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Gurland's Ratio for the Gamma Function

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Abstract—We consider the ratio $T(x,y) = \Gamma(x)\Gamma(y)/\Gamma^2((x+y)/2)$ and its properties related to convexity, logarithmic convexity, Schur-convexity, and complete monotonicity. Several new bounds and asymptotic expansions for T are derived. Sharp bounds for the function $x \mapsto x/(1-e^{-x})$ are presented, as well as bounds for the trigamma function. The results are applied to a problem related to the volume of the unit ball in $\mathbb{R}^n$ and also to the problem of finding the inverse of the function $x \mapsto T(1/x, 3/x)$, which is of importance in applied statistics. © 2005 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

The first result about the ratio of Gamma functions

$$T(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma^2((x+y)/2)}, \qquad x, y > 0, \tag{1}$$

appeared in 1956, in John Gurland's paper [1], where the following inequality was presented:

$$\frac{\Gamma(x)\Gamma(x+2\beta)}{\Gamma^2(x+\beta)} \geq 1 + \frac{\beta^2}{x}, \qquad x > 0, \quad x + 2\beta > 0. \tag{2}$$

Since then, there has appeared a considerable number of papers about Gurland's ratio and its properties. Recent papers [2,3] present some results regarding a more general ratio of even derivatives, $\Gamma^{(2n)}(x)\Gamma^{(2n)}(y)/\Gamma^{(2n)}((x+y)/2)$. Ratio (1) is related to also well-investigated Gautschi's ratio [4]

$$Q(x,\beta) = \frac{\Gamma(x+\beta)}{\Gamma(x)}, \qquad x > 0,$$

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where usually $\beta \in [0,1]$, see a survey in [5] or the bibliography in [6]. In fact, there is the following connection between the two ratios:

$$T(x, x+2\beta) = \frac{Q(x+\beta,\beta)}{Q(x,\beta)}. \tag{3}$$

Gurland's ratio appears in several places in the theory of Gamma function and related topics. For instance,

$$T(n, n+1) = \frac{1}{n\pi}\left(\frac{(2n)!!}{(2n-1)!!}\right) = \frac{1}{W_n},$$

where W_n is a form of Wallis product (see [7–9]). In a recent work [10], the Gurland's ratio $T(1+(k-1)/2, 1+(k+1)/2)$ appears in connection with volume of unit ball in $\mathbb{R}^k$. In probability theory and its applications, the Gurland's ratio $T(1/\gamma, 3/\gamma)$ appears in the form of ratio of the variance and squared absolute expectation of a generalized Gamma random variable with the shape parameter γ, cf. [11]; this ratio is also known as generalized Gaussian ratio [12], and has interesting applications in the domain of image recognition [12,13].

The original reason for interest in this ratio was its connection with the Cramér-Rao inequality in statistics (see, for example, [14, 32.3]), where it appears in a connection with the Gamma distribution. There is a number of articles [15–23] with the idea to use different versions of Cramér-Rao inequality or its generalizations [19,24–26] and to get improvements of (2). A survey of this early work can be found in [27].

However, Watson [28] for the case $\beta = 1/2$ and Boyd [29] for the general case noticed that (2) is a simple consequence of Gauss' formula for hypergeometric function (see [30, 14.11])

$$\frac{\Gamma(x)\Gamma(x+2\beta)}{\Gamma^2(x+\beta)} = F(-\beta,-\beta,x,1) = 1 + \sum_{k=1}^{\infty} \frac{((-\beta)_k)^2}{k!(x)_k}, \tag{4}$$

where $(z)_k = z(z+1)\cdots(z+k-1)$ and F is the hypergeometric function. The series is convergent whenever $x + 2\beta > 0$. If, in addition, $x > 0$, then all terms are nonnegative and, by retaining a finite number of terms in the series, we get (2) and its improvements.

The two apparently different techniques were related by Ruben [31], who explained the Cramér-Rao inequality and Bhattacharyya's [24] generalization from a viewpoint of approximations in a Hilbert space, and also showed that (4) can be derived from the Parseval identity for a suitably chosen orthonormal system. There is also a number of results derived from convexity, which will be discussed in Section 3.

Euler's formula for the Gamma function [32, 6.1.2] yields the following infinite product formula for T:

$$T(x,y) = \lim_{n\to+\infty} \frac{((x+y)/2)_n^2}{(x)_n(y)_n}. \tag{5}$$

We will show that this formula and its variations and improvements can be derived from inequalities, obtaining thus asymptotically infinitely sharp sequence of inequalities.

Another lower bound for T was found in [8], by means of majorization in expansion (5)

$$T(x,y) > 1 + \frac{(y-x)^2}{4} \sum_{k=0}^{+\infty} \frac{1}{(x+k)(y+k)}, \qquad x,y>0, \quad x \neq y. \tag{6}$$

In this paper, we investigate several properties of Gurland's ratio related to convexity (Section 2), we present some new results and sharp bounds for T via convexity of functions related to the Gamma function (Section 3), via convexity of functions related to the ratio Q (Section 4), and via bounds for the trigamma function (Section 5). In Section 6, we present a technique of turning inequalities into asymptotic expansions. In Section 7, we apply some of the results in preceeding sections to a problem related to the volume of unit ball in higher dimensions. Section 8 deals with the function $x \mapsto T(1/x, 3/x)$ and its inverse, which is of importance in applied statistics.

Throughout the paper, I will denote the interval $(0, +\infty)$.

2. MONOTONICITY, CONVEXITY, AND SCHUR-CONVEXITY

In the next lemma, we give several properties of the ratio T. For a convenience, we state here the well-known expansion for polygamma functions (see, for example, [32, 6.4.10]).

$$\Psi^{(n)}(x) = \frac{\mathrm{d}^{n+1}\log\Gamma(x)}{\mathrm{d}x^{n+1}} = (-1)^{n+1} n! \sum_{k=0}^{+\infty} \frac{1}{(x+k)^{n+1}}, \qquad n = 1, 2, \ldots. \tag{7}$$

LEMMA 1.

(i) *For any $\beta \in I$, the functions*

$$x \mapsto T(x, x+2\beta) \qquad \text{and} \qquad x \mapsto \log T(x, x+2\beta)$$

are completely monotonic on I.

(ii) *For any $\beta \in I$ the function $x \mapsto T(x, x+2\beta)$ is decreasing in $x \in I$ from $+\infty$ to 1.*

(iii) *For any $x \in I$, the function $\beta \mapsto T(x, x+2\beta)$ is increasing in $\beta \in I$ from 1 to $+\infty$.*

(iv) *The function $(x, y) \mapsto F(x, y)$ is Schur-convex on $I \times I$, that is, for any $x, y \in I$, such that $x < y$ and $0 < \varepsilon < (y-x)/2$,*

$$T(x+\varepsilon, y-\varepsilon) < T(x, y).$$

(v) *For any $\beta \in I$, the functions*

$$x \mapsto T(x, x+2\beta), \qquad x \mapsto \log T(x, x+2\beta), \qquad x \mapsto \log\log T(x, x+2\beta)$$

are convex on I.

PROOF. We will prove a more general statement than (i), about the function

$$F(x) = \frac{\Gamma(x)\Gamma(x+a+b)}{\Gamma(x+a)\Gamma(x+b)}, \qquad a, b > 0, \quad x \in I. \tag{8}$$

For $n \geq 0$, we have that

$$\begin{aligned}(\log F(x))^{(n+1)} &= \Psi^{(n)}(x) + \Psi^{(n)}(x+a+b) - \Psi^{(n)}(x+a) - \Psi^{(n)}(x+b) \\ &= a\left(\frac{\Psi^{(n)}(x+a+b) - \Psi^{(n)}(x+b)}{a} - \frac{\Psi^{(n)}(x+a) - \Psi^{(n)}(x)}{a}\right).\end{aligned} \tag{9}$$

By (7), $y \mapsto \Psi^{(n)}(y)$ is strictly convex in $y \in I$ for odd n, and hence, the ratio

$$\frac{\Psi^{(n)}(y+a) - \Psi^{(n)}(y)}{a} \tag{10}$$

is increasing with y (see, for example, [33, 16B.3.a]). For even n, the function $\Psi^{(n)}$ is concave, and the ratio (10) is decreasing. Thus, by (9) we conclude that the sign of $(\log F(x))^{(n+1)}$ is $(-1)^{n+1}$, for $n \geq 0$. The inequality $\log F(x) > 0$ is a consequence of convexity of $\log\Gamma(x)$. Hence, we proved that $\log F$ is a completely monotone function. From [34, p. 83] it follows that $f(g(x))$ is completely monotonic on I if f is completely monotonic on $\mathbb{R}$ and g' is completely monotonic on I. If we let here $f(x) = e^{-x}$ and $g(x) = -\log F(x)$, we see that complete monotonicity of $\log F$ implies the complete monotonicity of F. Now by letting $a = b = \beta \in I$ in (8), we get (i), and, automatically, the monotonicity part of (ii). Statements in (v) are consequences of (i). Indeed, by a result in [35], each completely monotone function is log-convex, so the functions $x \mapsto \log\log T(x, x+2\beta)$ and $x \mapsto \log T(x, x+2\beta)$ are convex. Since each log-convex function is also convex (see, for example, [27]), it follows that the function $x \mapsto T(x, x+2\beta)$ is convex.

To prove (iii), fix $x \in I$ and denote $G(\beta) = \log T(x, x+2\beta)$. Then $G'(\beta) = 2\Psi(x+2\beta) - 2\Psi(x+\beta) > 0$ because Ψ is increasing. Now, knowing that [32, 6.1.46], $\Gamma(x+a)/\Gamma(x+b) \sim x^{a-b}$ as $x \to +\infty$, it is easy to see that

$$\lim_{x\to+\infty} T(x, x+2\beta) = 1 \qquad \text{and} \qquad \lim_{\beta\to+\infty} T(x, x+2\beta) = +\infty.$$

Also, for any $\beta > 0$, we have

$$\lim_{x\to 0_+} T(x, x+2\beta) = \lim_{x\to 0_+} \frac{\Gamma(x)\Gamma(x+2\beta}{\Gamma^2(x+\beta)} = +\infty.$$

Schur convexity in (iv) can be shown directly. Under given assumptions, the inequality $T(x+\varepsilon, y-\varepsilon) < T(x,y)$ is equivalent to

$$\Gamma(x+\varepsilon)\Gamma(y-\varepsilon) < \Gamma(x)\Gamma(y),$$

which is the expression of Schur-convexity of the function $(x,y) \mapsto \Gamma(x)\Gamma(y)$. This function is Schur-convex because $\Gamma(x)$ is log-convex (see [33]). ∎

REMARK. There is a number of old and recent results about complete monotonicity of functions related to the Gamma function (see [6,36–39]). The statement in (i), together with its proof, gives an improvement of [37, Theorem 6], where it was shown that function (8) is completely monotonic.

The statement in (ii) is essentially obtained by Steinig [40], who used convexity of the Beta function to show that the function $x \mapsto T(x, x+1)$ is decreasing in $x \in I$.

3. BOUNDS FOR GURLAND'S RATIO VIA CONVEXITY

Bounds for T can be found from convexity or concavity of functions of the form $F(x) = \log\Gamma(x) + \log C(x)$ on I. Since $x \mapsto \log\Gamma(x)$ is already convex, $\log C(x)$ should be concave. A straightforward application of Jensen's inequality

$$F\left(\frac{x+y}{2}\right) \le \frac{F(x)+F(y)}{2}$$

yields

$$T(x,y) \ge \frac{C^2((x+y)/2)}{C(x)C(y)}, \qquad \text{for } x, y \in I, \tag{11}$$

if F is convex, and the opposite inequality if it is concave. If the convexity or concavity is strict, then the equality in (11) holds if and only if $x = y$. This observation yielded another set of results in connection with Gurland's ratio [41–44]. For example, the inequality

$$T(x,y) \ge \frac{x^x y^y}{((x+y)/2)^{x+y}} \tag{12}$$

was obtained in [43], using convexity of the function $x \mapsto \log\Gamma(x) - x\log x$ on $(0, +\infty)$; the same result was earlier obtained in [41] by other means.

The following upper bound was also obtained in [43]:

$$T(x,y) \le \frac{(x-1)^{x-1}(y-1)^{y-1}}{((x+y-2)/2)^{x+y-2}}, \qquad x > 1, \quad y > 1, \tag{13}$$

using the fact that for $x > 1$ the function $x \mapsto \log\Gamma(x) - (x-1)\log(x-1)$ is concave.

In [5], we found a class of log-convex or log-concave functions related to the Gamma function, and we applied these functions to finding bounds for Gautschi's ratio. They can be also applied to Gurland's ratio, as in the next theorem.

THEOREM 1. *For any* $x, y \in I$,

$$\frac{x^{x-1/2}y^{y-1/2}}{((x+y)/2)^{x+y-1}} \le T(x,y) \le \frac{x^{x-1/2}y^{y-1/2}}{((x+y)/2)^{x+y-1}} \exp\frac{(y-x)^2}{12xy(x+y)}, \tag{14}$$

with equality if and only if $x = y$.

PROOF. Let

$$F(x) = \log\Gamma(x) - \left(x - \frac{1}{2}\right)\log x, \qquad G(x) = \log\Gamma(x) - \left(x - \frac{1}{2}\right)\log x - \frac{1}{12x}.$$

From a result in [5] it follows that F is convex and G is concave on I, and then inequalities (14) follow from (11). ∎

An application of Theorem 1 will be given in Section 8. In Section 6 we will compare (14) with (12) and (13).

4. INEQUALITIES VIA CONVEXITY OF FUNCTIONS RELATED TO Q

From (7) for $n = 1$ it readily follows that the Gautschi ratio $Q(x, \beta)$ is log-concave. In general, suppose that the function

$$F(x) = \log Q(x,\beta) + \log D(x,\beta) \tag{15}$$

is convex with respect to $x \in I$, for a fixed $\beta \in (0,1)$. Then from Jensen's inequality

$$F(x) \le \beta F(x-1+\beta) + (1-\beta)F(x+\beta), \qquad x > 1-\beta, \tag{16}$$

we find, writing for simplicity $Q(x,\beta) = Q(x)$ and $D(x,\beta) = D(x)$,

$$Q(x)D(x) \le Q^{\beta}(x-1+\beta)Q^{1-\beta}(x+\beta)D^{\beta}(x-1+\beta)D^{1-\beta}(x+\beta). \tag{17}$$

Now note that

$$Q(x-1+\beta) = \frac{\Gamma(x-1+2\beta)}{\Gamma(x-1+\beta)} = \frac{x-1+\beta}{x-1+2\beta}\cdot\frac{\Gamma(x+2\beta)}{\Gamma(x+\beta)} = \frac{x-1+\beta}{x-1+2\beta}\cdot Q(x+\beta),$$

which finally yields, via (17) and relation (3),

$$T(x, x+2\beta) \ge \left(\frac{x-1+2\beta}{x-1+\beta}\right)^{\beta}\cdot\frac{D(x,\beta)}{D^{\beta}(x-1+\beta,\beta)D^{1-\beta}(x+\beta,\beta)}, \tag{18}$$

where $x > 1-\beta$. An upper bound may be found with the same function (15), but starting with Jensen's inequality

$$F(x+\beta) \le (1-\beta)F(x) + \beta F(x+1), \qquad x > 0, \tag{19}$$

instead of (16). In a similar way as above, we find

$$T(x, x+2\beta) \le \left(\frac{x+\beta}{x}\right)^{\beta}\cdot\frac{D^{\beta}(x+1,\beta)D^{1-\beta}(x,\beta)}{D(x+\beta,\beta)}, \qquad x \in I. \tag{20}$$

Obviously, we can apply an analogous procedure for a concave function F, which would yield the set of inequalities (18)–(20) with $\le$ and $\ge$ interchanged. In both cases, it makes sense to consider only a log-convex function D. The next theorem gives a convenient choice.

THEOREM 2. *For each $\beta > 0$ and $c \leq \min(\beta, 1)$, the function*

$$x \mapsto F(x) = \log Q(x,\beta) - c \log x = \log \Gamma(x+\beta) - \Gamma(x) - c \log x$$

is concave on $(0, +\infty)$.

If $c \geq \max(\beta, 1)$, the function F is convex on $(0, +\infty)$.

PROOF. Since $\log Q(x,\beta)$ is already concave, for the first part it suffices to assume that $0 < c \leq \beta$ and $c \leq 1$. First, we note that for $x > 0$,

$$\begin{aligned} F(x) &= \log \Gamma(x+\beta) - \log\left(\frac{x\Gamma(x)}{x}\right) - c \log x \\ &= \log \Gamma(x+\beta) - \log \Gamma(x+1) + (1-c) \log x. \end{aligned}$$

Further, by (7), we have that

$$\begin{aligned} F''(x) &= \sum_{k=0}^{+\infty} \frac{1}{(x+k+\beta)^2} - \frac{1}{(x+k+1)^2} - \frac{1-c}{x^2} \\ &= \sum_{k=0}^{+\infty} \frac{1}{(x+k+\beta)^2} - \frac{c}{(x+k+1)^2} - \frac{1-c}{(x+k)^2}, \end{aligned} \tag{21}$$

where we used the telescopic series

$$\sum_{k=0}^{+\infty} \frac{1}{(x+k)^2} - \frac{1}{(x+k+1)^2} = \frac{1}{x^2}.$$

Now, convexity and monotonicity of the function $x \mapsto x^{-2}$ yield, for $0 < c \leq \beta$ and $0 < c \leq 1$,

$$\frac{1}{(x+k+\beta)^2} \leq \frac{1}{(x+k+c)^2} \leq \frac{c}{(x+k+1)^2} + \frac{1-c}{(x+k)^2},$$

and therefore, all terms in (21) are nonpositive and so F is concave (strictly concave unless $\beta = c = 1$).

If $c > 1$, Jensen's inequality can be applied in the form

$$\frac{1}{(x+k+1)^2} \leq \frac{1}{c} \cdot \frac{1}{(x+k+c)^2} + \frac{c-1}{c} \cdot \frac{1}{(x+k)^2},$$

and, if also $c > \beta$, then all terms in (21) are positive and F is strictly convex. ∎

Now applying (18)–(20) with the convex function $x \mapsto \log Q(x,\beta) - \log x$ or with the concave function $x \mapsto \log Q(x,\beta) - \beta \log x$, we get the following two double bounds.

COROLLARY 1. *For any $\beta \in [0,1]$,*

$$\frac{(x-1+2\beta)^{\beta}(x+\beta)^{1-\beta}}{x} \leq T(x, x+2\beta) \leq \frac{(x+\beta)^{1+\beta}}{x(x+1)^{\beta}}, \tag{22}$$

$$\frac{(x+\beta)^{2\beta}}{x^{\beta(2-\beta)}(x+1)^{\beta^2}} \leq T(x, x+2\beta) \leq \frac{(x-1+2\beta)^{\beta}(x+\beta)^{\beta(1-\beta)}}{x^{\beta}(x-1+\beta)^{\beta(1-\beta)}}, \tag{23}$$

with equality on both sides if and only if $\beta = 0$ or $\beta = 1$. The left inequality in (22) and the right inequality in (23) hold for $x > 1-\beta$, and the other two inequalities hold for all $x \in I$. ∎

Several results about complete monotonicity of functions of the form $Q(x,\beta) \cdot U(x,\beta)$ are obtained in [36–38]. Since, as we already remarked, each completely monotone function is log-convex, the above procedure can be applied with those functions, in order to obtain bounds for T. For instance, from a result in [37] it follows that function (15), with $D(x,\beta) = \exp(-\beta\Psi(x+\beta/2))$, $\beta \in (0,1)$, is concave on I (see also [45]).

5. BOUNDS VIA TRIGAMMA FUNCTION

The trigamma function, the second logarithmic derivative of the Gamma function, can be expressed as the integral (cf. [32, 6.4.1])

$$\Psi'(x) = \int_0^{+\infty} \frac{t}{1-e^{-t}} e^{-xt}\, dt, \qquad x \in I. \tag{24}$$

If we find sharp bounds for the integrand in (24), we can find bounds for Ψ' and then, proceeding as in Section 3, we can get bounds for T. We illustrate the method with the following result.

LEMMA 2. *For any $t > 0$ we have that*

$$t + e^{-t}\left(1 + \frac{t}{2}\right) < \frac{t}{1-e^{-t}} < t + e^{-t}(1+t). \tag{25}$$

PROOF. For $t > 0$, the left inequality is equivalent to $f(t) = 1 - e^{-t} - te^{-t} > 0$, which is true since $f(0) = 0$ and $f'(t) = te^{-t} > 0$. The right inequality is equivalent to $g(t) = t + 2e^{-t} + te^{-t} - 2 > 0$, which is true because $g(0) = 0$ and $g'(t) = f(t) > 0$. ∎

From (25) and the representation (24), we obtain the following.

COROLLARY 2. *For $x \in (0, +\infty)$, we have that*

$$\frac{1}{x+1} + \frac{1}{x^2} + \frac{1}{2(x+1)^2} < \Psi'(x) < \frac{1}{x+1} + \frac{1}{x^2} + \frac{1}{(x+1)^2}. \tag{26}$$ ∎

This result gives sharp bounds for the trigamma function and can be easily generalized to polygamma functions (see [46] for references regarding inequalities for polygamma functions). We need (26) as a tool for the next theorem.

THEOREM 3. *For $x \in I$, let*

$$U(x) = \frac{\Gamma(x+1)}{(x+1)^x}, \qquad V(x) = \frac{\Gamma(x+1)}{(x+1)^{x+1/2}}.$$

The function $x \mapsto \log U(x)$ is concave on I and the function $x \mapsto \log V(x)$ is convex on I.

PROOF. It is straightforward to see that

$$(\log U(x))'' = \Psi'(x) - \frac{1}{x^2} - \frac{1}{(x+1)^2} - \frac{1}{x+1},$$

which is negative on I by Corollary 2. In the same way we conclude that

$$(\log V(x))'' = \Psi'(x) - \frac{1}{x^2} - \frac{1}{2(x+1)^2} - \frac{1}{x+1} > 0.$$ ∎

From Theorem 3, in the same way as in Section 3, we obtain the following bounds for T.

COROLLARY 3. *For any $x, y \in I$, we have*

$$\frac{4(x+1)^x (y+1)^y}{xy(x+y)^2\left((x+y)/2+1\right)^{x+y}} \le T(x,y) \le \frac{4(x+1)^{x+1/2}(y+1)^{y+1/2}}{xy(x+y)^2\left((x+y)/2+1\right)^{x+y}}. \tag{27}$$

6. TRANSFORMATIONS, ASYMPTOTIC EXPANSIONS, AND COMPARISONS

If a_n and b_n are two sequences, let $a_n \lesssim b_n$ stand for

$$a_n \leq b_n, \quad \text{for all } n, \qquad \text{and} \qquad \lim_{n\to+\infty}(a_n - b_n) = 0 \text{ monotonically in } n.$$

The notation $a_n \gtrsim b_n$ is equivalent to $b_n \lesssim a_n$.

In this section we will use the expression that appears in (5),

$$\rho_n(x,y) = \frac{((x+y)/2)_n^2}{(x)_n(y)_n} = \frac{(x+y)^2(x+y+2)^2\cdots(x+y+2n-2)^2}{2^{2n}x(x+1)\cdots(x+n-1)y(y+1)\cdots(y+n-1)}.$$

In the following theorem we introduce a transformation that sharpens inequalities for T and turns them into asymptotic expansions.

THEOREM 4. *Let B be any upper or lower bound for T, such that*

$$T(x,y) \leq (\geq) B(x,y), \qquad x \geq x_0, \quad y \geq y_0. \tag{28}$$

Then for every $n = 1, 2, \dots$ it holds

$$T(x,y) \leq (\geq) B(x+n, y+n)\rho_n(x,y). \tag{29}$$

Further, if the relative error in (28) decreases as both x and y increase for n, that is,

$$\left|\frac{B(x+n,y+n)}{T(x+n,y+n)} - 1\right| < \left|\frac{B(x,y)}{T(x,y)} - 1\right|, \qquad n = 1, 2, \dots, \tag{30}$$

then the transformed inequality (29) is sharper than the original inequality (28). Moreover, if $n_1 < n_2$, then inequality (30) with $n = n_2$ is sharper than the same inequality with $n = n_1$. In addition, if

$$\lim_{n\to+\infty}\left|\frac{B(x+n,y+n)}{T(x+n,y+n)} - 1\right| = 0, \tag{31}$$

then we have the asymptotic expansion

$$T(x,y) \lesssim \left(\gtrsim\right) B(x+n,y+n)\rho_n(x,y). \tag{32}$$

PROOF. From the recurrence formula $\Gamma(z+1) = z\Gamma(z)$, it follows that

$$T(x+n,y+n) = \frac{T(x,y)}{\rho_n(x,y)}, \qquad n = 1, 2, \dots. \tag{33}$$

Replacing x and y in (28) with $x+n$ and $y+n$, respectively, and using (33), we obtain (29). Now if (30) holds, then

$$\begin{aligned} |B(x+n,y+n)\rho_n(x,y) - T(x,y)| &= T(x,y)\cdot\left|\frac{B(x+n,y+n)}{T(x+n,y+n)} - 1\right| \\ &< |B(x,y) - T(x,y)|, \end{aligned} \tag{34}$$

which shows that inequality (29) is sharper than the original inequality (28); the same method shows that if $n_1 < n_2$, then (29) with $n = n_2$ is sharper than with $n = n_1$. The statement about asymptotic expansion also follows from (34). ∎

The first application of Theorem 4 can be a derivation of a more informative version of (5). Indeed, by Lemma 1(ii), the relative error in the inequality $T(x+n, y+n) \geq 1$ is decreasing with n, and hence,

$$T(x, y) \gtrsim \rho_n(x, y).$$

We will show that all inequalities stated or derived in previous sections can be turned into asymptotic expansions of form (32). To start with inequalities of Section 1, let $G_N(x, x+2\beta)$ be the N^{th} partial sum of the series (4), i.e,

$$G_N(x, x+2\beta) = 1 + \sum_{k=1}^{N} \frac{((-\beta)_k)^2}{k!(x)_k}. \tag{35}$$

The corresponding inequality is

$$T(x, x+2\beta) \geq G_N(x, x+2\beta) \tag{36}$$

and, for a β being fixed, the relative error is given by

$$R_N(x) = 1 - \frac{G_N(x, x+2\beta)}{T(x, x+2\beta)}. \tag{37}$$

LEMMA 3. *For $\beta > 0$, let R_N be defined by (37). If $0 < x < y$, then for each $N = 1, 2, \dots$, we have that $R_N(x) > R_N(y)$, with $\lim_{x\to+\infty} R_N(x) = 0$.*

PROOF. By Lemma 1, $\lim_{x\to+\infty} T(x, x+2\beta) = 1$ and also it is immediate to see that $\lim_{x\to+\infty} \cdot G_N(x, x+2\beta) = 1$, which proves the statement about the limit of R_N.

The inequality $R_N(x) > R_N(y)$ can be written in the form

$$\frac{G_N(y)}{G_N(x)} > \frac{T(y, y+2\beta)}{T(x, x+2\beta)} = \frac{G_\infty(y)}{G_\infty(x)}, \tag{38}$$

where $G_\infty = \lim_{N\to+\infty} G_N$ is the sum of series (4) and where the second argument of G_N is omitted for simplicity. So, the lemma will be proved if we prove that for fixed $x < y$, the ratio on the left-hand side of (38) is decreasing with respect to N. Let $a_k(x)$, $k = 1, 2, \dots$, be the k^{th} term in (35) and let $a_0(x) = 1$. Then we want to prove that

$$\frac{G_N(y) + a_{N+1}(y)}{G_N(x) + a_{N+1}(x)} < \frac{G_N(y)}{G_N(x)},$$

which is easily shown to be equivalent with

$$\sum_{k=0}^{N} \frac{a_k(x)}{a_{N+1}(x)} < \sum_{k=0}^{N} \frac{a_k(y)}{a_{N+1}(y)}. \tag{39}$$

Now, since

$$\frac{a_k(x)}{a_{N+1}(x)} = C_k \cdot (x+k) \cdots (x+N), \qquad k = 0, 1, \dots,$$

where $C_k > 0$ do not depend on x, we see that (39) holds termwise, and this ends the proof. ∎

From Lemma 3, we conclude that, for instance, Gurland's inequality (2) can be sharpened to

$$T(x, x+2\beta) \gtrsim \left(1 + \frac{\beta^2}{x+n}\right) \rho_n(x, x+2\beta).$$

Regarding inequality (6), let $E_N(x,y)$ denote the N^{th} partial sum of series in (6), so we consider the inequality $T(x,y) > 1 + E_N(x,y)$, i.e.,

$$T(x,y) > 1 + \frac{(y-x)^2}{4} \sum_{k=0}^{N} \frac{1}{(x+k)(y+k)}, \qquad x,y>0, \quad x \neq y. \tag{40}$$

It is easy to see that the relative error

$$R_N(x,y) = 1 - \frac{1+E_N(x,y)}{T(x,y)}$$

has the property that $R_N(x+n, y+n) \to 0$ as $n \to +\infty$. The monotonicity can be shown directly. In fact, it suffices to show that

$$R_N(x+1,y+1) < R_N(x,y), \qquad x,y \in I,$$

which is equivalent to the obviously true inequality

$$\frac{(y-x)^2}{4} \sum_{k=1}^{N+1} \frac{1}{(x+k)(y+k)} > -\frac{xy}{(x+N+1)(y+N+1)}.$$

So, this leads to another class of infinite product formulae. For example, for $N=1$, we get

$$T(x,y) \gtrsim \left(1 + \frac{(y-x)^2}{4(x+n)(y+n)}\right) \rho_n(x,y).$$

Let us now discuss inequalities in Sections 3–5, that are derived from convexity.

LEMMA 4. *Let*

$$R(x,y,\lambda;f) = \lambda f(x) + (1-\lambda) f(y) - f(\lambda x + (1-\lambda) y),$$

where x, y $(x<y)$ are real numbers, f is a function defined on the interval $[x,y]$, and $\lambda \in [0,1]$.

(i) *Let f_1 and f_2 be twice continuously differentiable functions defined on an interval I, and suppose that $f_1''(x) \le f_2''(x)$ for all $x \in I$. Then for all $x,y \in I$ and $\lambda \in (0,1)$ we have that*

$$R(x,y,\lambda;f_1) \le R(x,y,\lambda;f_2)$$

with the strict inequality if $f_1''(t) < f_2''(t)$ for some t in the interval with endpoints x, y.

(ii) *Let $\{f_n\}$ be a sequence of twice continuously differentiable functions defined on an interval I, and suppose that $\lim_{n\to+\infty} f_n''(x) = 0$ for all $x \in I$. Then for all $x,y \in I$ and $\lambda \in (0,1)$ we have that*

$$\lim_{n\to+\infty} R(x,y,\lambda;f_n) = 0.$$

PROOF. Both statements are proved in [47] as parts of Corollaries 1 and 2, respectively. ∎

In Sections 3–5, we derived several inequalities of form (28), starting from a convex or concave function F, and applying Jensen's inequality. It is not difficult to check that in all instances, $F''(x)$ monotonically converges to zero as $x \to +\infty$; by means of Lemma 4, it can be seen that both conditions (30) and (31) of Theorem 4 are satisfied. Therefore, all inequalities presented in Sections 3–5 can be turned into asymptotic expansions.

For example, with the function $x \mapsto Q(x,\beta)$, using the method described in Section 3, we get the double inequality

$$\left(1+\frac{\beta}{x}\right)^{\beta} \le T(x, x+2\beta) \le \left(1 + \frac{\beta}{x-1+\beta}\right)^{\beta}, \qquad x > 1-\beta, \quad 0 \le \beta \le 1,$$

and the corresponding simple expansions

$$\rho_n(x, x+2\beta) \cdot \left(1 + \frac{\beta}{x+n}\right)^{\beta} \lesssim T(x, x+2\beta) \lesssim \rho_n(x, x+2\beta) \cdot \left(1 + \frac{\beta}{x+n-1+\beta}\right)^{\beta},$$

where $x > 1 - \beta$ and $0 \le \beta \le 1$.

Moreover, by Lemma 4(i), two inequalities that can be obtained from the same form of Jensen's inequality for functions F and G, can be easily compared if $F''(x)$ and $G''(x)$ are in the same relation of order for all $x \in I$.

In this way, we can easily see that, for example, the left inequality in (14) is sharper (in the sense that it has smaller relative and absolute error) than inequality (12) for all $x, y \in I$. The right inequality in (14) is sharper than (13), for all $x, y > 1$. Comparing (14) with (27), we can see that the left inequality in (27) is sharper than the corresponding inequality in (14) for all $x, y \in I$. Right inequality in (27) is sharper than the right inequality in (14) on any interval in I where

$$\Psi'(x) - \frac{1}{x^2} - \frac{1}{(x+1)^2} - \frac{1}{x+1} > \Psi'(x) - \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{6x^3}.$$

This condition reduces to $-3x^3 + x^2 - x + 1 > 0$ and $x > 0$, which is satisfied for $x \in (0, c)$, where $c \approx 0.635$. So, for $x, y \in (0, c)$, the right inequality in (27) is sharper than the right inequality in (14), and (14) is sharper if $x, y > c$. If $x < c$ and $y > c$, the comparison is not possible in general.

Finally, we will show that the inequalities (23) are sharper than inequalities (22). Indeed, it suffices to show that $|G''(x)| < F''(x)$ for $x \ge 2$ and $\beta \in (0,1)$, where

$$F(x) = \log Q(x, \beta) - \log x, \qquad G(x) = \log Q(x, \beta) - \beta \log x.$$

To this end, it suffices to show that

$$\sum_{k=0}^{+\infty} \frac{2}{(x+k+\beta)^2} - \frac{1+\beta}{(x+k+1)^2} - \frac{1-\beta}{(x+k)^2} > 0,$$

which will be done if we show that

$$h(\beta) := \frac{2}{(y+\beta)^2} - \frac{1+\beta}{(y+1)^2} - \frac{1-\beta}{y^2} > 0,$$

for $\beta \in [0,1]$ and $y \ge 2$. It is easy to see that h is convex, $h(0) > 0$, $h(1) = 0$, and that the point of minimum of h is greater than 1 for $y > (3 + \sqrt{17})/2 \approx 1.78$. Hence, for $y \ge 1.78$, h is decreasing in $[0,1]$, and therefore, it is positive $(0,1)$.

7. AN INEQUALITY FOR THE VOLUME OF THE UNIT BALL

Several sequences related to the volume of the unit ball in $\mathbb{R}^k$,

$$\Omega_k = \frac{\pi^{k/2}}{\Gamma(1+k/2)}, \qquad k = 0, 1, 2, \dots, \tag{41}$$

have been a subject of research regarding their monotonicity and inequalities (see the bibliography in [10]). As a completion of earlier results, in [10] it was proved that

$$\left(1 + \frac{1}{k}\right)^{\alpha} \le \frac{\Omega_k^2}{\Omega_{k-1}\Omega_{k+1}} \le \left(1 + \frac{1}{k}\right)^{1/2}, \qquad k = 1, 2, \dots, \tag{42}$$

where $\alpha = 2 - \log \pi / \log 2$. From (41) it follows that

$$\frac{\Omega_k^2}{\Omega_{k-1}\Omega_{k+1}} = T\left(1 + \frac{k-1}{2}, 1 + \frac{k+1}{2}\right),$$

and results of previous sections can be applied to this particular case. The next theorem gives an improvement of (42).

THEOREM 5. *For any $k = 1, 2, \dots,$ it holds*

$$\left(1+\frac{1}{k+1}\right)^{1/2} \le \frac{\Omega_k^2}{\Omega_{k-1}\Omega_{k+1}} \le \left(1+\frac{1}{k+1}\right)^{1/2} \frac{(k+2)\sqrt{2}}{(k+3)\sqrt{k+1}}. \tag{43}$$

The left inequality of (43) is sharper than the left inequality in (42) for $k \ge 2$; the right inequality is sharper than the corresponding one in (42) for $k \ge 1$.

PROOF. Inequalities (43) follow from inequalities (22) of Corollary 1, with $x = 1 + (k-1)/2$ and $\beta = 1/2$. After some algebraic transformations, it can be seen that the left-hand side of (43) is greater than the left-hand side in (42) for $k \ge 2$ if and only if

$$k^{5-2\alpha} + 2k^{4-2\alpha} - (k+1)^{5-2\alpha} > 0, \qquad \alpha = \frac{\log \pi}{\log 2}, \quad k \ge 2,$$

which, after division by $k^{5-2\alpha}$ and substitution $t = 1/k$, becomes

$$f(t) := 1 + 2t - (1+t)^{5-2\alpha} > 0, \qquad \text{for } t \in (0, 1/2].$$

Now, since $f(0) = 0$, $f(1/2) \approx 0.01 > 0$, and f is concave, the assertion follows. The other part is equivalent to the statement that

$$k^5 + 7k^4 + 18k^3 + 22k^2 + 17k + 9 > 0,$$

which is true for any $k > 0$. ∎

8. THE FUNCTION $x \mapsto T(1/x, 3/x)$ AND ITS INVERSE

In this section we will investigate the function

$$F(x) = T\left(\frac{1}{x}, \frac{3}{x}\right) = \frac{\Gamma(1/x)\,\Gamma(3/x)}{\Gamma^2(2/x)}, \qquad x > 0. \tag{44}$$

This function plays a role in a statistical problem of estimation of the shape parameter of generalized Gamma random variable, where it is an important task to find the inverse of F (see [11–13]). Since this is analytically intractable, tabulated values of F have been used to find an approximation of x, with $y = F(x)$ given. This method, due to the shape of F (as described in Theorem 6), yields an error of a considerable magnitude. On the other hand, a straightforward use of numerical methods is not desirable in applications, because it has to be repeatedly done for each new value of y. In this section, we present a procedure for approximative accurate determination of x given $y = F(x)$, that can be executed by a formula that involves only simple calculations.

Let us define a function related to F as follows:

$$\begin{aligned} L(x) &= \log\Gamma(x) + \log\Gamma(3x) - 2\log\Gamma(2x) - \log\left(\frac{4}{3}\right), \qquad x > 0, \\ L(0) &= 0. \end{aligned} \tag{45}$$

Clearly, for $x > 0$ we have

$$L(x) = \log F\left(\frac{1}{x}\right) - \log\left(\frac{4}{3}\right). \tag{46}$$

THEOREM 6. *The function $x \mapsto F(x)$, defined by (44) is convex and monotonically decreasing in x, with*

$$\lim_{x\to 0_+} F(x) = +\infty, \qquad \lim_{x\to+\infty} F(x) = \frac{4}{3}.$$

The function $x \mapsto L(x)$ is continuously differentiable of any order, convex and monotonically increasing on $[0,+\infty)$, with $L'(0) = 0$.

PROOF. Using the recurrence relation for the Gamma function, we find that

$$\frac{\Gamma(x)\Gamma(3x)}{\Gamma^2(2x)} = \frac{4}{3}\cdot\frac{\Gamma(x+1)\Gamma(3x+1)}{\Gamma^2(2x+1)},$$

which shows continuity of L at zero. Further, we can express $\Gamma(2x)$ and $\Gamma(3x)$ by means of multiplication formulae [32] for the Gamma function, to get

$$\begin{aligned} L(x) = \left(3x-\frac{1}{2}\right)\log 3 - (4x-1)\log 2 \\ +\log\Gamma\left(x+\frac{1}{3}\right) + \log\Gamma\left(x+\frac{2}{3}\right) - 2\log\Gamma\left(x+\frac{1}{2}\right) - \log\left(\frac{4}{3}\right). \end{aligned} \tag{47}$$

Then

$$L'(x) = 3\log 3 - 4\log 2 + \Psi\left(x+\frac{1}{3}\right) + \Psi\left(x+\frac{2}{3}\right) - 2\Psi\left(x+\frac{1}{2}\right), \tag{48}$$

and

$$L'(0) = 3\log 3 - 4\log 2 + \Psi\left(\frac{1}{3}\right) + \Psi\left(\frac{2}{3}\right) - 2\Psi\left(\frac{1}{2}\right). \tag{49}$$

It is well known (see, for example, [32, 6.3.3]) that $\Psi(1/2) = -\gamma - 2\log 2$. Further, by means of the formula in [48, 1.7.3], it can be evaluated that $\Psi(1/3) = -\gamma - (3/2)\log 3 - \pi/(2\sqrt{3})$ and $\Psi(2/3) = -\gamma - (3/2)\log 3 + \pi/(2\sqrt{3})$, which, upon substitution in (49), gives $L'(0) = 0$. The existence and continuity of higher derivatives follows from (47) in the same manner. From (48), we find that

$$L''(x) = \Psi'\left(x+\frac{1}{3}\right) + \Psi'\left(x+\frac{2}{3}\right) - 2\Psi'\left(x+\frac{1}{2}\right) > 0,$$

by convexity of the function $x \mapsto \Psi'(x)$. Therefore, L is convex, and so $L'(x) > L'(0) = 0$ for $x > 0$, hence L is increasing. This implies that F is decreasing. To show convexity of F, we note that $\log F(x) = L(1/x) - \log(4/3)$, and, by (47) we find that

$$(\log F(x))'' = t^3\left(6\log 3 - 8\log 2 + \Psi'\left(t+\frac{1}{3}\right) + \Psi'\left(t+\frac{2}{3}\right) - 2\Psi'\left(t+\frac{1}{2}\right)\right) > 0,$$

where $t = 1/x$. So, F is log-convex, and, consequently, convex. Finally, by means of Stirling's formula, it is easy to see that

$$\lim_{x\to+\infty} L(x) = \lim_{x\to 0_+} F(x) = +\infty. \qquad \blacksquare$$

Due to the shape of F, as found in Theorem 6, we will apply two different techniques to find the unique solution $x = x(y)$ of the equation

$$y = F(x), \qquad x \in (0,+\infty), \quad y \in \left(\frac{4}{3}, +\infty\right). \tag{50}$$

The first method is based on the work in previous sections, and gives a satisfactory solution for y away of $4/3$. The second method is based on Lagrange's expansion (Taylor's expansion of the inverse) and works well for y close to $4/3$.

THEOREM 7. *The solution of (50) is given by*

$$x = V_1(y) + E_1(y), \tag{51}$$

where

$$\begin{aligned} V_1(y) &= \frac{2a}{b + \sqrt{b^2 - 4ac}}, \\ a &= 3\log 3 - 4\log 2, \\ b &= \log y - \log 2 + \frac{1}{2}\log 3, \\ c &= \frac{1}{36}, \end{aligned} \tag{52}$$

and where $E_1(y) < 0$ *for all* $y > 4/3$ *and, monotonically,*

$$\lim_{y\to+\infty} E_1(y) = 0.$$

PROOF. We will use the approximation of the right-hand side of (14), which, applied to $x = t$ and $y = 3t$ yields

$$T(t, 3t) \le \frac{3^{3t-1/2}}{2^{4t-1}} e^{1/36t}. \tag{53}$$

Denoting the expression on the right-hand side of (53) by $G(t)$, we have that

$$G\left(\frac{1}{x}\right) = F(x) + E_0(x),$$

where $E_0(x) > 0$ for $x > 0$. From Section 6, we know that $E_0(x)$ is increasing with x and $\lim_{x\to 0_+} E_0(x) = 0$. Let $x = x(y)$ be the solution of equation (50) and let $x_a = x_a(y)$ be the solution of equation $y = G(1/x)$. From the continuity and monotonicity of F it follows that

$$x(y) = x_a(y) + E_1(y),$$

where $E_1(y) < 0$ for all $y > 4/3$, and monotonically converges to zero as $y \to +\infty$. Further, it is easy to check that $x_a(y) = V_1(y)$, where V_1 is given by (52). ∎

THEOREM 8. *The solution of (50) is given by*

$$x = V_2\left(\sqrt{\log y - \log\left(\frac{4}{3}\right)}\right) + E_2(y), \tag{54}$$

where

$$\begin{aligned} V_2(t) &= \frac{1}{c_1 t + c_2 t^2 + c_3 t^3}, \\ c_1 &= \sqrt{\frac{2}{L''(0)}} = 0.7796968009, \\ c_2 &= -\frac{1}{3}\cdot\frac{L'''(0)}{L''^2(0)} = 0.8885012277, \\ c_3 &= \frac{\sqrt{2}}{36}\cdot\frac{5L'''^2(0) - 3L''(0)L^{(\mathrm{iv})}(0)}{(L''(0))^{7/2}} = 0.5819804921, \end{aligned}$$

and $E_2(y) = O(y - 4/3)$ *as* $y \to 4/3$.

PROOF. Let $u = u(t)$ be the solution of equation

$$t = \sqrt{L(u)}. \tag{55}$$

Then u can be represented by its Maclaurin's expansion, as

$$u(t) = P_3(t) + R(t) = c_1 t + c_2 t^2 + c_3 t^3 + R(t), \qquad (t \to 0),$$

where $R(t) = O(t^4)$. Due to relation (46), the solution of equation (50) is given by

$$x = x(y) = \frac{1}{P_3(t) + R(t)},$$

where $t = \sqrt{\log y - \log(4/3)}$. Further,

$$\begin{aligned} \frac{1}{P_3(t) + R(t)} &= \frac{1}{P_3(t)}\left(1 - \frac{R(t)}{P_3(t)}\right) + O\left(\frac{R^2(t)}{P_3(t)}\right) \\ &= \frac{1}{P_3(t)} + O\left(t^2\right) = \frac{1}{P_3(t)} + O\left(y - \frac{4}{3}\right), \end{aligned}$$

which is (54), with coefficients c_i, $i = 1, 2, 3$, to be determined. Differentiating (55) with respect to t, we get

$$u'(t) = \frac{2\sqrt{L(u)}}{L'(u)}.$$

The value at zero can be found by L'Hôpital's rule

$$(u'(0))^2 = \lim_{u \to 0} \frac{4L(u)}{L'^2(u)} = \frac{2}{L''(0)},$$

and so,

$$c_1 = u'(0) = \sqrt{\frac{2}{L''(0)}}.$$

Let

$$g(u) := \frac{4L(u)}{L'^2(u)}, \qquad g(0) = \frac{2}{L''(0)}.$$

Then

$$g'(u) = \frac{4L'^2(u) - 8L'(u)L''(u)}{L'^3(u)},$$

and $g'(0)$ can be found with two applications of L'Hôpital's rule

$$g'(0) = -\frac{4}{3} \cdot \frac{L'''(0)}{L''^2(0)}.$$

Differentiating the relation $u'(t) = \sqrt{g(u)}$ with respect to t, we get

$$u''(t) = \frac{g'(u)}{2}, \tag{56}$$

that is,

$$c_2 = \frac{u''(t)}{2} = -\frac{1}{3} \cdot \frac{L'''(0)}{L''^2(0)}.$$

Further, it is not difficult to check that

$$g''(u) = -\frac{2g(u)L'''(u) + 3L''(u)g'(u)}{L'(u)},$$

and at $u = 0$ this is also of the form $0/0$. After an application of L'Hôpital's rule, we get

$$g''(0) = -\frac{5g'(0)Q'''(0) + 2g(0)L^{(\mathrm{iv})} + 3L''(0)g''(0)}{L''(0)},$$

which, after substitution of expressions for $g(0)$ and $g'(0)$ yields

$$g''(0) = \frac{5L'''^2(0) - 3L''(0)L^{(\mathrm{iv})}(0)}{3L''^3(0)}.$$

On the other hand, by differentiation of (56) with respect to t, we get

$$u'''(t) = \frac{g''(u)\sqrt{g(u)}}{2},$$

and, finally,

$$c_3 = \frac{u'''(0)}{6} = \frac{\sqrt{2}}{36} \cdot \frac{5L'''^2(0) - 3L''(0)L^{(\mathrm{iv})}(0)}{(L''(0))^{7/2}}.$$

To find numerical values for coefficients, we need to find

$$L^{(n)}(0) = \Psi^{(n-1)}\left(\frac{1}{3}\right) + \Psi^{(n-1)}\left(\frac{2}{3}\right) - 2\Psi^{(n-1)}\left(\frac{1}{2}\right), \qquad n = 2, 3, 4.$$

This was done using the built-in Psi function in the program Maple. ∎

Error Analysis and Numerical Data

The absolute error $|E_1(F(x))|$ is increasing function of x, with $\lim_{x\to 0_+} |E_1(F(x))| = 0$ and $\lim_{x\to+\infty} |E_1(F(x))| = +\infty$. For the error $E_2(F(x))$ it can be found that $\lim_{x\to 0_+} E_2(F(x)) = 0$ and numerical data reveal that it increases to 0.0246 at $x \approx 0.187$, and then decreases to zero at $x \approx 1.8865$, wherefrom it remains negative and converges to zero as $x \to +\infty$. The two error curves intersect at $x_0 \approx 1.10445$, with $|E_1(F(x_0))| = E_2(F(x_0)) \approx 0.00559$. The value of F at that point is $y_0 = F(x_0) \approx 1.908$. Therefore, the optimal algorithm would be to use formula (54) for $y \in (4/3, y_0]$ and formula (51) for $y \in [y_0, +\infty)$. The maximal absolute error in this procedure is then less than $6 \cdot 10^{-3}$.

Some numerical values are presented in Table 1.

Table 1. Approximative solutions x_a of the equation $y = F(x)$, for selected values of x. The values for x_a are calculated with formula (51) for $x \le 1$ and with (54) for $x > 1$.

x	$y = F(x)$	x_a
0.001	$2.2024244 \cdot 10^{227}$	0.001
0.01	$6.12305 \cdot 10^{22}$	0.01
0.1	216.8266253	0.1000000416
0.3	6.661052616	0.3000099079
0.5	3.333333	0.5001227336
0.7	2.484640412	0.7006302314
1	2	1.003483771
2	1.570796326	2.000398257
5	1.389497325	5.002165999
10	1.350375904	10.00115967
20	1.338122716	20.00042994
100	1.333546439	100.0000262

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