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Some existence results for impulsive nonlinear fractional differential equations with mixed boundary conditions

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ABSTRACT

This paper investigates the existence and uniqueness of solutions for an impulsive mixed boundary value problem of nonlinear differential equations of fractional order $\alpha \in (1, 2]$. Our results are based on some standard fixed point theorems. Some examples are presented to illustrate the main results.

1. Introduction

Boundary value problems for nonlinear fractional differential equations have recently been addressed by several researchers. The interest in the study of differential equations of fractional order lies in the fact that fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. With this advantage, the fractional-order models become more realistic and practical than the classical integer-order models, in which such effects are not taken into account. As a matter of fact, fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. [1–4]. For some recent development on the topic, see [5–19] and the references therein.

Impulsive differential equations, which provide a natural description of observed evolution processes, are regarded as important mathematical tools for the better understanding of several real world problems in applied sciences. The theory of impulsive differential equations of integer order has found extensive applications in realistic mathematical modelling of a wide variety of practical situations and has emerged as an important area of investigation in recent years. For the general theory and applications of impulsive differential equations, we refer the reader to the references [20–23]. On the other hand, the impulsive boundary value problems for nonlinear fractional differential equations have not been addressed so extensively and many aspect of these problems are yet to be explored. For some recent work on impulsive differential equations of fractional order, see [24–31] and the references therein.

In this paper, we investigate the existence and uniqueness of solutions for a mixed boundary value problem of nonlinear impulsive differential equations of fractional order $\alpha \in (1, 2]$ given by

$$\begin{cases} {}^CD^\alpha u(t) = f(t, u(t)), \quad 1 < \alpha \le 2, t \in J', \\ \triangle u(t_k) = I_k(u(t_k)), \qquad \triangle u'(t_k) = I_k^*(u(t_k)), \quad k = 1, 2, \ldots, p, \\ Tu'(0) = -au(0) - bu(T), \qquad Tu'(T) = cu(0) + du(T), \quad a, b, c, d \in \mathbb{R}, \end{cases} \tag{1.1}$$

where ${}^CD^\alpha$ is the Caputo fractional derivative, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $I_k, I_k^* \in C(\mathbb{R}, \mathbb{R})$, $J = [0, T] (T > 0)$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_p < t_{p+1} = T$, $J' = J \setminus \{t_1, t_2, \ldots, t_p\}$, $\triangle u(t_k) = u(t_k^+) - u(t_k^-)$, $u(t_k^+)$ and $u(t_k^-)$ denote the right and the left limits of $u(t)$ at $t = t_k (k = 1, 2, \ldots, p)$, respectively and $\triangle u'(t_k)$ have a similar meaning for $u'(t)$.

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Here we remark that the boundary conditions in (1.1) interpolate between Neumann ($a = b = c = d = 0$) and Dirichlet ($a, d \to \infty$ with finite values of b and c) boundary conditions. Notice that Zaremba boundary conditions ($u(0) = 0, u'(T) = 0$) can be considered as mixed boundary conditions with $a \to \infty, c = d = 0$. For more details on Zaremba boundary conditions, see [32–34].

2. Preliminaries

Let $J_0 = [0, t_1]$, $J_1 = (t_1, t_2], \ldots, J_{p-1} = (t_{p-1}, t_p]$, $J_p = (t_p, T]$, and we introduce the spaces: $PC(J, \mathbb{R}) = \{u : J \to \mathbb{R} \mid u \in C(J_k),\ k = 0, 1, \ldots, p,$ and $u(t_k^+)$ exist, $k = 1, 2, \ldots, p, \}$ with the norm $\|u\| = \sup_{t\in J} |u(t)|$, and $PC^1(J, \mathbb{R}) = \{u : J \to \mathbb{R} \mid u \in C^1(J_k),\ k = 0, 1, \ldots, p,$ and $u(t_k^+), u'(t_k^+)$ exist, $k = 1, 2, \ldots, p, \}$ with the norm $\|u\|_{PC^1} = \max\{\|u\|, \|u'\|\}$. Obviously, $PC(J, \mathbb{R})$ and $PC^1(J, \mathbb{R})$ are Banach spaces.

Definition 2.1. A function $u \in PC^1(J, \mathbb{R})$ with its Caputo derivative of order α existing on J is a solution of (1.1) if it satisfies (1.1).

We need the following known results to prove the existence of solutions for (1.1).

Theorem 2.1 ([35]). *Let E be a Banach space. Assume that Ω is an open bounded subset of E with $\theta \in \Omega$ and let $T : \overline{\Omega} \to E$ be a completely continuous operator such that*

$$\|Tu\| \le \|u\|, \quad \forall u \in \partial\Omega.$$

Then T has a fixed point in $\overline{\Omega}$.

Theorem 2.2 ([35]). *Let E be a Banach space. Assume that $T : E \to E$ is a completely continuous operator and the set $V = \{u \in E \mid u = \mu Tu, 0 < \mu < 1\}$ is bounded. Then T has a fixed point in E.*

Lemma 2.1. *For a given $y \in C[0, T]$, a function u is a solution of the impulsive mixed boundary value problem*

$$\begin{cases} {}^C D^{\alpha} u(t) = y(t), \quad 1 < \alpha \le 2, t \in J', \\ \triangle u(t_k) = I_k(u(t_k)), \qquad \triangle u'(t_k) = I_k^*(u(t_k)), \quad k = 1, 2, \ldots, p, \\ Tu'(0) = -au(0) - bu(T), \qquad Tu'(T) = cu(0) + \mathrm{d}u(T), \quad a, b, c, d \in \mathbb{R}, \end{cases} \tag{2.1}$$

if and only if u is a solution of the impulsive fractional integral equation

$$u(t) = \begin{cases} \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s + \frac{(b+d)T + (ad-bc)t}{\Lambda T} \int_{t_p}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s \\ \displaystyle\quad - \frac{(b+1)T + (a+b)t}{\Lambda} \int_{t_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\mathrm{d}s + \mathscr{A}, \quad t \in J_0; \\ \displaystyle\int_{t_k}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s + \frac{(b+d)T + (ad-bc)t}{\Lambda T} \int_{t_p}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s \\ \displaystyle\quad - \frac{(b+1)T + (a+b)t}{\Lambda} \int_{t_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\mathrm{d}s + \sum_{i=1}^{k} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s + I_i(u(t_i)) \right] \\ \displaystyle\quad + \sum_{i=1}^{k-1} (t_k - t_i) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\mathrm{d}s + I_i^*(u(t_i)) \right] \\ \displaystyle\quad + \sum_{i=1}^{k} (t - t_k) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\mathrm{d}s + I_i^*(u(t_i)) \right] + \mathscr{A}, \quad t \in J_k,\ k = 1, 2, \ldots, p, \end{cases} \tag{2.2}$$

where

$$\begin{aligned} \mathscr{A} = {} & \frac{(b+d)T + (ad-bc)t}{\Lambda T} \sum_{i=1}^{p} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s + I_i(u(t_i)) \right] \\ & + \frac{(b+d)T + (ad-bc)t}{\Lambda T} \sum_{i=1}^{p-1} (t_p - t_i) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\mathrm{d}s + I_i^*(u(t_i)) \right] \\ & - \sum_{i=1}^{p} \left[\frac{(1-d)(T+at) + b(c+1)t}{\Lambda} + \frac{[(b+d)T + (ad-bc)t]t_p}{\Lambda T} \right] \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\mathrm{d}s + I_i^*(u(t_i)) \right] \end{aligned}$$

and

$$\Lambda = (b+1)(c+d) - (a+b)(d-1) \neq 0.$$

Proof. Let u be a solution of (2.1). Then, for $t \in J_0$, there exist constants $c_1, c_2 \in \mathbb{R}$ such that

$$u(t) = I^{\alpha}y(t) - c_1 - c_2 t = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\mathrm{d}s - c_1 - c_2 t. \tag{2.3}$$

$$u'(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}y(s)\mathrm{d}s - c_2.$$

For $t \in J_1$, then there exist constants $d_1, d_2 \in \mathbb{R}$, such that

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}y(s)\mathrm{d}s - d_1 - d_2(t-t_1),$$

$$u'(t) = \frac{1}{\Gamma(\alpha-1)}\int_{t_1}^t (t-s)^{\alpha-2}y(s)\mathrm{d}s - d_2.$$

Then we have

$$u(t_1^-) = \frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}y(s)\mathrm{d}s - c_1 - c_2 t_1, \qquad u(t_1^+) = -d_1,$$

$$u'(t_1^-) = \frac{1}{\Gamma(\alpha-1)}\int_0^{t_1} (t_1-s)^{\alpha-2}y(s)\mathrm{d}s - c_2, \qquad u'(t_1^+) = -d_2.$$

In view of $\triangle u(t_1) = u(t_1^+) - u(t_1^-) = I_1(u(t_1))$, and $\triangle u'(t_1) = u'(t_1^+) - u'(t_1^-) = I_1^*(u(t_1))$, we have

$$-d_1 = \frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}y(s)\mathrm{d}s - c_1 - c_2 t_1 + I_1(u(t_1)),$$

$$-d_2 = \frac{1}{\Gamma(\alpha-1)}\int_0^{t_1} (t_1-s)^{\alpha-2}y(s)\mathrm{d}s - c_2 + I_1^*(u(t_1)).$$

Consequently,

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}y(s)\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}y(s)\mathrm{d}s \\ &\quad + \frac{t-t_1}{\Gamma(\alpha-1)}\int_0^{t_1} (t_1-s)^{\alpha-2}y(s)\mathrm{d}s + I_1(u(t_1)) + (t-t_1)I_1^*(u(t_1)) - c_1 - c_2 t, \quad t \in J_1. \end{aligned}$$

By a similar process, we can get

$$\begin{aligned} u(t) &= \int_{t_k}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s + \sum_{i=1}^{k}\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s + I_i(u(t_i))\right] \\ &\quad + \sum_{i=1}^{k-1}(t_k-t_i)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)\mathrm{d}s + I_i^*(u(t_i))\right] \\ &\quad + \sum_{i=1}^{k}(t-t_k)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)\mathrm{d}s + I_i^*(u(t_i))\right] - c_1 - c_2 t, \quad t \in J_k,\ k = 1, 2, \ldots, p. \end{aligned} \tag{2.4}$$

Using the mixed boundary conditions $Tu'(0) = -au(0) - bu(T)$ and $Tu'(T) = cu(0) + \mathrm{d}u(T)$, we find that

$$\begin{aligned} c_1 &= \frac{1}{[(b+1)(c+d) - (a+b)(d-1)]}\left\{-(b+d)\int_{t_p}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s \right. \\ &\quad + (b+1)T\int_{t_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)\mathrm{d}s - (b+d)\sum_{i=1}^{p}\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s + I_i(u(t_i))\right] \\ &\quad - (b+d)\sum_{i=1}^{p-1}(t_p-t_i)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)\mathrm{d}s + I_i^*(u(t_i))\right] \\ &\quad \left. + \sum_{i=1}^{p}[(1-d)T + (b+d)t_p]\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)\mathrm{d}s + I_i^*(u(t_i))\right]\right\}, \end{aligned}$$

$$c_2 = \frac{1}{[(b+1)(c+d)-(a+b)(d-1)]T}\left\{(bc-ad)\int_{t_p}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s\right.$$
$$+(a+b)T\int_{t_p}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)\mathrm{d}s+(bc-ad)\sum_{i=1}^{p}\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s+I_i(u(t_i))\right]$$
$$+(bc-ad)\sum_{i=1}^{p-1}(t_p-t_i)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)\mathrm{d}s+I_i^*(u(t_i))\right]$$
$$\left.+\sum_{i=1}^{p}[[b(c+1)+a(1-d)]T-(bc-ad)t_p]\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)\mathrm{d}s+I_i^*(u(t_i))\right]\right\}.$$

Substituting the values of c_1 and c_2 in (2.3) and (2.4), we get (2.2). The converse of the lemma follows by a direct computation. This completes the proof. □

3. Main results

For the sake of convenience, we set

$$\lambda_1(t)=\frac{(b+d)T+(ad-bc)t}{\Lambda T},\qquad \lambda_2(t)=\frac{(b+1)T+(a+b)t}{\Lambda},$$
$$\lambda_3(t)=\frac{(1-d)(T+at)+b(c+1)t}{\Lambda},\qquad \lambda_4=\frac{|ad|+|bc|}{T|\Lambda|},\qquad \lambda_5=\frac{|a+b|}{|\Lambda|},$$

where $\Lambda=(b+1)(c+d)-(a+b)(d-1)\neq 0$.

Define the operator $T: PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ as

$$Tu(t)=\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))\mathrm{d}s+\lambda_1(t)\int_{t_p}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))\mathrm{d}s$$
$$-\lambda_2(t)\int_{t_p}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u(s))\mathrm{d}s+\sum_{i=1}^{k}\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))\mathrm{d}s+I_i(u(t_i))\right]$$
$$+\sum_{i=1}^{k-1}(t_k-t_i)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u(s))\mathrm{d}s+I_i^*(u(t_i))\right]$$
$$+\sum_{i=1}^{k}(t-t_k)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u(s))\mathrm{d}s+I_i^*(u(t_i))\right]$$
$$+\lambda_1(t)\sum_{i=1}^{p}\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))\mathrm{d}s+I_i(u(t_i))\right]$$
$$+\lambda_1(t)\sum_{i=1}^{p-1}(t_p-t_i)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u(s))\mathrm{d}s+I_i^*(u(t_i))\right]$$
$$-\sum_{i=1}^{p}[\lambda_3(t)+\lambda_1(t)t_p]\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u(s))\mathrm{d}s+I_i^*(u(t_i))\right]. \tag{3.1}$$

Lemma 3.1. *The operator $T: PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ is completely continuous.*

Proof. Observe that T is continuous in view of the continuity of f, I_k and I_k^*. Let $\Omega\subset PC(J,\mathbb{R})$ be bounded. Then, there exist positive constants $L_i>0$ $(i=1,2,3)$ such that $|f(t,u)|\le L_1$, $|I_k(u)|\le L_2$ and $|I_k^*(u)|\le L_3$, $\forall u\in\Omega$. Thus, $\forall u\in\Omega$, we have

$$|Tu(t)|\le\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))|\mathrm{d}s+|\lambda_1(t)|\int_{t_p}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))|\mathrm{d}s$$
$$+|\lambda_2(t)|\int_{t_p}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|\mathrm{d}s+\sum_{i=1}^{k}\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))|\mathrm{d}s+|I_i(u(t_i))|\right]$$
$$+\sum_{i=1}^{k-1}(t_k-t_i)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|\mathrm{d}s+|I_i^*(u(t_i))|\right]$$

$$
\begin{aligned}
&+\sum_{i=1}^{k}(t-t_k)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|\mathrm{d}s+|I_i^*(u(t_i))|\right]\\
&+|\lambda_1(t)|\sum_{i=1}^{p}\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))|\mathrm{d}s+|I_i(u(t_i))|\right]\\
&+|\lambda_1(t)|\sum_{i=1}^{p-1}(t_p-t_i)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|\mathrm{d}s+|I_i^*(u(t_i))|\right]\\
&+\sum_{i=1}^{p}\big[|\lambda_3(t)|+|\lambda_1(t)|t_p\big]\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|\mathrm{d}s+|I_i^*(u(t_i))|\right]\\
\le{}& L_1\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s+|\lambda_1(t)|L_1\int_{t_p}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s+|\lambda_2(t)|L_1\int_{t_p}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\mathrm{d}s\\
&+\sum_{i=1}^{p}\left[L_1\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s+L_2\right]+\sum_{i=1}^{p-1}T\left[L_1\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\mathrm{d}s+L_3\right]\\
&+\sum_{i=1}^{p}T\left[L_1\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\mathrm{d}s+L_3\right]+|\lambda_1(t)|\sum_{i=1}^{p}\left[L_1\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s+L_2\right]\\
&+|\lambda_1(t)|\sum_{i=1}^{p-1}T\left[L_1\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\mathrm{d}s+L_3\right]+\sum_{i=1}^{p}[|\lambda_3(t)|+T|\lambda_1(t)|]T\left[L_1\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\mathrm{d}s+L_3\right]\\
\le{}&\frac{(1+p)(1+|\lambda_1(t)|)T^{\alpha}L_1}{\Gamma(\alpha+1)}+\frac{[(2p-1)(1+T|\lambda_1(t)|)+p|\lambda_3(t)|+|\lambda_2(t)|]T^{\alpha-1}L_1}{\Gamma(\alpha)}\\
&+(1+|\lambda_1(t)|)pL_2+[(2p-1)(1+T|\lambda_1(t)|)+p|\lambda_3(t)|]L_3. \qquad (3.2)
\end{aligned}
$$

Since $t\in[0,T]$, therefore there exists a positive constant L, such that $\|Tu\|\le L$, which implies that the operator T is uniformly bounded.

On the other hand, for any $t\in J_k$, $0\le k\le p$, we have

$$
\begin{aligned}
|(Tu)'(t)|\le{}&\int_{t_k}^{t}\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|\mathrm{d}s+\lambda_4\int_{t_p}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))|\mathrm{d}s\\
&+\lambda_5\int_{t_p}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|\mathrm{d}s+\sum_{i=1}^{p}\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|\mathrm{d}s+|I_i^*(u(t_i))|\right]\\
&+\lambda_4\sum_{i=1}^{p}\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))|\mathrm{d}s+|I_i(u(t_i))|\right]\\
&+\lambda_4\sum_{i=1}^{p-1}(t_p-t_i)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|\mathrm{d}s+|I_i^*(u(t_i))|\right]\\
&+\sum_{i=1}^{p}\left|\frac{(a+b)T+(bc-ad)(T-t_p)}{T\Lambda}\right|\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|\mathrm{d}s+|I_i^*(u(t_i))|\right]\\
\le{}& L_1\int_{t_k}^{t}\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\mathrm{d}s+\lambda_4L_1\int_{t_p}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s+\lambda_5L_1\int_{t_p}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\mathrm{d}s\\
&+\sum_{i=1}^{p}\left[L_1\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\mathrm{d}s+L_3\right]+\lambda_4\sum_{i=1}^{p}\left[L_1\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s+L_2\right]\\
&+\lambda_4\sum_{i=1}^{p-1}\left[L_1\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\mathrm{d}s+L_3\right]+\sum_{i=1}^{p}(\lambda_5+T\lambda_4)\left[L_1\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\mathrm{d}s+L_3\right]\\
\le{}&\frac{(1+p)\lambda_4T^{\alpha}L_1}{\Gamma(\alpha+1)}+[(1+p)(1+\lambda_5)+(1+pT)\lambda_4]\frac{T^{\alpha-1}L_1}{\Gamma(\alpha)}\\
&+p\lambda_4L_2+[p+(1+pT)\lambda_4+(1+p)\lambda_5]L_3:=\bar{L}.
\end{aligned}
$$

Hence, for $t_1, t_2 \in J_k,\ t_1 < t_2,\ 0 \le k \le p$, we have

$$|(Tu)(t_2) - (Tu)(t_1)| \le \int_{t_1}^{t_2} |(Tu)'(s)|ds \le \bar{L}(t_2 - t_1),$$

which implies that T is equicontinuous on all $J_k,\ k = 0, 1, 2, \ldots, p$. Thus, by the Arzela–Ascoli Theorem, the operator $T : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is completely continuous. □

Theorem 3.1. *Let* $\lim_{u\to 0} \frac{f(t,u)}{u} = 0$, $\lim_{u\to 0} \frac{I_k(u)}{u} = 0$ *and* $\lim_{u\to 0} \frac{I_k^*(u)}{u} = 0$, *then the problem* (1.1) *has at least one solution.*

Proof. Since $\lim_{u\to 0} \frac{f(t,u)}{u} = 0$, $\lim_{u\to 0} \frac{I_k(u)}{u} = 0$ and $\lim_{u\to 0} \frac{I_k^*(u)}{u} = 0$, therefore there exists a constant $r > 0$ such that $|f(t,u)| \le \delta_1|u|$, $|I_k(u)| \le \delta_2|u|$ and $|I_k^*(u)| \le \delta_3|u|$ for $0 < |u| < r$, where $\delta_i > 0$ $(i = 1, 2, 3)$ satisfy the inequality

$$\sup_{t\in J}\left\{\frac{(1+p)(1+|\lambda_1(t)|)T^{\alpha}\delta_1}{\Gamma(\alpha+1)} + \frac{[(2p-1)(1+T|\lambda_1(t)|) + p|\lambda_3(t)| + |\lambda_2(t)|]T^{\alpha-1}\delta_1}{\Gamma(\alpha)} + (1+|\lambda_1(t)|)p\delta_2 + [(2p-1)(1+T|\lambda_1(t)|) + p|\lambda_3(t)|]\delta_3\right\} \le 1. \tag{3.3}$$

Let us set $\Omega = \{u \in PC(J, \mathbb{R}) \mid \|u\| < r\}$ and take $u \in PC(J, \mathbb{R})$ such that $\|u\| = r$, that is, $u \in \partial\Omega$. Then, by the process used to obtain (3.2), we have

$$|Tu(t)| \le \sup_{t\in J}\left\{\frac{(1+p)(1+|\lambda_1(t)|)T^{\alpha}\delta_1}{\Gamma(\alpha+1)} + \frac{[(2p-1)(1+T|\lambda_1(t)|) + p|\lambda_3(t)| + |\lambda_2(t)|]T^{\alpha-1}\delta_1}{\Gamma(\alpha)} + (1+|\lambda_1(t)|)p\delta_2 + [(2p-1)(1+T|\lambda_1(t)|) + p|\lambda_3(t)|]\delta_3\right\}\|u\|. \tag{3.4}$$

Thus, it follows that $\|Tu\| \le \|u\|$, $u \in \partial\Omega$. Therefore, by Theorem 2.1, the operator T has at least one fixed point, which in turn implies that the problem (1.1) has at least one solution $u \in \overline{\Omega}$. □

Theorem 3.2. *Assume that there exist positive constants* L_i $(i = 1, 2, 3)$ *such that*

$$|f(t,u)| \le L_1, \qquad |I_k(u)| \le L_2, \qquad |I_k^*(u)| \le L_3, \quad \text{for } t \in J,\ u \in \mathbb{R} \text{ and } k = 1, 2, \ldots, p. \tag{3.5}$$

Then the problem (1.1) *has at least one solution.*

Proof. Let us show that the set $V = \{u \in PC(J, \mathbb{R}) \mid u = \mu Tu,\ 0 < \mu < 1\}$ is bounded. Let $u \in V$, then $u = \mu Tu,\ 0 < \mu < 1$. For any $t \in J$, we have

$$\begin{aligned} u(t) = {} & \int_{t_k}^{t} \frac{\mu(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds + \lambda_1(t)\int_{t_p}^{T} \frac{\mu(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds \\ & - \lambda_2(t)\int_{t_p}^{T} \frac{\mu(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds + \sum_{i=1}^{k}\mu\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds + I_i(u(t_i))\right] \\ & + \sum_{i=1}^{k-1}\mu(t_k-t_i)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds + I_i^*(u(t_i))\right] \\ & + \sum_{i=1}^{k}\mu(t-t_k)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds + I_i^*(u(t_i))\right] \\ & + \lambda_1(t)\sum_{i=1}^{p}\mu\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds + I_i(u(t_i))\right] \\ & + \lambda_1(t)\sum_{i=1}^{p-1}\mu(t_p-t_i)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds + I_i^*(u(t_i))\right] \\ & - \sum_{i=1}^{p}\mu\left[\lambda_3(t) + \lambda_1(t)t_p\right]\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds + I_i^*(u(t_i))\right]. \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6) and employing the procedure used to obtain (3.2), we obtain

$$\begin{aligned}
|u(t)| &= \mu|Tu(t)| \le \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))|\mathrm{d}s + |\lambda_1(t)| \int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))|\mathrm{d}s \\
&\quad + |\lambda_2(t)| \int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|\mathrm{d}s + \sum_{i=1}^{k}\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))|\mathrm{d}s + |I_i(u(t_i))|\right] \\
&\quad + \sum_{i=1}^{k-1}(t_k-t_i)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|\mathrm{d}s + |I_i^*(u(t_i))|\right] \\
&\quad + \sum_{i=1}^{k}(t-t_k)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|\mathrm{d}s + |I_i^*(u(t_i))|\right] \\
&\quad + |\lambda_1(t)| \sum_{i=1}^{p}\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))|\mathrm{d}s + |I_i(u(t_i))|\right] \\
&\quad + |\lambda_1(t)| \sum_{i=1}^{p-1}(t_p-t_i)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|\mathrm{d}s + |I_i^*(u(t_i))|\right] \\
&\quad + \sum_{i=1}^{p}\left[|\lambda_3(t)| + |\lambda_1(t)|t_p\right]\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|\mathrm{d}s + |I_i^*(u(t_i))|\right] \\
&\le \max_{t\in J}\left\{\frac{(1+p)(1+|\lambda_1(t)|)T^{\alpha}L_1}{\Gamma(\alpha+1)} + \frac{[(2p-1)(1+T|\lambda_1(t)|)+p|\lambda_3(t)|+|\lambda_2(t)|]T^{\alpha-1}L_1}{\Gamma(\alpha)}\right. \\
&\quad \left. +\ (1+|\lambda_1(t)|)pL_2 + [(2p-1)(1+T|\lambda_1(t)|)+p|\lambda_3(t)|]L_3\right\} := M,
\end{aligned}$$

which implies that $\|u\| \le M$ for any $t \in J$. So, the set V is bounded. Thus, by Theorem 2.2, the operator T has at least one fixed point. Hence the problem (1.1) has at least one solution. □

Theorem 3.3. *Assume that there exist positive constants K_i $(i = 1, 2, 3)$ such that*

$$|f(t,u)-f(t,v)| \le K_1|u-v|, \qquad |I_k(u)-I_k(v)| \le K_2|u-v|, \qquad |I_k^*(u)-I_k^*(v)| \le K_3|u-v|,$$

for $t \in J$, $u, v \in \mathbb{R}$ and $k = 1, 2, \ldots, p$.

Then the problem (1.1) has a unique solution if

$$\begin{aligned}
\mathcal{H} &= \max_{t\in J}\left\{\frac{(1+p)(1+|\lambda_1(t)|)T^{\alpha}K_1}{\Gamma(\alpha+1)} + \frac{[(2p-1)(1+T|\lambda_1(t)|)+p|\lambda_3(t)|+|\lambda_2(t)|]T^{\alpha-1}K_1}{\Gamma(\alpha)}\right. \\
&\quad \left. +\ (1+|\lambda_1(t)|)pK_2 + [(2p-1)(1+T|\lambda_1(t)|)+p|\lambda_3(t)|]K_3\right\} < 1. \qquad (3.7)
\end{aligned}$$

Proof. For $u, v \in PC(J,\mathbb{R})$, we have

$$\begin{aligned}
&|(Tu)(t)-(Tv)(t)| \\
&\le \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))-f(s,v(s))|\mathrm{d}s + |\lambda_1(t)| \int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))-f(s,v(s))|\mathrm{d}s \\
&\quad + |\lambda_2(t)| \int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))-f(s,v(s))|\mathrm{d}s \\
&\quad + \sum_{i=1}^{k}\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))-f(s,v(s))|\mathrm{d}s + |I_i(u(t_i))-I_i(v(t_i))|\right] \\
&\quad + \sum_{i=1}^{k-1}(t_k-t_i)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))-f(s,v(s))|\mathrm{d}s + |I_i^*(u(t_i))-I_i^*(v(t_i))|\right] \\
&\quad + \sum_{i=1}^{k}(t-t_k)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))-f(s,v(s))|\mathrm{d}s + |I_i^*(u(t_i))-I_i^*(v(t_i))|\right]
\end{aligned}$$

$$
\begin{aligned}
&+|\lambda_1(t)|\sum_{i=1}^{p}\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))-f(s,v(s))|\mathrm{d}s+|I_i(u(t_i))-I_i(v(t_i))|\right]\\
&+|\lambda_1(t)|\sum_{i=1}^{p-1}(t_p-t_i)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))-f(s,v(s))|\mathrm{d}s+|I_i^*(u(t_i))-I_i^*(v(t_i))|\right]\\
&+\sum_{i=1}^{p}\left[|\lambda_3(t)|+|\lambda_1(t)|t_p\right]\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))-f(s,v(s))|\mathrm{d}s+|I_i^*(u(t_i))-I_i^*(v(t_i))|\right]\\
\le&\left\{\frac{(1+p)(1+|\lambda_1(t)|)T^{\alpha}K_1}{\Gamma(\alpha+1)}+\frac{[(2p-1)(1+T|\lambda_1(t)|)+p|\lambda_3(t)|+|\lambda_2(t)|]T^{\alpha-1}K_1}{\Gamma(\alpha)}\right.\\
&\left.+\ (1+|\lambda_1(t)|)pK_2+[(2p-1)(1+T|\lambda_1(t)|)+p|\lambda_3(t)|]K_3\right\}\|u-v\|.
\end{aligned}
$$

Thus, we obtain $\|Tu-Tv\|\le\mathcal{H}\|u-v\|$, where $\mathcal{H}$ is given by (3.7). As $\mathcal{H}<1$, therefore, T is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle. This completes the proof. □

4. Examples

Example 4.1. For $1<\alpha\le 2$, consider the following fractional order impulsive mixed boundary value problem

$$
\begin{cases}
{}^{C}D^{\alpha}u(t)=2\ln(1+u^2(t))-\cos u(t)+1, & 0<t<T,\ t\neq t_1,\ 0<t_1<T,\\
\triangle u(t_1)=e^{u^3(t_1)}-1, \qquad \triangle u'(t_1)=(1+u^2(t_1))^{\frac{1}{3}}-1, &\\
Tu'(0)=-2u(0)-3u(T), \qquad Tu'(T)=4u(0)+5u(T). &
\end{cases}
\tag{4.1}
$$

Here $p=1,\ a=2,\ b=3,\ c=4, d=5,\ f(t,u)=2\ln(1+u^2)-\cos u+1,\ I_1(u(t_1))=e^{u^3}-1,\ I_1^*(u(t_1))=(1+u^2)^{\frac{1}{3}}-1$. Clearly

$$
\begin{aligned}
&\lim_{u\to 0}\frac{f(t,u)}{u}=\lim_{u\to 0}\frac{2\ln(1+u^2)-\cos u+1}{u}=\lim_{u\to 0}\frac{2\ln(1+u^2)}{u}+\lim_{u\to 0}\frac{1-\cos u}{u}=0,\\
&\lim_{u\to 0}\frac{I_1(u)}{u}=\lim_{u\to 0}\frac{e^{u^3}-1}{u}=\lim_{u\to 0}u^2=0,\\
&\lim_{u\to 0}\frac{I_1^*(u)}{u}=\lim_{u\to 0}\frac{(1+u^2)^{\frac{1}{3}}-1}{u}=\lim_{u\to 0}\frac{u}{3}=0.
\end{aligned}
$$

Furthermore, in this case, δ_i $(i=1,2,3)$ given by (3.3) satisfy the inequality:

$$
\frac{T^{\alpha-1}\delta_1}{\Gamma(\alpha+1)}\left(3T+\frac{\alpha}{16}(16+21T)\right)+\frac{3}{2}\delta_2+\left(1+\frac{3}{4}T\right)\delta_3\le 1.
$$

Thus all the assumptions of Theorem 3.1 hold. Hence, the conclusion of Theorem 3.1 applies and the impulsive fractional mixed boundary value problem (4.1) has at least one solution.

Example 4.2. Consider the impulsive fractional mixed boundary value problem given by

$$
\begin{cases}
{}^{C}D^{\alpha}u(t)=\dfrac{(1+\cos t^2)e^{-u^2(t)}\sin 2t}{3+u^2(t)}, \quad 0<t<1,\ t\neq\dfrac{1}{5},\\
\triangle u\left(\dfrac{1}{5}\right)=4\sin(6+e^{2u\left(\frac{1}{5}\right)}), \qquad \triangle u'\left(\dfrac{1}{5}\right)=\dfrac{5+4e^{-u^4\left(\frac{1}{5}\right)}}{3+\sin^2 u\left(\frac{1}{5}\right)},\\
u'(0)=-\dfrac{1}{3}u(0)-\dfrac{2}{3}u(1), \qquad u'(1)=\dfrac{1}{5}u(0)+\dfrac{4}{5}u(1).
\end{cases}
\tag{4.2}
$$

Here $1<\alpha\le 2,\ p=1,\ T=1,\ a=1/3,\ b=2/3,\ c=1/5, d=4/5$, and $|f(t,u)|=|\frac{(1+\cos t^2)e^{-u^2}\sin 2t}{3+u^2}|\le\frac{2}{3},\ |I_1(u)|=|4\sin(6+e^{2u})|\le 4$ and $|I_1^*(u)|=|\frac{5+4e^{-u^4}}{3+\sin^2 u}|\le 3$.

In this case, $L_1=\frac{2}{3},\ L_2=4,\ L_3=3$ and

$$M = \max_{t\in J}\left\{\frac{(1+p)(1+|\lambda_1(t)|)T^\alpha L_1}{\Gamma(\alpha+1)} + \frac{[(2p-1)(1+T|\lambda_1(t)|)+p|\lambda_3(t)|+|\lambda_2(t)|]T^{\alpha-1}L_1}{\Gamma(\alpha)}\right.$$
$$\left. +\ (1+|\lambda_1(t)|)pL_2 + [(2p-1)(1+T|\lambda_1(t)|)+p|\lambda_3(t)|]L_3\right\} = \frac{2(26+27\alpha)}{21\Gamma(\alpha+1)} + \frac{191}{14}.$$

Thus the hypothesis of Theorem 3.2 is satisfied. Therefore, by Theorem 3.2, the impulsive fractional mixed boundary value problem (4.2) has at least one solution.

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