ACADEMIC PRESS

Available at WWW.MATHEMATICSWEB.ORG POWERED BY SCIENCE DIRECT®

Journal of Complexity 19 (2003) 730–743

Journal of COMPLEXITY

http://www.elsevier.com/locate/jco

Worst case complexity of multivariate Feynman–Kac path integration ☆

Marek Kwas[a,b,*,1] and Youming Li[c,d,1]

[a] *Department of Computer Science, Columbia University, New York, NY 10027, USA*
[b] *Institute of Applied Mathematics and Mechanics, University of Warsaw, Ul. Banacha 2, 02-097 Warsaw, Poland*
[c] *Department of Computer Science, University of Kentucky, Lexington, KY 40506, USA*
[d] *Department of Mathematics and Computer Science, Georgia Southern University, Statesboro, GA 30460, USA*

Received 2 May 2002; accepted 6 January 2003

Abstract

We study the multivariate Feynman–Kac path integration problem. This problem was studied in Plaskota et al. (J. Comp. Phys. 164 (2000) 335) for the univariate case. We describe an algorithm based on uniform approximation, instead of the L_2-approximation used in Plaskota et al. (2000). Similarly to Plaskota et al. (2000), our algorithm requires extensive precomputing. We also present bounds on the complexity of our problem. The lower bound is provided by the complexity of a certain integration problem, and the upper bound by the complexity of the uniform approximation problem. The algorithm presented in this paper is almost optimal for the classes of functions for which uniform approximation and integration have roughly the same complexities.

MSC: 65D99; 81C35

Keywords: Multivariate Feynman–Kac path integration; Wiener measure; Multivariate approximation; Worst case complexity

☆ The original version of this paper was based on the M. S. Degree Thesis defended by the first author at the University of Warsaw in May, 2001. At that stage the estimate of the combinatorial complexity was super-polynomial in ε^{-1}. The recent results and contribution of the second author allowed to improve that estimate essentially.

* Corresponding author. Institute of Applied Mathematics and Mechanics, University of Warsaw, Ul. Banacha 2, 02-097 Warsaw, Poland.

E-mail address: mkwas@hydra.mimuw.edu.pl (M. Kwas).

[1] Both authors are in part supported by the National Science Foundation.

0885-064X/03/$ - see front matter © 2003 Elsevier Inc. All rights reserved.
doi:10.1016/S0885-064X(03)00048-7

1. Introduction

Path integrals are defined as integrals over an infinite dimensional space equipped with a probability measure. A path integral is called Wiener integral if the respective measure is the Wiener measure w on the space $\mathscr{C}$ of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^d$. In case $d \geqslant 2$ we add the term "multivariate". The multivariate Feynman–Kac path integral (3) is the solution of the initial value problem for the multivariate heat equation (1) and (2).

Various computational methods have been developed for the univariate case where $d = 1$ and the Feynman–Kac path integral is the solution of the heat equation with one space variable. Most of them are stochastic methods. First the path integral is approximated by a multivariate integral over $\mathbb{R}^n$ (with *large* n), and then this integral is approximately computed by using randomized methods such as a Monte Carlo method.

A new approach was proposed in [5] for the univariate case $d = 1$. A deterministic algorithm based on L_2-approximation was constructed and the complexity of the Feynman–Kac path integration was estimated. We recall that the complexity is the minimal information and combinatory cost of any algorithm that solves the problem with an error at most ε. The upper bound was given by the complexity of the L_2-approximation problem. The lower bound was given by the complexity of a certain weighted integration problem. We refer to [8] for a thorough discussion of the complexity of continuous problems.

Our paper is a continuation of [5]. We present an extension of results from [5] to the multivariate case by providing an algorithm for the multivariate Feynman–Kac path integration problem. The structure of our algorithm is similar to the structure of the algorithm from [5]. However, our algorithm is based on uniform approximation, which allows applications for arbitrary d. Indeed, an algorithm based on L_2-approximation, such as that from [5], can be used for $d = 1$ only. Both the information and combinatory costs of our algorithm are roughly of order $\varepsilon^{-\alpha(F)}$ for a certain $\alpha(F) > 0$ dependent on a given class F of functions. This upper bound is derived from the complexity of uniform approximation for the class F. When the class F is a ball of $C^r(\mathbb{R}^d)$, the space of r times continuously differentiable functions, then we have $\alpha(F) = d/r$. Our algorithm also requires precomputation of large number of certain coefficients given as multivariate integrals. This process, similarly as for the algorithm from [5], is computationally difficult.

We also study the complexity of multivariate Feynman–Kac path integration. The complexity is bounded, analogously as in [5], from below by the complexity of multivariate integration, and from above by complexity of the uniform approximation (in [5] it was the L_2-approximation). Moreover, our algorithm is almost optimal for classes F for which the complexities of the multivariate integration and the uniform approximation re of the same order. This holds for F being balls of $C^r(\mathbb{R}^d)$, see Section 4.2.

2. Motivating example

Although our paper deals with path integrals, we first motivate our approach by presenting a few facts concerning the heat equation. Denote $\Omega = \mathbb{R}^d \times [0, \infty)$. Let us consider the initial value problem for the heat equation

$$\frac{\partial z}{\partial t}(\mathbf{u}, t) = \frac{1}{2}\Delta z(\mathbf{u}, t) + V(\mathbf{u})z(\mathbf{u}, t) \quad \text{for } (\mathbf{u}, t) \in \operatorname{int} \Omega, \tag{1}$$

$$z(\mathbf{u}, 0) = v(\mathbf{u}). \tag{2}$$

Here $v, V : \mathbb{R}^d \to \mathbb{R}$ are the initial value function and the potential function, respectively. As usual, Δ denotes the Laplacian, i.e., $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial u_i^2}$, where $\mathbf{u} = [u_1, \ldots, u_d]^T$. We assume that v and V belong to a class F that is chosen such that the solution z of (1) and (2) exists. For simplicity global properties of v and V are not distinguished.

For a given fixed point $(\mathbf{u}^*, t^*) \in \Omega$ and arbitrary functions v, V from the class F, we want to compute an ε-approximation of the exact solution $z^* = z_{v,V}(\mathbf{u}^*, t^*)$, i.e., to compute $a = a_{v,V}(\mathbf{u}^*, t^*)$ such that $|z^* - a| \leqslant \varepsilon$.

We assume that evaluations of the functions v and V at finitely many points are allowed. We want to determine the minimal number of such values $n(\varepsilon, F) = n(\varepsilon, F, \mathbf{u}^*, t^*)$ that is sufficient to compute an ε-approximation.

This problem can obviously be solved by using classical algorithms such as finite element or finite difference methods. For instance, suppose that $\frac{\partial}{\partial t} z(\mathbf{u}, t)$ is approximated by the first forward difference $(z(\mathbf{u}, t + \tau) - z(\mathbf{u}, t))/\tau$, and $\frac{\partial^2}{\partial \mathbf{u}_i^2} z(\mathbf{u}, t)$ is approximated by the second central difference $(z(\mathbf{u} + \mathbf{h}_i, t) - 2z(\mathbf{u}, t) + z(\mathbf{u} - \mathbf{h}_i, t))/h^2$ where $(\mathbf{h}_i)_j = \delta_{ij} h$. For functions v and V that are bounded and four times continuously differentiable, the solution z is twice continuously differentiable with respect to t and four times continuously differentiable with respect to $\mathbf{u}$, and we can compute an ε-approximation with cost of order $\varepsilon^{-(d/2+1)}$.

In the next section we see that the solution of the heat equation can be formulated as a particular path integral.

3. Multivariate Feynman–Kac path integration

It turns out that for a certain class F, defined in the next paragraph, the solution z of (1) and (2) is given by the famous Feynman–Kac formula

$$z(\mathbf{u}, t) = \int_{\mathscr{C}} v(\mathbf{x}(t) + \mathbf{u}) \exp\left(\int_0^t V(\mathbf{x}(s) + \mathbf{u})\, ds \right) w(d\mathbf{x}). \tag{3}$$

Here, $\mathscr{C}$ is the set of continuous functions $\mathbf{x} : \mathbb{R}_+ \to \mathbb{R}^d$ such that $\mathbf{x}(0) = \mathbf{0}$. The path integral (3) is with respect to the d-dimensional Wiener measure w, see [1] or [6]. We recall that w is a Gaussian measure, whose coordinate process, i.e., stochastic process

formed by the point functionals L_t, where $L_t x = x(t)$, is a Wiener process. This means that the mean element m and the covariance function C of w for $d = 1$ are

$$m(t) = \int_{\mathscr{C}} x(t)w(dx) = 0 \quad \forall t, \quad C(s,t) = \int_{\mathscr{C}} x(s)x(t)w(dx) = \min\{s,t\} \quad \forall s,t.$$

For $d \geqslant 1$, instead of point functionals we consider operators $L_t : \mathscr{C} \to \mathbb{R}^d$ of the form $L_t \mathbf{x} = \mathbf{x}(t)$. Such operators form a d-dimensional Wiener process, i.e., a stochastic process in $\mathbb{R}^d$, whose coordinates are independent Wiener processes. It means that w is still a Gaussian measure with zero mean element and the covariance function (the matrix function in this case) $C : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^d \times \mathbb{R}^d$,

$$C(s,t) = \int_{\mathscr{C}} \mathbf{x}(s)\mathbf{x}^T(t)w(d\mathbf{x}) = \min\{s,t\} I_{\mathbb{R}^d}.$$

We now precisely state the conditions for the class F, which guarantee the existence of the path integral (3). Let F be a normed linear space of functions $f : \mathbb{R}^d \to \mathbb{R}$. We assume that for every $\mathbf{u} \in \mathbb{R}^d$, the functional $L_{\mathbf{u}} : F \to \mathbb{R}$ defined by $L_{\mathbf{u}} f = f(\mathbf{u})$ is continuous, and for arbitrary $a, t \in \mathbb{R}_+$ we have

$$\int_{\mathscr{C}} ||L_{\mathbf{x}(t)}||_F \exp\left(a \int_0^t ||L_{\mathbf{x}(s)}||_F \, ds\right) w(d\mathbf{x}) < \infty. \tag{4}$$

By the Fernique theorem, see e.g. [2], condition (4) holds if there exists $\alpha < 2$ such that $||L_{\mathbf{x}}||_F = O(||\mathbf{x}||^{\alpha})$ for $||\mathbf{x}||$ approaching infinity, see [5] for details. Here $||\mathbf{x}||^2 = \sum_{i=1}^d x_i^2$ is the Euclidean norm in $\mathbb{R}^d$.

As in the previous section we want to compute an ε-approximation of z given by (3). Similarly as there, let $n^{\mathrm{FK}}(\varepsilon, F)$ denote the minimal n sufficient to compute an ε-approximation of $z_{v,V}(\mathbf{u}^*, t^*)$ given by (3). Obviously $n^{\mathrm{FK}}(\varepsilon, F) = n(\varepsilon, F)$ if both the differential equation problem (1), (2) and the right-hand side of (3) are well defined for the class F. We will call $n^{\mathrm{FK}}(\varepsilon, F)$ the (*information*) *worst case complexity* of the multivariate Feynman–Kac path integral. The quantity $n^{\mathrm{FK}}(\varepsilon, F)$ is strongly related to the complexity of the multivariate Feynman–Kac path integration, and it will be discussed in Section 4.3.

The main objective of our paper is to check whether the estimate mentioned in the previous section can be improved and how the exponent of ε^{-1} depends on the class F. In particular, we want to study its dependence on the smoothness of the functions from F and the dimension d. The case $d = 1$ was considered in [5], where it was shown that Feynman–Kac path integration is related to the integration and L_2-approximation. In this paper we study the case $d \geqslant 1$. Our approach will be slightly different and based on the fact that Feynman–Kac path integration is also related to uniform approximation. Hence, even for $d = 1$, our result will be different than the one in [5].

4. Algorithm for approximation of multivariate Feynman–Kac path integral

We derive an algorithm that solves the multivariate Feynman–Kac path integration in a way similar to [5]. Without lost of generality we assume that $\mathbf{u} = \mathbf{0}$ in (3), since we can shift the initial value and potential functions otherwise.

4.1. First step: towards uniform approximation

First we expand the function $\exp(\cdot)$ into the power series

$$S(v, V) = z(\mathbf{0}, t) = \int_{\mathscr{C}} v(\mathbf{x}(t)) \exp\left(\int_0^t V(\mathbf{x}(s))\, ds\right) w(d\mathbf{x}) = \sum_{k=0}^{\infty} S_{k+1}(v, V),$$

where

$$S_{k+1}(v, V) = \frac{1}{k!} \int_{\mathscr{C}} v(\mathbf{x}(t)) \left(\int_0^t V(\mathbf{x}(s))\, ds\right)^k w(d\mathbf{x}).$$

For $k = 0$ the quantity $S_1(v, V)$ is the expectation $\mathbb{E}v(W_t)$, where W_t, for $t \in \mathbb{R}_+$, is the d-dimensional Wiener process. Thus we obtain

$$S_{1,d}(v, V) = \int_{\mathbb{R}^d} v(\mathbf{z}) g_{1,d}(\mathbf{z})\, d\mathbf{z},$$

where

$$g_{1,d}(\mathbf{z}) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{||\mathbf{z}||^2}{2t}\right).$$

Here, as before, $||\cdot||$ denotes the Euclidean norm in $\mathbb{R}^d$.

For $k > 0$, we rewrite the kth power of the inner integral of $S_{k+1}(v, V)$ as a k-variate integral and, by symmetry, we switch it to the integral over the k-dimensional simplex

$$\begin{aligned} S_{k+1}(v, V) &= \int_{\mathscr{C}} v(\mathbf{x}(t)) \int_{0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant t} V(\mathbf{x}(t_1)) \cdots V(\mathbf{x}(t_k))\, dt_1 \ldots dt_k w(d\mathbf{x}) \\ &= \int_{0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant t} \int_{\mathscr{C}} v(\mathbf{x}(t)) V(\mathbf{x}(t_1)) \cdots V(\mathbf{x}(t_k)) w(d\mathbf{x})\, dt_1 \ldots dt_k. \end{aligned}$$

The inner integral

$$\int_{\mathscr{C}} v(\mathbf{x}(t)) \prod_{i=1}^{k} V(\mathbf{x}(t_i)) w(d\mathbf{x})$$

is the expectation $\mathbb{E}[v(\mathrm{W}_t) \prod_{i=1}^{k} V(\mathrm{W}_{t_i})]$. Since the t_i's are increasingly ordered and dominated by t, then using the known form of finite dimensional distributions of the d-dimensional Wiener process, the expectation $\mathbb{E}[v(\mathrm{W}_t) \prod_{i=1}^{k} V(\mathrm{W}_{t_i})]$ is equal to

$$\int_{\mathbb{R}^{(k+1)d}} v(\mathbf{z}_{k+1}) \prod_{i=1}^{k} V(\mathbf{z}_i) f_{k+1,d}(t_1, \ldots, t_k, t, \mathbf{z}_1, \ldots, \mathbf{z}_{k+1})\, d\mathbf{z}_1 \ldots d\mathbf{z}_{k+1},$$

with $\mathbf{z}_i \in \mathbb{R}^d$ and

$$f_{k+1,d}(t_1, \ldots, t_k, t, \mathbf{z}_1, \ldots, \mathbf{z}_{k+1}) = ((2\pi)^{k+1} t_1(t_2 - t_1)\cdots(t - t_k))^{-d/2} \times \exp\left(-\frac{1}{2}\left(\frac{||\mathbf{z}_1||^2}{t_1} + \frac{||\mathbf{z}_2 - \mathbf{z}_1||^2}{t_2 - t_1} + \cdots + \frac{||\mathbf{z}_{k+1} - \mathbf{z}_k||^2}{t - t_k}\right)\right).$$

Hence, after changing the order of the integrals, we finally obtain

$$S_{k+1}(v, V) = \int_{\mathbb{R}^{(k+1)d}} v(\mathbf{z}_{k+1}) \prod_{i=1}^{k} V(\mathbf{z}_i) g_{k+1,d}(\mathbf{z}_1, \ldots, \mathbf{z}_{k+1})\, d\mathbf{z}_1 \ldots d\mathbf{z}_{k+1},$$

where

$$g_{k+1,d}(\mathbf{z}_1, \ldots, \mathbf{z}_{k+1}) = \int_{0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant t} f_{k+1,d}(t_1, \ldots, t_k, t, \mathbf{z}_1, \ldots, \mathbf{z}_{k+1})\, dt_1 \ldots dt_k. \quad (5)$$

The following lemma will be useful for further considerations.

Lemma 1.

$$||g_{k+1,d}||_{L_1(\mathbb{R}^{(k+1)d})} = \frac{t^k}{k!} \quad \textit{for } k \geqslant 0,$$

here, by convention, $0^0 = 1$.

Proof. Since $g_{1,d}$ is the density of a probability measure on $\mathbb{R}^d$ the result for $g_{1,d}$ is obvious. For $k \geqslant 1$, by the non-negativity of the integrand, we can apply the Fubini theorem. For fixed t_i's, $f_{k+1,d}$ is the density of a probability measure on $\mathbb{R}^{(k+1)d}$. Thus the whole integral is equal the volume of the simplex $\{(t_1, \ldots, t_k) \in \mathbb{R}^k\colon 0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant t\}$, which is $t^k/k!$. □

Observe that $g_{k+1,d}$ does not depend on v and V. The only dependence of $S_{k+1}(v, V)$ on v and V belonging to the class F is through the product

$$h_{k+1,d}(\mathbf{z}_1, \ldots, \mathbf{z}_{k+1}) = v(\mathbf{z}_{k+1}) \prod_{i=1}^{k} V(\mathbf{z}_i). \quad (6)$$

As in [5] we remark that $h_{k+1,d}$ belongs to the tensor product space $\overbrace{F \otimes \cdots \otimes F}^{k+1}$. Therefore the function $h_{k+1,d}$ can be approximated by an algorithm for tensor product spaces.

The Smolyak algorithm, see [9], based on the L_2-approximation of v and V in the univariate case was used in [5]. This algorithm can be also applied based on the algorithm for the approximation v and V in the d-variate case. However, the Smolyak algorithm based on L_2-approximation, when applied to $S_{k+1}(v, V)$, requires that $g_{k+1,d}$ belongs to $L_2(\mathbb{R}^{(k+1)d})$. This holds for $d = 1$, and so

L_2-approximation could have been applied in [5]. For $d \geqslant 2$, it can be checked that $g_{k+1,d}$ does *not* belong to $L_2(\mathbb{R}^{(k+1)d})$ and thus we have to base our approach on uniform approximation, which requires that $g_{k+1,d}$ belongs only to $L_1(\mathbb{R}^{(k+1)d})$. By Lemma 1 this holds for arbitrary d. Furthermore, the L_1 norm of $g_{k+1,d}$ is super-exponentially small for large k and for arbitrary d. In the following section we derive a uniform approximation algorithm for functions of the form (6).

4.2. Second step: uniform approximation

So far we have assumed that the class F is a normed linear space of functions $\mathbb{R}^d \to \mathbb{R}$ for which (4) holds and function evaluations are continuous linear functionals. We need to add more constraints on F. First, we assume that F is continuously embedded into $L_\infty(\mathbb{R}^d)$. That is, $F \subset L_\infty(\mathbb{R}^d)$ and there exists a positive K such that

$$||f||_{L_\infty(\mathbb{R}^d)} \leqslant K||f||_F \quad \forall f \in F. \tag{7}$$

Second, we assume that the complexity $n^{\mathrm{APP}}(\varepsilon, F)$ of uniform approximation of functions from the class F satisfies

$$n^{\mathrm{APP}}(\varepsilon, F) = O(\varepsilon^{-\alpha(F)}) \quad \text{as } \varepsilon \to 0 \tag{8}$$

for some positive constant $\alpha(F)$. Moreover, we assume the existence of an almost optimal algorithm that uses only function values. These conditions hold for many classes F.

We illustrate these assumptions by two examples.

Example 1. Consider the Sobolev space $W^r_\infty(\mathbb{R}^d)$ with $r > 0$, equipped with the following norm:

$$||f|| = \max_{|\alpha| \leqslant r} ||f^{(\alpha)}||_\infty.$$

Obviously, $W^r_\infty(\mathbb{R}^d)$ can be continuously embedded into $L_\infty(\mathbb{R}^d)$, and function evaluations are continuous. It is known that $\alpha(W^r_\infty(\mathbb{R}^d)) = d/r$ and, by [4], there exists an almost optimal algorithm that uses function values only.

Example 2. Following [7], we consider the class F of 2π-periodic functions $f : \mathbb{R}^d \to \mathbb{R}$ satisfying the condition

$$\forall f \in F \quad \forall j = 1, \ldots, d \quad \exists \varphi_j \in L_\infty([-2\pi, 2\pi]^d)$$

$$f(\mathbf{x}) = \frac{1}{2\pi} \int_0^{2\pi} \varphi_j(x_1, \ldots, x_j - t, \ldots, x_d) P_r(t)\, dt, \tag{9}$$

where $r > 1$ and

$$P_r(t) = 1 + 2 \sum_{k=0}^{\infty} k^{-r} \cos(kt - r\pi/2).$$

We define the norm in the class F as

$$||f||_F = \sum_{j=1}^{d} ||\varphi_j||_{L_\infty([-2\pi,2\pi]^d)},$$

where, for given f, the φ_j are the functions from representation (9). It is easy to see that for $f \in F$ and $\mathbf{z} \in \mathbb{R}^d$ and arbitrary $j \in \{1, \ldots, d\}$ we have

$$|f(\mathbf{z})| \leqslant ||f||_{L_\infty(\mathbb{R}^d)} = ||f|_{[0,2\pi]^d}||_{L_\infty([0,2\pi]^d)} \leqslant C||\varphi_j||_{L_\infty([-2\pi,2\pi]^d)},$$

with $C = \frac{1}{2\pi}\int_0^{2\pi} |F_r(t)|\, dt$. Hence

$$|f(\mathbf{z})| \leqslant ||f||_{L_\infty(\mathbb{R}^d)} \leqslant \frac{C}{d}||f||_F,$$

from which our assumptions are satisfied. From [7], we know the existence of the algorithm that computes uniform approximation of functions from F with the cost of order $\varepsilon^{-d/r}$. Hence, in this case also $\alpha(F) = d/r$.

Now we outline the algorithm for uniform approximation of elements of the tensor product space. This is well-known Smolyak's construction whose applicability to our case follows from [3].

Lemma 2. *Assume that $h_{k+1,d}$ is defined by* (6), *$v, V \in F$ and F satisfies* (7) *and* (8). *Then for $\varepsilon > 0$ and $k \geqslant 0$, there exists an algorithm*

$$U_{\varepsilon,k+1}(h_{k+1,d}) = \sum_{i=1}^{n(\varepsilon,k+1)} h_{k+1,d}(\mathbf{t}_{i,\varepsilon,1}, \ldots, \mathbf{t}_{i,\varepsilon,k+1})\zeta_{i,\varepsilon,k+1}, \tag{10}$$

where $n(\varepsilon, k+1) \in \mathbb{N}_+$, $\mathbf{t}_{i,\varepsilon,j} \in \mathbb{R}^d$, $i = 1, \ldots, n(\varepsilon, k+1)$, $j = 1, \ldots, d$, *and* $\zeta_{i,\varepsilon,k+1} \in L_\infty(\mathbb{R}^{(k+1)d})$, *such that*

$$||h_{k+1,d} - U_{\varepsilon,k+1}(h_{k+1,d})||_{L_\infty(\mathbb{R}^{(k+1)d})} \leqslant \varepsilon||v||_F||V||_F^k, \tag{11}$$

and

$$n(\varepsilon, k+1) \leqslant c_0\left(c_1 + c_2\frac{\ln 1/\varepsilon}{k}\right)_+^{(\alpha(F)+1)k} \varepsilon^{-\alpha(F)} \tag{12}$$

for some $c_i \in \mathbb{R}$. Here a_+ denotes $\max\{a, 0\}$, *and when $k = 0$, the right-hand side of* (12) *is defined to be $c_0\varepsilon^{-\alpha}$.*

Proof. We first prove the following fact using [3, Remark 1].

Let $S_i : F \to L_\infty(\mathbb{R}^d)$, $i = 1, \ldots, n$, be continuous linear operators. Let H be a Banach space containing $F^{\otimes n}$ as dense subspace. Additionally, we assume that $||\cdot||_H$ satisfies

$$||v_1 \otimes \cdots \otimes v_n||_H = \prod_{i=1}^{n} ||v_i||_F,$$

where, as usual, $(v_1 \otimes \cdots \otimes v_n)(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \prod_{i=1}^{n} v_i(\mathbf{x}_i)$. Then, the norm of the operator $S_1 \otimes \cdots \otimes S_n$, defined as $(S_1 \otimes \cdots \otimes S_n)(v_1 \otimes \cdots \otimes v_n)(\mathbf{x}_1, \ldots \mathbf{x}_n) = \prod_{i=1}^{n} S_i(v_i)(\mathbf{x}_i)$, satisfies

$$||S_1 \otimes \cdots \otimes S_n||_{H \to L_\infty(\mathbb{R}^{nd})} \leqslant \prod_{i=1}^{n} ||S_i||. \tag{13}$$

Here $||\cdot||$ is the ordinary operator norm.

Indeed, by [3, Theorem 1] we see that (13) holds for functionals S_i, i.e., when the range spaces are $\mathbb{R}$ instead of $L_\infty(\mathbb{R}^d)$. Write $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_d)$, then by [3, Remark 1] we have

$$\begin{aligned}
&||S_1 \otimes \cdots \otimes S_n||_{H \to L_\infty(\mathbb{R}^{nd})} \\
&\quad = \sup_{||\xi||_H \leqslant 1} ||S_1 \otimes \cdots \otimes S_n(\xi)||_{L_\infty(\mathbb{R}^{nd})} \\
&\quad = \sup_{||\xi||_H \leqslant 1} \sup_{f_j \in L_1(\mathbb{R}^d), ||f_j||_1 \leqslant 1} \int_{\mathbb{R}^{nd}} (S_1 \otimes \cdots \otimes S_n(\xi))(\mathbf{x}) f_1(\mathbf{x}_1) \cdots f_n(\mathbf{x}_n)\, d\mathbf{x} \\
&\quad = \sup_{||\xi||_H \leqslant 1} \sup_{||f_j||_1 \leqslant 1} (T_{f_1} \otimes \cdots \otimes T_{f_n})(\xi),
\end{aligned}$$

where $T_{f_i} : F \to \mathbb{R}$ are linear functionals defined as

$$T_{f_i}(v_i) = \int_{\mathbb{R}^d} S_i(v_i)(\mathbf{x}_i) f_i(\mathbf{x}_i)\, d\mathbf{x}_i.$$

We can now use the result mentioned above for linear functionals to complete the proof of (13).

$$\begin{aligned}
&||S_1 \otimes \cdots \otimes S_n||_{H \to L_\infty(\mathbb{R}^{nd})} \\
&\quad = \sup_{||f_j||_1 \leqslant 1} \sup_{||\xi||_H \leqslant 1} (T_{f_1} \otimes \cdots \otimes T_{f_n})(\xi) \\
&\quad = \sup_{||f_j||_1 \leqslant 1} ||T_{f_1} \otimes \cdots \otimes T_{f_n}|| \leqslant \sup_{||f_j||_1 \leqslant 1} \prod_{i=1}^{n} ||T_{f_i}|| \\
&\quad \leqslant \sup_{||f_j||_1 \leqslant 1} \prod_{i=1}^{n} ||S_i||\, ||f_i||_1 = \prod_{i=1}^{n} ||S_i||.
\end{aligned}$$

Now, assume that $n = k + 1$ and H is, for instance, the projective tensor product, see [3], of n copies of F. We can then apply the Smolyak algorithm for H to approximate the embedding operator into $L_\infty(\mathbb{R}^{nd})$, in particular, to approximate the function $h_{k+1,d}(\mathbf{z}_1, \ldots, \mathbf{z}_{k+1})$. By the fact proven above and [9, Theorem 1] the cost bound (12) holds for our case. □

4.3. Final step: algorithm, error and cost analysis

We are now ready to present the final construction step of the algorithm for approximating $S(v, V)$. We first consider approximating $S_{k+1}(v, V)$ by the algorithm

$$\phi_{\varepsilon,k+1}(v, V) = \int_{\mathbb{R}^{(k+1)d}} (U_{\varepsilon,k+1} h_{k+1,d}) g_{k+1,d}(\mathbf{z}_1, \ldots, \mathbf{z}_{k+1})\, d\mathbf{z}_1 \ldots d\mathbf{z}_{k+1},$$

where $h_{k+1,d}$ is defined by (6) and $g_{k+1,d}$ by (5). Using (10) we can rewrite $\phi_{\varepsilon,k+1}$ as

$$\phi_{\varepsilon,k+1}(v, V) = \sum_{i=1}^{n(\varepsilon,k+1)} h_{k+1,d}(\mathbf{t}_{i,\varepsilon,1}, \ldots, \mathbf{t}_{i,\varepsilon,k+1}) \boldsymbol{\zeta}_{i,\varepsilon,k+1}, \tag{14}$$

where

$$\boldsymbol{\zeta}_{i,\varepsilon,k+1} = \int_{\mathbb{R}^{(k+1)d}} \zeta_{i,\varepsilon,k+1}(\mathbf{z}_1, \ldots, \mathbf{z}_{k+1}) g_{k+1,d}(\mathbf{z}_1, \ldots, \mathbf{z}_{k+1})\, d\mathbf{z}_1 \ldots d\mathbf{z}_{k+1}. \tag{15}$$

The following lemma gives an estimate of the error of $\phi_{\varepsilon,k+1}$.

Lemma 3. *The error of $\phi_{\varepsilon,k+1}$ satisfies*

$$|S_{k+1}(v, V) - \phi_{\varepsilon,k+1}(v, V)| \leqslant \varepsilon \frac{||V||_F^k ||v||_F t^k}{k!}.$$

Proof.

$$\begin{aligned}
&|S_{k+1}(v, V) - \phi_{\varepsilon,k+1}(v, V)| \\
&\quad \leqslant \int_{\mathbb{R}^{(k+1)d}} |(h_{k+1,d} - U_{\varepsilon,k+1} h_{k+1,d}) g_{k+1,d}(\mathbf{z}_1, \ldots, \mathbf{z}_{k+1})|\, d\mathbf{z}_1 \ldots d\mathbf{z}_{k+1} \\
&\quad \leqslant ||h_{k+1,d} - U_{\varepsilon,k+1} h_{k+1,d}||_{L_\infty(\mathbb{R}^{(k+1)d})} ||g_{k+1,d}||_{L_1(\mathbb{R}^{(k+1)d})} \\
&\quad \leqslant \varepsilon \frac{||V||_F^k ||v||_F t^k}{k!}. \qquad \square
\end{aligned}$$

To define the final algorithm that computes an ε-approximation of $S(v, V)$, we need to restrict ourselves to the following class of functions. Let $\boldsymbol{\beta} = (\beta_1, \beta_2) \in \mathbb{R}^2$ and $\beta_i > 0$. Define

$$F_{\boldsymbol{\beta}} = \{(f_1, f_2) \in F \times F\colon\ ||f_1||_F \leqslant \beta_1, ||f_2||_F \leqslant \beta_2\}, \tag{16}$$

where, as before, F stands for the normed linear space of functions defined over $\mathbb{R}^d$, continuously embedded into $L_\infty(\mathbb{R}^d)$ and satisfying (4). For $(v, V) \in F_{\boldsymbol{\beta}}$, we now define

$$\Phi_\varepsilon(v, V) = \sum_{k=0}^{\infty} \phi_{\varepsilon_{k+1},k+1}(v, V), \tag{17}$$

where

$$\varepsilon_{k+1} = \frac{\varepsilon k!}{\beta_1 \beta_2^k t^k 2^{k+1}}. \tag{18}$$

By Lemma 3 we have

$$|\Phi_\varepsilon(v, V) - S(v, V)| \leqslant \sum_{k=0}^{\infty} \varepsilon_{k+1} \frac{||v||_F ||V||_F^k t^k}{k!} \leqslant \sum_{k=0}^{\infty} \frac{\varepsilon}{2^{k+1}} = \varepsilon, \tag{19}$$

and hence the absolute error of Φ_ε is at most ε. Formally Φ_ε is defined as an infinite sum. However, observe that ε_{k+1} grows super-exponentially fast to infinity with k. As we have already remarked $\phi_{\varepsilon,k+1} \equiv 0$ is the zero algorithm for $\varepsilon_{k+1} \geqslant K$ and we have only finitely many non-zero terms in (17). Moreover, it is easy to see that the number of non-zero terms in (17) depends logarithmically on ε^{-1}.

The cost of Φ_ε is estimated in the following theorem.

Theorem 1. *Let $n(\Phi_\varepsilon)$ denote the (information) cost of the algorithm Φ_ε. Then for F satisfying* (4), (7) *and* (8) *we have*

$$n(\Phi_\varepsilon) \leqslant K_0 C(\varepsilon) \varepsilon^{-\alpha(F)}, \tag{20}$$

where

$$K_0 = 2c_0(2t\beta_1)^{\alpha(F)},$$

$$C(\varepsilon) \leqslant 2 + \sum_{k=1}^{\infty} \frac{K_1^{k\alpha(F)}}{((k+1)!)^{\alpha(F)}} \times \left(K_2 + c_2 \frac{\ln(\varepsilon^{-1}) + \ln(2\beta_1) - \ln((k+1)!)}{k} \right)_+^{(\alpha(F)+1)k}$$

with

$$K_1 = 2t\beta_2, \quad \textit{and} \quad K_2 = c_1 + c_2 \ln(2\beta_2).$$

Moreover

$$n(\Phi_\varepsilon) = O(\varepsilon^{-\alpha(F)-\delta}) \quad \forall \delta > 0. \tag{21}$$

Proof. We only sketch the proof because of its similarity to the proof of Theorem 1 from [5]. First, we have

$$n(\Phi_\varepsilon) \leqslant \sum_{k=0}^{\infty} n(\varepsilon_{k+1}, k+1).$$

Using (12) and (18), we obtain the bound in (20). To prove (21), it suffices to show that $C(\varepsilon) = O(\varepsilon^{-\delta})$ holds for all positive δ. Using the Stirling formula

$$k! = \sqrt{2\pi k} \left(\frac{k}{e}\right)^k e^{\theta(k)/(12k)} \quad \text{for some } 0 < \theta(k) < 1,$$

we see that there exists a constant $K_3 > 0$, such that

$$C(\varepsilon) \leqslant 2 + \sum_{k=1}^{\infty} \frac{K_3^{2k(\alpha(F)+1)}}{k^{k\alpha(F)}} \left(1 + \frac{\ln(\varepsilon^{-1})}{k}\right)^{2k(\alpha(F)+1)},$$

and the same argumentation as in [5] yields (21). □

Remark 1. In Theorem 1 the information cost of the algorithm Φ_ε was determined. The combinatorial cost of Φ_ε satisfies

$$n^{\mathrm{comb}}(\Phi_\varepsilon) \leqslant \sum_{k=0}^{\infty} 2(k+1)n(\varepsilon_{k+1}, k+1). \tag{22}$$

It is easy to see that by using similar technique as in the proof of Theorem 1, we can obtain the estimate $n^{\mathrm{comb}}(\Phi_\varepsilon) = O(\varepsilon^{-\alpha(F)-\delta})$ for any positive δ. Thus the total cost of our algorithm also satisfies this estimate.

5. Complexity of multivariate Feynman–Kac path integration

The analysis of the complexity of the multivariate Feynman–Kac path integration is quite similar to the one presented in [5]. We can bound the complexity of the multivariate Feynman–Kac path integration by complexities of two auxiliary problems—uniform approximation and weighted integration. Note, that although the problem that we are dealing with is non-linear, the problems establishing bounds of its complexity are linear. Let us define these problems precisely.

The uniform approximation problem APP is defined as follows. For an arbitrary function $f \in F$, we compute an ε-approximation of $A : F \to L_\infty(\mathbb{R}^d)$ defined as

$$Af = f \quad \forall f \in F.$$

In this case ε-approximation is given as

$$C_n f = \sum_{i=1}^{n} f(\mathbf{t}_{i,n}) g_{i,n},$$

where $\mathbf{t}_{i,n} \in \mathbb{R}^d$ and $g_{i,n} \in F$ are fixed, and

$$e^{\mathrm{APP}}(C_n, F) = \sup_{f \in F} \frac{||Af - C_n f||_{L_\infty(\mathbb{R}^d)}}{||f||_F} \leqslant \varepsilon.$$

The weighted integration problem INT is defined as follows. For an arbitrary function $f \in F$, we compute an ε-approximation of $I : F \to \mathbb{R}$ defined as

$$If = (2\pi t)^{-d/2} \int_{\mathbb{R}^d} f(\mathbf{u}) \exp(-||\mathbf{u}||/(2t))\, d\mathbf{u}.$$

Now, ε-approximation is given as

$$B_n f = \sum_{i=1}^{n} f(\boldsymbol{\tau}_{i,n}) a_{i,n},$$

where $\boldsymbol{\tau}_{i,n} \in \mathbb{R}^d$ and $a_{i,n} \in \mathbb{R}$ and

$$e^{\mathrm{INT}}(B_n, F) = \sup_{f \in F} \frac{|If - B_n f|}{||f||_F} \leqslant \varepsilon.$$

Similarly to Section 2, we define $n^{\mathrm{INT}}(\varepsilon, F)$ and $n^{\mathrm{APP}}(\varepsilon, F)$ as the minimal n such that algorithms B_n and C_n of the forms mentioned above give ε-approximations of INT and APP, respectively. Define $p^*(F)$ and $q^*(F)$ as

$$p^*(F) = \sup\{p\colon\ \exists B_n\ e^{\mathrm{INT}}(B_n, F) = O(n^{-p}) \text{ as } n \to \infty\},$$

$$q^*(F) = \sup\{q\colon\ \exists C_n\ e^{\mathrm{APP}}(C_n, F) = O(n^{-q}) \text{ as } n \to \infty\}.$$

Obviously, the quantities $p^*(F)$ and $q^*(F)$ yield the bounds

$$\Omega(\varepsilon^{-1/p^*(F)+\delta}) = n^{\mathrm{INT}}(\varepsilon, F) = O(\varepsilon^{-1/p^*(F)-\delta}), \tag{23}$$

$$\Omega(\varepsilon^{-1/q^*(F)+\delta}) = n^{\mathrm{APP}}(\varepsilon, F) = O(\varepsilon^{-1/q^*(F)-\delta}) \tag{24}$$

for any positive δ.

5.1. Lower bound

It is easy to see that computing of an ε-approximation of the multivariate Feynman–Kac path integral is not easier than solving the problem INT. Indeed, by setting $V \equiv 0$ in formula (3) and using the Wiener measure property we obtain

$$S(v, 0) = Iv.$$

Since $(v, V) \in F_{\boldsymbol{\beta}}$, we have $||v||_F \leqslant \beta_1$. Thus, by setting $V \equiv 0$, the multivariate Feynman–Kac path integration can be reduced to the rescaled INT problem. Obviously, this scaling has no influence on the asymptotic dependence of the complexity of the problem INT on ε^{-1}. Then by (23) we obtain

$$\Omega(\varepsilon^{-1/p^*(F)+\delta}) = n^{\mathrm{FK}}(\varepsilon, F_{\boldsymbol{\beta}}) \quad \forall \delta > 0.$$

5.2. Upper bound

Obviously, the cost of every particular algorithm that computes an ε-approximation of multivariate Feynman–Kac path integration is an upper bound of the complexity of this problem. Thus, we have

$$n^{\mathrm{FK}}(\varepsilon, F_{\boldsymbol{\beta}}) \leqslant n(\Phi_\varepsilon). \tag{25}$$

Observe that each algorithm C_n for the approximation problem APP can be used as the basic algorithm for Smolyak's construction $U_{\varepsilon,k+1}$ from Lemma 2 leading to $\phi_{\varepsilon,k+1}$ and Φ_ε, see Sections 4.1 and 4.2 for details. By Theorem 1, we then obtain

$n(\Phi_\varepsilon) = O(\varepsilon^{-\alpha(F)-\delta})$. Thus, by (24) and (25), we finally obtain

$$n^{\mathrm{FK}}(\varepsilon, F_{\boldsymbol{\beta}}) = O(\varepsilon^{-1/q^*(F)-\delta}) \quad \forall \delta > 0.$$

This proves that the approximation problem APP is essentially no harder than the multivariate Feynman–Kac path integration.

From Sections 5.1 and 5.2 we finally obtain the lower and upper bounds for the information complexity of multivariate Feynman–Kac path integration, namely

$$\Omega(\varepsilon^{-1/p^*(F)+\delta}) = n^{\mathrm{FK}}(\varepsilon, F_{\boldsymbol{\beta}}) = O(\varepsilon^{-1/q^*(F)-\delta}) \quad \forall \delta > 0.$$

By Remark 1 such an estimate holds for the complexity of our problem. Thus our result is analogous with the result from [5].

We now illustrate the theoretical results of the previous sections by considering the two examples from Section 4.2.

Example 3. We consider the classes F defined in Examples 1 and 2. It turns out that for both of them the complexities of the problems INT and APP are of order $\varepsilon^{-d/r}$. Thus the algorithm Φ_ε, based on the almost optimal algorithms of the uniform approximation for these classes, computes an ε-approximation of the multivariate Feynman–Kac path integral with cost of order $\varepsilon^{-d/r}$ and is almost optimal for these classes.

Acknowledgments

The first author wishes to express sincere thanks to his advisor H. Woźniakowski for numerous hours of consultation and for invaluable remarks and suggestions. The authors are also grateful to J. F. Traub, G. W. Wasilkowski and A. G. Werschulz for their valuable comments.

References

[1] I. Karatzas, S.E. Shreve, Brownian Motion and Stochastic Calculus, Springer, Berlin, 1988.
[2] H.-H. Kuo, Gaussian Measures in Banach Spaces, Springer, Berlin, 1995.
[3] Y. Li, Applicability of Smolyak's algorithm to certain Banach spaces of multivariate functions, J. Complexity 18 (2002) 792–814.
[4] Y. Li, G.W. Wasilkowski, Worst case complexity of weighted integration and approximation over $\mathbb{R}^d$, J. Complexity 18 (2002) 330–345.
[5] L. Plaskota, G.W. Wasilkowski, H. Woźniakowski, A new algorithm and worst case complexity for Feynman–Kac path integration, J. Comp. Phys. 164 (2000) 335–353.
[6] D. Revuz, M. Yor, Continuous Martingales and Brownian Motion, Springer, Berlin, 1991.
[7] V.N. Temlyakov, On approximate recovery of functions with bounded mixed derivative, J. Complexity 9 (1993) 41–59.
[8] J.F. Traub, G.W. Wasilkowski, H. Woźniakowski, Information-Based Complexity, Academic Press, New York, 1988.
[9] G.W. Wasilkowski, H. Woźniakowski, Explicit cost bounds for multivariate tensor product problems, J. Complexity 11 (1995) 1–56.