM 3.1. *Suppose* $\min(m,n) \geqslant 2$ *and* $\mathbf{V}$ *is a subspace of* $M_{m,n}(\mathbb{U}_+)$. *Then* $\mathbf{V}$ *is a rank*-1 *space if and only* $\mathbf{V}$ *is a factor space.*

Proof. Suppose $\mathbf{V}$ is a rank-1 space.

Case 1. Suppose there exist A, B in $\mathbf{V}$ having no common nonzero multiple. Since $\mathbf{V}$ is a rank-1 space, Lemma 3.1 implies that A, B have a common (say left) factor. Then A, B have no common right factor (in this case). Let X be any nonzero member of $\mathbf{V}$. Again by Lemma 3.1, X, A have a common factor and so do X, B. If X, A had no common left factor, then neither would X, B. (If $X \lambda B$, then $B \lambda A$ implies $X \lambda A$.) But then $X^t \lambda B^t$, $X^t \lambda A^t$, and so $A^t \lambda B^t$, a contradiction. Hence $X \lambda A$ for all $0 \neq X$ in $\mathbf{V}$. Therefore $\mathbf{V}$ is a left factor space by Lemma 3.2. If A, B had a common right factor, then (symmetrically) $\mathbf{V}$ would be a right factor space.

Case 2. For every A and B in $\mathbf{V}$ there exist α, β in $\mathbb{U}_+$, not both 0, such that $\alpha A = \beta B$. Therefore by Lemma 2.4.1, for some D in $M_{m,n}(\mathbb{U}_+)$, not necessarily in $\mathbf{V}$, $\mathbf{V} \subseteq \{\sigma D : \sigma \in \mathbb{U}_+\}$. Thus $\mathbf{V}$ is simultaneously a left factor space and a right factor space.

The converse is immediate. ∎

4. RANK PRESERVING LINEAR OPERATORS

Suppose $\mathbb{S}$ is a nonnegative semidomain and $M(\mathbb{S}) = M_{m,n}(\mathbb{S})$. If $T: M(\mathbb{S}) \to M(\mathbb{S})$ and $T(\alpha X + \beta Y) = \alpha T(X) + \beta T(Y)$ for all α, β in $\mathbb{S}$ and all X, Y in $M(\mathbb{S})$, then T is a *linear operator* on M. For X in $M(\mathbb{S})$ we write $r(X)$ for $r_{\mathbb{S}}(X)$ and $\rho(X)$ for $r_{\mathbb{R}}(X)$, as in previous sections. We also write $\mathbf{x}_j$ for the jth column of X, and $\mathbf{x}^i$ for its ith row.

LEMMA 4.1. *If T preserves $\mathbb{S}$-rank* 1, *then*

$$r(T(X)) \leqslant r(X) \qquad \text{for all} \quad X \text{ in } M(\mathbb{S}).$$

Proof. We may assume $X \neq 0$. If $r(X) = k$ then $X = AB$, where A is $m \times k$, and k is the least such index. Now $AB = \sum_{j=1}^{k} \mathbf{a}_j \mathbf{b}^j$, but T preserves $\mathbb{S}$-rank 1, so there exist $\mathbf{u}_j$ and $\mathbf{v}_j$ such that $T(\mathbf{a}_j \mathbf{b}^j) = \mathbf{u}_j \mathbf{v}_j^t$. Let $U = [\mathbf{u}_1, \dots, \mathbf{u}_k]$ and $V^t = [\mathbf{v}_1, \dots, \mathbf{v}_k]$; then $T(X) = UV$ and hence $r(T(X)) \leqslant k$. ■

Since $\rho(Y) \leqslant r(Y)$ for all Y, we have the inequality

$$\rho(T(X)) \leqslant r(T(X)) \leqslant r(X) \tag{4.1}$$

for all X in M when T preserves $\mathbb{S}$-rank 1.

LEMMA 4.2.

(a) *T preserves real rank* 1 *over $M(\mathbb{U}_+)$ if and only if T preserves $\mathbb{U}_+$-rank* 1.

(b) *If T preserves real rank* 2 *over $M(\mathbb{U}_+)$, then T preserves $\mathbb{U}_+$-rank* 2.

Proof. (a): This follows from Lemma 2.4.3.

(b): Suppose $r(X) = 2$; then $\rho(X) = 2$ by Lemma 2.4.3's corollary. Therefore $\rho(T(X)) = 2$ and hence $r(T(X)) = 2$ by the inequality (4.1). ■

LEMMA 4.3. *If T preserves real ranks* 1, 2, *and* 3 *for all X in $M(\mathbb{F}_+)$, then T preserves $\mathbb{F}_+$-ranks* 1, 2, *and* 3.

Proof. Suppose $r(X) = 3$. Then $\rho(X) \leqslant 3$ but $\rho(X) \nleqslant 2$ by Lemma 2.4.4, so $\rho(X) = 3$. Therefore $\rho(T(X)) = 3$, and hence the inequality (4.1) implies $r(T(X)) = 3$. The rest follows from Lemma 4.2. ■

In the following lemma "rank" can be interpreted as either real rank or $\mathbb{S}$-rank by letting ρ play the role of r in the proof.

LEMMA 4.4. *Suppose $\mathbb{S}$ is a nonnegative semidomain, U is in $M_{m,n}(\mathbb{S})$, and $T(X)=UX$ for all X in $M_{n,k}(\mathbb{S})$.*

(a) *If T preserves rank 1, then $T(X)=0$ only if $X=0$.*
(b) *If $k \geqslant 2$ and T preserves rank 1 and 2, then T is injective on $M_{n,k}(\mathbb{S})$.*

Proof. If $UX=0$ and $X \neq 0$, then U would have a zero column and so T would reduce the rank of some rank-1 matrix. That proves part (a). We now turn to (b). Suppose $T(A)=T(B)$. Then for all j, $U\mathbf{a}_j = U\mathbf{b}_j \equiv \mathbf{z}_j$. If $\mathbf{z}_j = \mathbf{0}$, then $\mathbf{a}_j = \mathbf{0} = \mathbf{b}_j$ by part (a). If $\mathbf{z}_j \neq \mathbf{0}$, let $Y=[\mathbf{a}_j, \mathbf{b}_j, \mathbf{0}, \dots, \mathbf{0}]$. Then $r(T(Y)) = 1$, but T preserves rank 2, so $r(Y)=1$. Therefore $\mathbf{a}_j = \alpha\mathbf{c}$ and $\mathbf{b}_j = \beta\mathbf{c}$ for some $\alpha, \beta, \mathbf{c}$. Hence $\alpha(U\mathbf{c}) = \mathbf{z}_j = \beta(U\mathbf{c})$ but $\mathbf{z}_j \neq \mathbf{0}$. Therefore $\alpha = \beta$ and $\mathbf{a}_j = \mathbf{b}_j$. ■

We use the notation E_{ij} for the $m \times n$ matrix whose ijth entry is 1 and whose other entries are all 0. We'll let $\mathbf{e}_i$ denote the ith column of I_m, the $m \times m$ identity matrix, and $\mathbf{f}_j$ the jth column of I_n. Then $E_{ij} = \mathbf{e}_i \mathbf{f}_j^t$.

THEOREM 4.1. *Suppose $\mathbb{S}$ consists of the nonnegative elements of a unique factorization domain in $\mathbb{R}$, T is a linear operator on $M_{m,n}(\mathbb{S})$, and* $\min(m,n) \geqslant 2$. *Then the following are equivalent:*

(a) *T preserves ranks 1 and 2.*

(b) *T is injective, and there exists matrices U, V over $\mathbb{S}$ such that either*

(1) *$T(X)=UXV$ for all X in $M_{m,n}(\mathbb{S})$, or*
(2) *$T(X)=UX^tV$ for all X in $M_{m,n}(\mathbb{S})$, possibly $m \neq n$.*

[Here, T need not be a (U,V)-operator because U or V need not be invertible.]

Proof. Suppose that (a) holds. Then $\mathbf{V}_i \equiv \{T(\mathbf{e}_i\mathbf{y}^t) : \mathbf{y} \in \mathbb{S}^N\}$ and $\mathbf{V}^j \equiv \{T(\mathbf{x}\mathbf{f}_j^t) : \mathbf{x} \in \mathbb{S}^m\}$ are rank-1 spaces. Therefore each $\mathbf{V}_i$ and $\mathbf{V}^j$ is a factor space by Theorem 3.1.

Case 1: $\mathbf{V}_1$ is a left factor space. If some $\mathbf{V}^j$ were a left factor space, choose $k \neq j$ and $i \neq 1$. Then $\{T(E_{1j}), T(E_{1k}), T(E_{ij})\}$ would have a common left factor by Lemma 3.2. But then $r(E_{1k}+E_{ij})=2$ and $r(T(E_{1k}+E_{ij})) = 1$, a contradiction. Hence all $\mathbf{V}^j$ are right factor spaces. Similarly, all $\mathbf{V}_i$ are left factor spaces. Therefore there exist nonzero vectors $\mathbf{x}_i$, $\mathbf{z}_j$, $\mathbf{p}_{ij}$, and $\mathbf{q}_{ij}$

such that for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$

$$\mathbf{x}_i \mathbf{p}_{ij}^t = T(E_{ij}) = \mathbf{q}_{ij} \mathbf{z}_j^t.$$

Fix i. Since $T(E_{ij}) \neq 0$, $\mathbf{x}_i$ and $\mathbf{q}_{ij}$ have a common nonzero scalar multiple for all $j \geqslant 1$. Therefore (Lemma 2.4.1) they have a common vector factor $\mathbf{f}_{ij}$. Let $\Gamma_i = \{\mathbf{f}_{ij}: 1 \leqslant j \leqslant n\}$. Then $\mathbf{x}_i$ is a common nonzero scalar multiple of $\mathbf{f}_{ir}$ and $\mathbf{f}_{is}$ for all $1 \leqslant r, s \leqslant n$. Consequently by Lemma 2.4.1, Γ_i has a common vector factor; call it $\mathbf{w}_i$. Since $\mathbf{w}_i$ is a factor of $\mathbf{f}_{ij}$ and $\mathbf{f}_{ij}$ is a factor of $\mathbf{q}_{ij}$, it follows that $\mathbf{q}_{ij} = a_{ij}\mathbf{w}_i$ for some nonzero scalar a_{ij}. Therefore there exist scalars a_{ij} and vectors $\mathbf{w}_i, \mathbf{z}_j$ such that for all $1 \leqslant i \leqslant m$ and all $1 \leqslant j \leqslant n$,

$$T(E_{ij}) = a_{ij} \mathbf{w}_i \mathbf{z}_j^t. \tag{4.2}$$

Let $A = [a_{ij}]$. We are going to show that $r(A) = 1$. No $a_{ij} = 0$, because T preserves rank 1. Therefore $\rho(A) \geqslant 1$. Let $W = [\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m]$ and $Z^t = [\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n]$. Let C be any 2×2 submatrix of A. We will show that $\det C = 0$, and hence that $\rho(A) = 1$. We have

$$C = \begin{bmatrix} a_{su} & a_{sv} \\ a_{tu} & a_{tv} \end{bmatrix} \quad \text{and} \quad \rho(C) > 1.$$

Let $E = E_{su} + E_{sv} + E_{tu} + E_{tv}$, $G = (1/a_{su})E_{su} + (1/a_{sv})E_{sv} + (1/a_{tu})E_{tu} + (1/a_{tv})E_{tv}$, and $\alpha = a_{su}a_{sv}a_{tu}a_{tv}$. Then αG is in $M_{m,n}(S)$ and $1 \leqslant r(\alpha G) \leqslant 2$. Also, $r(W(\alpha E)Z) \leqslant 1$ because $\alpha E = \alpha(\mathbf{e}_s + \mathbf{e}_t)(\mathbf{f}_u + \mathbf{f}_v)^t$. But $T(\alpha G) = W(\alpha E)Z$, and T preserves $\mathbb{S}$-ranks 0, 1, and 2. Therefore $r(\alpha G) = 1$; hence $\rho(G) = 1$. But

$$\det \begin{bmatrix} 1/a_{su} & 1/a_{sv} \\ 1/a_{tu} & 1/a_{tv} \end{bmatrix} = -\frac{1}{\alpha} \det C,$$

so $\det C = 0$. Therefore $\rho(A) = 1$. Hence $r(A) = 1$ by Lemma 2.4.3. So $a_{ij} = a_i b_j$ for some a_i, b_j in $\mathbb{S}$. Let $U = [a_1\mathbf{w}_1, a_2\mathbf{w}_2, \dots, a_m\mathbf{w}_m]$ and $V^t = [b_1\mathbf{z}_1, b_2\mathbf{z}_2, \dots, b_n\mathbf{z}_n]$; thus

$$T(X) = UXV \qquad \text{for all} \quad X \text{ in } M_{m,n}(\mathbb{S}). \tag{4.3}$$

Then T is injective by Lemma 4.4.

Case 2: $\mathbf{V}_1$ *is a right factor space.* Arguments parallel to those for case 1 show that

$$T(X) = UX^tV \qquad \text{for all} \quad X \text{ in } M_{m,n}(\mathbb{S}) \tag{4.4}$$

and that T is injective.

Suppose (b) holds. Forms (1) and (2), the inequality (2.2.1), and the equation (2.2.2) imply that for all X in $M_{m,n}(\mathbb{S})$

$$r(T(X)) \leqslant r(X). \tag{4.5}$$

But T is also injective, so T preserves rank 1. Suppose Y is any rank-2 member of $M_{m,n}(\mathbb{S})$. If (1) holds, let $X = Y$; otherwise let $X = Y^t$. Then $X = [\mathbf{c},\mathbf{d}][\mathbf{x},\mathbf{y}]^t$, where $r([\mathbf{c},\mathbf{d}]) = 2$ and $r([\mathbf{x},\mathbf{y}]) = 2$. In either case (1) or (2),

$$T(Y) = UXV = [U\mathbf{c}, U\mathbf{d}]\begin{bmatrix}\mathbf{x}^tV\\ \mathbf{y}^tV\end{bmatrix}.$$

Now $r(T(Y)) = 1$ or 2 by (4.5) and the fact that T preserves rank 1. If $r(T(Y)) = 1$, then by Lemma 3.1, $r([U\mathbf{c}, U\mathbf{d}]) = 1$ or $r([V^t\mathbf{x}, V^t\mathbf{y}]) = 1$. Without loss of generality, suppose the former holds. Then for some $\mathbf{z} \neq \mathbf{0}$ and $\alpha + \beta \neq 0$, $[U\mathbf{c}, U\mathbf{d}] = \mathbf{z}[\alpha, \beta]$. Consequently $U(\beta\mathbf{c}) = U(\alpha\mathbf{d})$. But U is injective because T is. Therefore $\beta\mathbf{c} = \alpha\mathbf{d}$. Since $r(X) = 2$, neither $\mathbf{c}$ nor $\mathbf{d}$ can be $\mathbf{0}$, so $\alpha \neq 0$ and $\beta \neq 0$. Therefore (Lemma 2.4.1) $\mathbf{c},\mathbf{d}$ have a common factor and hence $r([\mathbf{c},\mathbf{d}]) = 1$, contrary to assumption. Therefore $r(T(Y)) = 2$ and T preserves rank 2. ■

EXAMPLE 4.1. Suppose $\mathbb{S}$ is an arbitrary nonnegative semidomain. Let $T_k(X) = (\sum_{i,j} x_{ij})A$ for all X in $M_{m,n}(\mathbb{S})$, where $r(A) = k$. Then T_k preserves rank k, but T isn't injective. Thus (a) of Theorem 4.1 cannot be relaxed by requiring that T preserve rank 1 or that T preserve rank 2.

EXAMPLE 4.2. Suppose $\mathbb{S}$ is an arbitrary nonnegative semidomain. Let $T(E_{12}) = E_{22}$, $T(E_{22}) = E_{12}$, and $T(E_{ij}) = E_{ij}$ for all other i, j. Extend T to $M_{m,n}(\mathbb{S})$ by linearity. Let $A = E_{11} + E_{12}$ and $B = E_{11} + E_{22}$; then $r(A) = 1$ but $r(T(A)) = 2$. Also $r(B) = 2$ but $r(T(B)) = 1$. Nevertheless, T is injective. In fact, T is bijective. Thus injectivity alone does not ensure that ranks 1,2 will be preserved by a linear operator.

EXAMPLE 4.3. Suppose $\mathbb{S}$ is an arbitrary semidomain in $\mathbb{R}$. Let

$$W=\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix},$$

and define a linear operator T on $M=M_{3,3}(\mathbb{S})$ by $T(X)=WX$ for all X in M. Then T preserves all ranks by Lemma 4.2 and the inequality (4.1). Nevertheless T is not a (U,V)-operator on M, because T is not surjective, since

$$T(X)\neq\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}\qquad\text{for all}\quad X\text{ in }M.$$

Curiously enough, this example doesn't generalize. We shall see (Theorem 4.2) that when $\mathbb{S}=\mathbb{U}_+$ and $\min(m,n)\geqslant 4$, then T's preserving ranks 1,2,4 is equivalent to its being a (U,V)-operator and to its preserving all ranks.

EXAMPLE 4.4. Let A be the matrix of Example 2.4.2 and $B=2\oplus I_{m-1}$; then let $T(X)=X(B\oplus 0_{m,n-m})$ for all $m\times n$ matrices X over $\mathbb{Z}_+$ with $n\geqslant 3$. The operator T preserves $\mathbb{Z}_+$-ranks 1 and 2 by Lemma 4.2. But if $X=A\oplus 0_{m-3,n-3}$ then $r(T(X))=2$ while $r(X)=3$. Therefore, preserving ranks 1,2 isn't always sufficient to preserve rank 3.

EXAMPLE 4.5. Let

$$M=\begin{bmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{bmatrix}$$

and Λ be as in Example 2.3.1. The linear operator T on $M_{4,4}(\mathbb{R}_+)$ defined by $T\colon X\to MX$ preserves $\mathbb{R}_+$-ranks 1,2,3 by Lemma 4.3. But as we observed in Example 2.3.1, the $\mathbb{R}_+$-rank of Λ is 4. Nevertheless the $\mathbb{R}_+$-rank of $T(\Lambda)$ is not 4, because

$$M\Lambda=\begin{bmatrix}2&0&2\\1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&0&0\\0&1&1&0\\0&0&1&1\end{bmatrix}.$$

Therefore a linear operator on the 4×4 matrices over $\mathbb{F}_+$ can preserve $\mathbb{F}_+$-ranks 1, 2, 3, but not 4.

The next sequence of lemmas is needed to prove the main theorem.

LEMMA 4.5. *Suppose A is in $M_{m,k}(\mathbb{S})$, $4 \leqslant k \leqslant m$, and $\mathbf{a}_j$ is the jth column of A. If $\alpha\mathbf{a}_i \geqslant \mathbf{a}_j$ (entrywise) for some α in $\mathbb{S}$ and some $i \neq j$, then there exists a $k \times n$ matrix X such that $r(AX) \leqslant 3$ and $r(X) = 4$.*

Proof. Suppose without loss of generality that $\alpha \neq 0$ and that $A = [\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \dots]$, where $\alpha\mathbf{w} \geqslant \mathbf{u}$. Let

$$W = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & \alpha & \alpha & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then let $X = W \oplus 0$ be a $k \times n$ matrix. Now $r(W) = 4$ by Example 2.3.1. Therefore $r(X) = 4$, but

$$AX = A\left(\begin{bmatrix} B \\ 0 \end{bmatrix} [C, 0] \right) = [\mathbf{u} + \mathbf{v}, \alpha\mathbf{w} - \mathbf{u}, \mathbf{u} + \mathbf{x}][C, 0]$$

and $\alpha\mathbf{w} - \mathbf{u}$ is in $\mathbb{S}^m$. Therefore $r(AX) < r(X)$. ■

Recall the homomorphism $M \to M^*$ sending $M_{m,n}(\mathbb{S})$ onto $M_{m,n}(\mathbb{B})$, where $\mathbb{B}$ is the 2-element Boolean algebra as in Section 2.3.

LEMMA 4.6. *Suppose A is in $M_{m,m}(\mathbb{S})$, $m \geqslant 4$, and $T(X) = AX$ for all X in $M_{m,n}(\mathbb{S})$. Then T preserves rank 4 only if A^* is a permutation matrix.*

Proof. Since A is a square nonnegative matrix, there exists a permutation matrix P such that $PAP^t = W$, where

$$W = \begin{bmatrix} B_{11} & B_{12} & B_{13} & \cdots & B_{1q} \\ 0 & B_{22} & B_{23} & \cdots & B_{2q} \\ 0 & 0 & B_{33} & \cdots & B_{3q} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & B_{qq} \end{bmatrix},$$

$B_{ij} = 0$ if $i > j$, and each B_{ii} is square and is either a 1×1 zero matrix or such that $B_{ii}^{d_i}$ is the direct sum of d_i square matrices, each having only positive entries in all sufficiently high powers. This is the Frobenius normal form of a nonnegative matrix (see e.g. Seneta [10, pp. 14–16, 21–22], or Berman and

Plemmons [2, pp. 32, 35, 39]). Therefore, for some positive integer k

$$W^k = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1p} \\ 0 & C_{22} & \cdots & C_{2p} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & C_{pp} \end{bmatrix},$$

where each $C_{ij} = 0$ if $i > j$, and each C_{ii} is either a 1×1 zero matrix or a square matrix all of whose entries are positive. If $i < j$, then $C_{ij} = 0$ or each entry in C_{ij} is positive. Now T preserves rank 4, so each of its powers including T^k must do so. Therefore if $\mathbf{c}_i$ and $\mathbf{c}_j$ are columns of W^k $(i \neq j)$, then $\alpha \mathbf{c}_i \not\geqslant \mathbf{c}_j$ for all $\alpha > 0$, by Lemma 4.5. Therefore $\mathbf{c}_i^* \not\leqslant \mathbf{c}_j^*$ for all $i \neq j$, so $(W^k)^*$ is the $m \times m$ identity matrix. Therefore A^* is a permutation matrix. ■

Any operator defined on $M_{m,n}(\mathbb{S})$ can be extended to $M_{m,n}(\mathbb{R})$ by linearity. The extension is unique because the E_{ij} are in $M_{m,n}(\mathbb{S})$.

LEMMA 4.7. *Suppose* $\mathbb{S} = \mathbb{U}_+$, T *is a linear operator on* $M_{m,n}(\mathbb{S})$, *and* $\min(m,n) \geqslant 4$. *If T preserves* $\mathbb{S}$*-ranks* 1, 2, *and* 4, *then the extension of T is a* (U,V)*-operator on* $M_{m,n}(\mathbb{R})$.

Proof. The operator T is injective and has the form (1) or (2) given in Theorem 4.1. If (2) holds, then $T^2(X) = WXZ$ where $W = UV^t$ and $Z = U^tV$. The operator T^2 preserves $\mathbb{S}$-rank 4 because T does so. Therefore W^* and Z^* are permutation matrices by Lemma 4.5. If $m > n$ then $\rho(W) < m$. But $W = P\operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_m)$ for some permutation matrix P and nonzero α_j, a contradiction. If $m < n$, we obtain a contradiction to Z^* being a permutation matrix, similarly. Hence $m = n$, so we may assume (1) holds. Then, as above, U^* is a permutation matrix, so $\rho(U) = m$. Hence U is invertible over $\mathbb{R}$. Similarly V is invertible over $\mathbb{R}$. ■

Example 4.5 shows that the converse of Lemma 4.7 is false.

THEOREM 4.2. *Suppose* $\mathbb{S}$ *consists of the nonnegative elements of a unique factorization domain in* $\mathbb{R}$, *and T is a linear operator on* $M_{m,n}(\mathbb{S})$. *If* $\min(m,n) \geqslant 4$, *then the following are equivalent*:

(a) *T preserves* $\mathbb{S}$*-ranks* 1, 2, *and* 4.
(b) *T is a* (U,V)*-operator on* $M_{m,n}(\mathbb{S})$.
(c) *T preserves all* $\mathbb{S}$*-ranks*.

Proof. Suppose (a) holds. Then by Lemma 4.7 and Lemma 4.6, T is a (U,V)-operator on $M_{m,n}(\mathbb{R})$ with U,V over $\mathbb{S}$, and U^*,V^* are permutation matrices. If some nonzero entry in V, say k, weren't a unit in $\mathbb{S}$, then for some permutation matrix P and diagonal matrix M, $VP=(I_{j-1}\oplus k\oplus I_{n-j})M$. Hence, by Lemma 2.4.6 there is an $m\times n$ matrix X with rank 4 such that $r(XVP)\leqslant 3$. Therefore $r(UXV)\leqslant 3$. Thus T reduces the rank of X [of X^t if $T(X)=UX^tV$]. This contradiction proves that every nonzero entry in V is a unit in $\mathbb{S}$. But V^* is a permutation matrix, so V is invertible over $\mathbb{S}$. Similarly U is invertible over $\mathbb{S}$, and hence (b) holds. The definitions imply that (b) implies (c) directly. That (c) implies (a) is immediate. ∎

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