Journal of Mathematical Analysis and Applications **239,** 440–448 (1999)
Article ID jmaa.1999.6580, available online at http://www.idealibrary.com on IDEAL®

A Nonlinear Mercerian Theorem

K. J. Falconer

Mathematical Institute, University of St. Andrews, North Haugh, St. Andrews, Fife, KY16955 United Kingdom

Submitted by Joseph D. Ward

Received July 15, 1998

We show that convergence of $x(t)$ as $t \to \infty$ may be deduced from the limiting behavior of certain functions involving $x(t)$ and its Cesàro averages. Dynamical systems methods are used to derive this "Mercerian-type" result.

We consider averages of a continuous function $x\colon [0,\infty) \to \mathbb{R}$. Let $\phi\colon [0,\infty) \to [0,\infty)$ be continuous and strictly increasing to infinity. The Cesàro averages of x with respect to ϕ are defined by

$$y(t) = \frac{1}{\phi(t)} \int_0^t x(u)\, d\phi(u) \qquad (t > 0) \tag{1}$$

so that $y\colon (0,\infty) \to \mathbb{R}$ is also continuous. [In the most common case $\phi(t) = t$, so $y(t) = t^{-1}\int_0^t x(u)\, du$.] We are interested in the relationship between the limiting behavior of $x(t)$ and $y(t)$ as $t \to \infty$. The basic "Abelian" result, that if $x(t) \to c$ for some $c \in \mathbb{R}$ then $y(t) \to c$, is easy to verify. The direct converse is false, but there is a well-known theorem of "Tauberian" type, see [1], [4], that $y(t) \to c$ implies $x(t) \to c$ provided that x satisfies a Tauberian condition:

$$\lim_{\lambda \searrow 1} \liminf_{t \to \infty} \inf_{t \le u \le \phi^{-1}(\lambda\phi(t))} \left[x(u) - x(t) \right] \ge 0. \tag{2}$$

The classical "Mercerian" theorem states that if, for some $0 < \lambda < 1$, the linear combination $\lambda x(t) + (1-\lambda) y(t) \to c$, then $x(t) \to c$ without any additional condition. This theorem has given rise to many other results of similar character, such as in [1], [2], [5], [6]. However, these are *linear*

0022-247X/99 $30.00

Mercerian theorems, which have as hypotheses a linear relationship between x and its image under some linear transformation.

Here we consider *nonlinear* analogues. We use a dynamical systems method to show that the convergence of almost any function of $x(t)$ and $y(t)$ is enough to imply convergence of $x(t)$. For example, our theorem shows that if $x(t)^3 - y(t)^2 \to 4$ then $x(t) \to 2$. A specific instance of the theorem has applications to fractal dimensions and Minkowski measurability, see [3].

Comparing (1) at t_1 and t gives, for $t_1 > t$,

$$\begin{aligned} y(t_1) - y(t) &= \frac{1}{\phi(t_1)} \int_t^{t_1} x(u)\, d\phi(u) + \frac{1}{\phi(t_1)} \int_0^t x(u)\, d\phi(u) - y(t) \\ &= \frac{1}{\phi(t_1)} \int_t^{t_1} x(u)\, d\phi(u) + y(t) \left(\frac{\phi(t) - \phi(t_1)}{\phi(t_1)} \right) \\ &= \frac{1}{\phi(t_1)} \int_t^{t_1} [x(u) - y(t)]\, d\phi(u). \end{aligned} \tag{3}$$

[Note that if ϕ is absolutely continuous then differentiating (1) gives $(\phi y)' = \phi' x$ so

$$y'(t) = \frac{\phi'(t)}{\phi(t)} (x(t) - y(t)); \tag{4}$$

however, using (3) avoids the requirement of absolute continuity.] In particular, if $x(t) > y(t)$ then $y(t)$ is increasing at t and if $x(t) < y(t)$ then $y(t)$ is decreasing at t. We think of $\mathbf{x}(t) = (x(t), y(t))$ as the position at time t of a point moving in the x–y plane, so this means that when $\mathbf{x}(t)$ is above the diagonal line $D = \{(x, x) : x \in \mathbb{R}\}$ it is moving downwards and when $\mathbf{x}(t)$ is below D it is moving upwards, an observation crucial to our approach.

A further useful consequence of (3) is that, if there exists $t_0 > 0$ and $\epsilon > 0$ such that

$$|x(t) - y(t)| \geq \epsilon \qquad \text{for all } t \geq t_0, \tag{5}$$

then $y(t)$ must be unbounded. To see this assume that $x(t) - y(t) \geq \epsilon$ for $t \geq t_0$. Then (3), together with the inequality $1 - x \geq -\frac{1}{2} \log x$ for $\frac{1}{2} \leq x$

≤ 1, implies that

$$y(t_1) - y(t) \ge \frac{(\phi(t_1) - \phi(t))\epsilon}{2\phi(t_1)} = \frac{\epsilon}{2}\left(1 - \frac{\phi(t)}{\phi(t_1)}\right)$$
$$\ge \frac{\epsilon}{4}(\log \phi(t_1) - \log \phi(t)) \tag{6}$$

for $t_1 \ge t \ge t_0$ with t_1 sufficiently close to t. Using compactness and summing over sufficiently small intervals gives (6) for all $t_1 \ge t \ge t_0$. A similar argument applies if $x(t) - y(t) \le -\epsilon$ for $t \ge t_0$.

We will assume a limiting relationship as $t \to \infty$ between $x(t)$ and its average $y(t)$ given in terms of a continuous function $F: \mathbb{R}^2 \to \mathbb{R}$. We will require that F satisfies a condition at the points of intersection of the diagonal D and a level set $F^{-1}(c) = \{(x, y): F(x, y) = c\}$. We say that F is *good* at $\mathbf{x}_0 = (x_0, y_0) \in \mathbb{R}^2$ if, given $F(\mathbf{x}_0) = c$, there is a neighborhood of $\mathbf{x}_0$ in which the level set $F^{-1}(c)$ is a continuously differentiable curve with slope $\lambda(\mathbf{x}_0)$ at $\mathbf{x}_0$ with $\lambda(\mathbf{x}_0) \ne 0$. [We allow $\lambda(\mathbf{x}_0) = \infty$ corresponding to a vertical tangent.]

By the implicit function theorem, a continuously differentiable function F is good at $\mathbf{x}_0$ if $\partial F(\mathbf{x}_0)/\partial x \ne 0$, so "almost all" continuously differentiable functions are good at any given finite set of points. Thus a great many functions F satisfy the requirements of the following theorem.

THEOREM. *Let $F: \mathbb{R}^2 \to \mathbb{R}$ be continuous, let c be a real number such that $F^{-1}(c) \cap D$ is finite with $F(x, x)$ bounded away from c as $x \to \pm\infty$, and assume that F is good at all $\mathbf{x}_0 \in F^{-1}(c) \cap D$. Let x: $[0, \infty) \to \mathbb{R}$ be continuous and let y be defined by* (1). *If*

$$F(x(t), y(t)) \to c \qquad \text{as } t \to \infty \tag{7}$$

then either

(a) *for all sufficiently large t we have either $x(t) \ge y(t) \to \infty$ or $x(t) \le y(t) \to -\infty$, or*

(b) *$x(t) \to x_0$ as $t \to \infty$ for some $x_0 \in \mathbb{R}$ satisfying $F(x_0, x_0) = c$.*

Note that in many situations, part (a) of the conclusion may be excluded by examining F or $x(t)$.

To prove the theorem we regard $\mathbf{x}(t) = (x(t), y(t))$ as a nonautonomous dynamical system in the plane with t as time. We first prove a lemma which describes the trajectories $\mathbf{x}(t)$ near good points of $F^{-1}(c) \cap D$. There are essentially different cases depending on the local form of $F^{-1}(c)$ near the good point $\mathbf{x}_0 = (x_0, x_0)$; these are illustrated in Fig. 1.

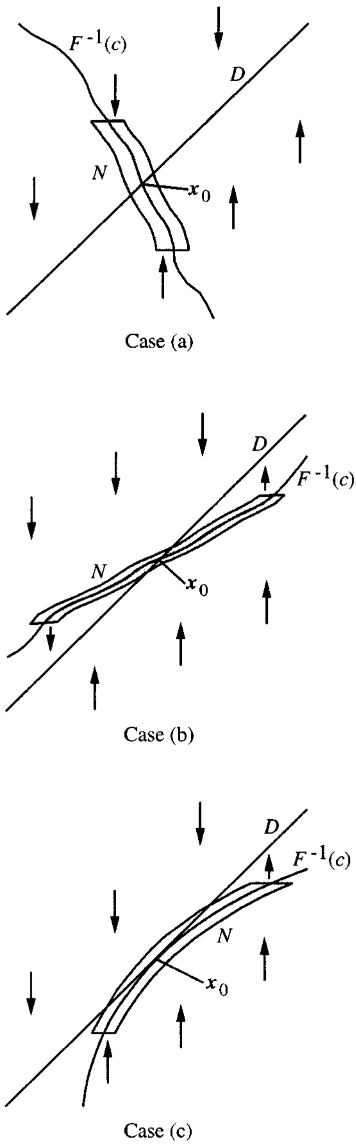

FIG. 1. The neighborhood N of $\mathbf{x}_0$ in the three cases. The arrows indicate the vertical component of motion of the trajectory $\mathbf{x}(t)$.

We write $D^- = \{(x, y) \in \mathbb{R}^2 : x > y\}$ and $D^+ = \{(x, y) \in \mathbb{R}^2 : x < y\}$ for the open half-spaces below and above the diagonal D. Similarly, we write L_a for the horizontal line with equation $x = a$ and L_a^- and L_a^+ for the open half-spaces below and above L_a.

Case (a): At $\mathbf{x}_0$ the curve $F^{-1}(c)$ crosses D from $D^- \cap L_{x_0}^-$ to $D^+ \cap L_{x_0}^+$. This always occurs if $1 < \lambda(\mathbf{x}_0) \leq \infty$ or $-\infty < \lambda(\mathbf{x}_0) < 0$ and may occur if $\lambda(\mathbf{x}_0) = 1$. Intuitively, the lemma shows in this case that $\mathbf{x}_0$ is "stable" in the sense that if $\mathbf{x}(t)$ is close to $\mathbf{x}_0$ at a sufficiently large t then the trajectory remains close to $\mathbf{x}_0$ for all subsequent t.

Case (b): At $\mathbf{x}_0$ the curve $F^{-1}(c)$ crosses D from $D^+ \cap L_{x_0}^-$ to $D^- \cap L_{x_0}^+$. This always occurs if $0 < \lambda(\mathbf{x}_0) < 1$ and may occur if $\lambda(\mathbf{x}_0) = 1$. In this case the lemma implies that either $\mathbf{x}(t)$ approaches $\mathbf{x}_0$ fairly rapidly or eventually moves away from $\mathbf{x}_0$ for good.

Case (c): At $\mathbf{x}_0$ the curve $F^{-1}(c)$ touches D but does not cross D. This requires $\lambda(\mathbf{x}_0) = 1$. Here the lemma implies that when t is large $\mathbf{x}(t)$ always approaches and departs from near $\mathbf{x}_0$ on the same side of D.

LEMMA. *Suppose* (7) *holds for some* $c \in \mathbb{R}$. *Let* $\mathbf{x}_0 \in F^{-1}(c) \cap D$ *and suppose that F is good at* $\mathbf{x}_0$. *Then there is a base of closed neighborhoods* $\mathcal{N}$ *of* $\mathbf{x}_0$ *as follows*:

In Case (*a*): *for all* $N \in \mathcal{N}$ *there exists* $t_N > 0$ *such that if* $t_1 \geq t_N$ *and* $\mathbf{x}(t_1) \in N$ *then* $\mathbf{x}(t) \in N$ *for all* $t \geq t_1$.

In Case (*b*): *for all* $N \in \mathcal{N}$ *there exists* $t_N > 0$ *such that if* $t_1 \geq t_N$ *and* $\mathbf{x}(t_1) \notin N$ *then* $\mathbf{x}(t) \notin N$ *for all* $t \geq t_1$.

In Case (*c*): *for all* $N \in \mathcal{N}$ *there exists* $t_N > 0$ *and* $\delta_N > 0$ *such that* either *every entry of the trajectory* $\mathbf{x}(t)$ *to N after time* t_N *is across* $D_- \cap L_{x_0 - \delta_N}$ *and every exit from N is across* $D_- \cap L_{x_0 + \delta_N}$, or *every entry of the trajectory* $\mathbf{x}(t)$ *to N after time* t_N *is across* $D_+ \cap L_{x_0 + \delta_N}$ *and every exit from N is across* $D_- \cap L_{x_0 - \delta_N}$.

Proof of Lemma. Write $F^{-1}(c)_\epsilon$ for the closed ϵ-neighborhood of $F^{-1}(c)$, that is the union of all closed discs of radius ϵ with centres in $F^{-1}(c)$. Let $S_{x_0, \delta}$ denote the closed horizontal strip $\{(x, y) : x_0 - \delta \leq y \leq x_0 + \delta\}$. The required neighborhoods of $\mathbf{x}_0$ will be the connected components of $F^{-1}(c)_\epsilon \cap S_{x_0, \delta}$ that contain $\mathbf{x}_0$, where δ is sufficiently small and $\epsilon = \epsilon(\delta)$ is chosen suitably. Thus the neighborhoods will be narrow "bands" around part of $F^{-1}(c)$ through $\mathbf{x}_0$.

Since F is good at $\mathbf{x}_0 = (x_0, x_0)$, the slope of $F^{-1}(c)$ near $\mathbf{x}_0$ is nonzero, so by choosing appropriate δ_0 and η_0 sufficiently small, there are coordinate rectangles $R = [x_0 - \eta_0, x_0 + \eta_0] \times [x_0 - \delta_0, x_0 + \delta_0]$ and $R' = [x_0 + \frac{1}{3}\eta_0, x_0 + \frac{1}{3}\eta_0] \times [x_0 - \delta_0, x_0 + \delta_0]$ in which $F^{-1}(c)$ is of the following form: the set $F^{-1}(c) \cap R$ has a single connected component which is a differentiable curve crossing R from bottom to top, with slope bounded away from 0, with $F^{-1}(c) \cap R \subset R'$, and such that $F^{-1}(c) \cap D \cap R$ is the single point $\mathbf{x}_0$.

Thus for all $0 < \delta \leq \frac{1}{2}\delta_0$ the set $F^{-1}(c) \cap R$ intersects each of the lines $L_{x_0 - \delta}$ and $L_{x_0 + \delta}$ at a single point inside R' that is not on D. With $F^{-1}(c)_\epsilon$ as the ϵ-neighborhood of $F^{-1}(c)$, we choose $\epsilon = \epsilon(\delta) < \min\{\frac{1}{2}\delta_0, \frac{1}{3}\eta_0\}$ sufficiently small to ensure that $F^{-1}(c)_\epsilon \cap R'' \cap L_{x_0 - \delta}$ and $F^{-1}(c)_\epsilon \cap R'' \cap L_{x_0 + \delta}$ are closed subintervals of $L_{x_0 - \delta}$ and $L_{x_0 + \delta}$, respectively, which do not intersect D, where $R'' = [x_0 - \frac{2}{3}\eta_0, x_0 + \frac{2}{3}\eta_0] \times [x_0 - \delta_0, x_0 + \delta_0]$.

For each such $\delta \leq \frac{1}{2}\delta_0$ we set $N = N(\delta) = F^{-1}(c)_\epsilon \cap R'' \cap S_{x_0, \delta}$, where $S_{x_0, \delta}$ is the strip $\{(x, y): x_0 - \delta \leq y \leq x_0 + \delta\}$. Thus N has the form of a 'band' about $F^{-1}(c) \cap R \cap S_{x_0, \delta}$ stretching from its *floor* $F^{-1}(c)_\epsilon \cap R'' \cap L_{x_0 - \delta}$ to its *ceiling* $F^{-1}(c)_\epsilon \cap R'' \cap L_{x_0 + \delta}$. In fact N is the connected component of $F^{-1}(c)_\epsilon \cap S_{x_0, \delta}$ containing $\mathbf{x}_0$.

We take $\mathscr{N}$ to be the collection of all neighborhoods $\{N(\delta)\}$ constructed in this way for $0 < \delta \leq \frac{1}{2}\delta_0$. For $N \in \mathscr{N}$ we write δ_N for the value of δ such that $N = N(\delta_N)$.

Using (7) and the compactness of R, we may choose t_N such that for all $t \geq t_N$ for which $\mathbf{x}(t) \in R$ we have $\mathbf{x}(t) \in F^{-1}(c)_\epsilon$ [where $\epsilon = \epsilon(\delta_N)$]. In particular, if the trajectory $\mathbf{x}(t)$ enters or leaves N at time $t \geq t_N$ it must do so by crossing either the floor or ceiling of N.

We use this fact to check that these basic neighborhoods N of $\mathbf{x}_0$ satisfy the conclusion of the lemma in each of the three cases.

Case (a): If $N \in \mathscr{N}$ and $\mathbf{x}(t_1) \in N$ for some $t_1 \geq t_N$, the trajectory $\mathbf{x}(t)$ cannot enter N without crossing its floor in an downward direction or its ceiling in a upward direction, which is impossible, since the floor is in D^- and the ceiling is in D^+, see Fig. 1(a). We conclude that $\mathbf{x}(t)$ remains in N for all $t \geq t_1$.

Case (b): If $N \in \mathscr{N}$ and $\mathbf{x}(t_1) \notin N$ where $t_1 \geq t_N$, the trajectory $\mathbf{x}(t)$ cannot enter N without crossing its floor in an upward direction or its ceiling in a downward direction, which is impossible, since the floor is in D^+ and the ceiling is in D^-, see Fig. 1(b). We conclude that $\mathbf{x}(t) \notin N$ for all $t \geq t_1$.

Case (c): If the curve $F^{-1}(c)$ lies in D_- near $\mathbf{x}_0$, then for all $N \in \mathscr{N}$ the floor of N is a subinterval of $D_- \cap L_{x_0 - \delta}$ and the ceiling of N is a subinterval of $D_- \cap L_{x_0 + \delta_N}$ and the conclusion follows. Similarly, if the curve $F^{-1}(c)$ lies in D_+ near $\mathbf{x}_0$, we reach the alternative conclusion. ∎

Proof of Theorem. We split the proof into two cases: when $y(t)$ is eventually monotonic and when $y(t)$ is oscillatory.

Case (i). There exists t_1 such that $\mathbf{x}(t) = (x(t), y(t))$ does not strictly cross the diagonal line D for $t \geq t_1$. By the remark after (3), $y(t)$ is monotonic for $t \geq t_1$. For such t, if $y(t)$ is unbounded and increasing then $(x(t), y(t))$ is below D, so $x(t) \geq y(t) \to \infty$. Similarly if $y(t)$ is unbounded and eventually decreasing then $x(t) \leq y(t) \to -\infty$, leading to conclusion (a) of the theorem.

If $y(t)$ is bounded, there exists $x_0 \in \mathbb{R}$ such that $y(t) \to x_0$ as $t \to \infty$. We claim that $F(x_0, x_0) = c$. Assume not: by continuity of F there exist $\delta > 0$ and a coordinate square S with side length $4\epsilon > 0$ and center (x_0, x_0), such that $|F(x, y) - c| \geq \delta > 0$ for all $(x, y) \in S$, so by (7) there exists $t_2 \geq t_1$ such that $(x(t), y(t)) \notin S$ for all $t \geq t_2$. Since $y(t) \to x_0$ this would require that, for all sufficiently large t, we have $|x(t) - x_0| \geq 2\epsilon$ and thus $|x(t) - y(t)| \geq \epsilon$, which by the remark at (5) would contradict the boundedness of $y(t)$.

Thus for some x_0 with $F(x_0, x_0) = c$, we have $y(t) \to x_0$, so $\operatorname{dist}(\mathbf{x}(t), L_{x_0}) \to 0$ where L_{x_0} is the horizontal line $y = x_0$. Since F is good at $\mathbf{x}_0 = (x_0, x_0) \in D$, the point $\mathbf{x}_0$ is an isolated point of $L_{x_0} \cap F^{-1}(c)$. Since $F(x(t), y(t)) \to c$, either $x(t) \to x_0$ or $\liminf_{t \to \infty} |x(t) - x_0| > 0$, an alternative which is again disallowed by (5).

Case (ii). The point $\mathbf{x}(t) = (x(t), y(t))$ goes strictly above D and also strictly below D for arbitrarily large t. Write S for the set of limit points of $\mathbf{x}(t)$ on D, that is

$$S = \left\{\mathbf{x} \in D\text{: there exists } t_i \nearrow \infty \text{ with } \mathbf{x}(t_i) \to \mathbf{x}\right\}.$$

Since $F(\mathbf{x}(t)) \to c$ and $F(x, x)$ is bounded away from c for large x, we have $S \subset F^{-1}(c) \cap D$, so S is nonempty and (by hypothesis) is finite.

By hypothesis, F is good at all $\mathbf{x}_0 = (x_0, x_0) \in S$. Suppose case (a) pertains at some $\mathbf{x}_0 \in S$ and let $\mathscr{N}$ be the base of neighborhoods of $\mathbf{x}_0$ given by the lemma. For all $N \in \mathscr{N}$ we have $\mathbf{x}(t) \in N$ for arbitrarily large t and so, by case (a) of the lemma, for all sufficiently large t. Thus $\mathbf{x}(t) \to \mathbf{x}_0$, giving $x(t) \to x_0$.

Now suppose case (b) applies at some $\mathbf{x}_0 \in S$ and let $\mathscr{N}$ be the base of neighborhoods of $\mathbf{x}_0$ given by the lemma. If $\mathbf{x}(t) \nrightarrow \mathbf{x}_0$ then there exists $N \in \mathscr{N}$ such that $\mathbf{x}(t) \notin N$ for arbitrarily large t and so, by case (b) of the lemma, for all sufficiently large t, contradicting that $\mathbf{x}_0$ is a limit point of $\mathbf{x}(t)$. Thus again $\mathbf{x}(t) \to \mathbf{x}_0$ and $x(t) \to x_0$.

Otherwise, case (c) pertains to all the points S. Let $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, and for each $i = 1, \ldots, n$ let N_i be a basic neighborhood of $\mathbf{x}_i$ as given by the lemma, satisfying case (c) of the conclusion. We may choose these neighborhoods small enough to have pairwise disjoint projections onto the y-axis. Using (7) there is a time $t_0 \geq \max_{i=1,\ldots,n}\{t_{N_i}\}$ such that, if $t \geq t_0$, the trajectory $\mathbf{x}(t)$ cannot cross D outside $\bigcup_{i=1}^{n} N_i$. Moreover, by case (c) of the lemma, $\mathbf{x}(t)$ enters and leaves each N_i on the *same* side of D. Thus $y(t)$ is monotonic for all $t \geq t_0$ such that $\mathbf{x}(t) \notin \bigcup_{i=1}^{n} N_i$. In particular there exists some integer i and time $t_1 \geq t_0$ such that $\mathbf{x}(t) \in N_i$ for all $t \geq t_1$, so in fact S consists of a single point, $\mathbf{x}_0 = (x_0, x_0)$ say, at which case (c) holds.

Let N be a member of the base of neighborhoods $\mathscr{N}$ of $\mathbf{x}_0$ given by the lemma. Again by (7) there exists $t_2 \geq t_N$ such that $\mathbf{x}(t)$ does not cross D outside N when $t \geq t_2$, and therefore $y(t)$ is monotonic outside N. Thus, by case (c) of the lemma, if $\mathbf{x}(t)$ leaves N after time t_2 it cannot re-enter N, contradicting that $\mathbf{x}_0 \in S$. We conclude that for all $N \in \mathscr{N}$ we have $\mathbf{x}(t) \in N$ for all sufficiently large t, so $\mathbf{x}(t) \to \mathbf{x}_0$ and $x(t) \to x_0$. ∎

Note that some condition on F on the diagonal D is necessary for the validity of the theorem. For example, if $F^{-1}(c)$ has horizontal tangents at points on D then $\mathbf{x}(t)$ could be asymptotic to a loop stradling the diagonal D. In particular the result fails if $F(x, y) = y$ when $F^{-1}(c) = L_c$; the hypothesis (7) reduces to $y(t) \to c$ and we are back with the Tauberian situation, requiring a condition such as (2) for the convergence of $x(t)$. Similarly, if $F^{-1}(c) \cap D$ contains an interval then the limit set of $\mathbf{x}(t)$ could be a subinterval of D. (Some results related to the case where $y(t)/x(t) \sim c$ are discussed in [1, Chapt. 5].)

The conditions on F can certainly be weakened or varied. For instance if $F^{-1}(c) \cap D$ contains a single point (x_0, x_0) and the horizontal line L_{x_0} contains no interval of the form $[x_0 - \delta, x_0]$ or $[x_0, x_0 + \delta]$ for all $\delta > 0$, then the same conclusion follows. If it is known that $x(t)$ [and thus $y(t)$] is bounded then the requirements that $F^{-1}(c) \cap D$ is finite and that $F(x, x)$ is bounded away from c for large x may be dropped, since we may work in a bounded region of the plane with the local finiteness of $F^{-1}(c) \cap D$ following from the goodness of F at all $\mathbf{x}_0 \in F^{-1}(c) \cap D$.

The theorem may be used to deduce other asymptotic behavior of functions. For example, taking $\alpha > -1$ and setting $\phi(t) = t^{\alpha+1}$ and $f(t) = x(t)t^{\alpha}$, hypothesis (7) becomes

$$F\left(f(t)t^{-\alpha}, (\alpha + 1)t^{-\alpha-1} \int_0^t f(u)\, du \right) \to c. \tag{8}$$

Provided F satisfies the conditions of the theorem, we may conclude that if f is continuous and satisfies (8) then $f(t)t^{-\alpha}$ converges to $-\infty$, to ∞ or to some x_0 satisfying $F(x_0, x_0) = c$.

ACKNOWLEDGMENT

I thank Professor N. H. Bingham for a talk that inspired this investigation and for many helpful comments.

REFERENCES

1. N. H. Bingham, C. M. Goldie, and J. L. Teugels, ''Regular Variation,'' Cambridge Univ. Press, Cambridge, UK, 1987.
2. N. H. Bingham and A. Inoue, The Drasin-Shea-Jordan theorem for Fourier and Hankel transforms, *Quart. J. Math.* **48** (1997), 279–307.
3. K. J. Falconer, On the Minkowski measurability of fractals, *Proc. Amer. Math. Soc.* **123** (1995), 1115–1124.
4. J. Karamata, Sur les théorèms inverses des procédé de sommabilité, *Act. Sci. Ind.* **450** (1937), Hermann, Paris.
5. R. E. A. C. Paley and N. Wiener, ''Fourier transforms in the complex domain,'' Colloq. Publ. **19**, American Mathematical Society, Washington, DC, 1934.
6. H. R. Pitt, ''Tauberian Theorems,'' Oxford Univ. Press, London, 1958.