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The Computational Complexity of Asymptotic Problems I: Partial Orders

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The class of partial orders is shown to have 0–1 laws for first-order logic and for inductive fixed-point logic, a logic which properly contains first-order logic. This means that for every sentence in one of these logics the proportion of labeled (or unlabeled) partial orders of size n satisfying the sentence has a limit of either 0 or 1 as n goes to ∞. This limit, called the *asymptotic probability* of the sentence, is the same for labeled and unlabeled structures. The computational complexity of the set of sentences with asymptotic probability 1 is determined. For first-order logic, it is *PSPACE*-complete. For inductive fixed-point logic, it is *EXPTIME*-complete.

1. INTRODUCTION

Let $\mathscr{C}$ be a class of structures of the same similarity type and suppose $\mathscr{C}$ is closed under isomorphism. Define $\mathscr{C}_n$ to be the set of structures in $\mathscr{C}$ with universe $n = \{0, 1, ..., n-1\}$. Structures in $\bigcup_{n \geq 0} \mathscr{C}_n$ are called the *labeled structures* of $\mathscr{C}$. For a sentence φ in some given logic, let $\mu_n^{\mathscr{C}}(\varphi)$ be the fraction of structures in $\mathscr{C}_n$ which satisfy φ and $\nu_n^{\mathscr{C}}(\varphi)$ be the fraction of isomorphism types (or *unlabeled structures*) in $\mathscr{C}_n$ which satisfy φ. Define the *labeled asymptotic probability* of φ, denoted $\mu^{\mathscr{C}}(\varphi)$, to be the limit of $\mu_n^{\mathscr{C}}(\varphi)$ as n increases if that limit exists (otherwise it is undefined). Define the *unlabeled asymptotic probability* of φ, denoted $\nu^{\mathscr{C}}(\varphi)$, similarly. We say that $\mathscr{C}$ has a *labeled* 0–1 *law* for a given logic if for every sentence φ in the logic, $\mu^{\mathscr{C}}(\varphi)$ is defined and equal to either 0 or 1; similarly, $\mathscr{C}$ has a *unlabeled* 0–1 *law* for a given logic if for every sentence φ in the logic, $\nu^{\mathscr{C}}(\varphi)$ is defined and equal to either 0 or 1. Also, we will say that a sentence φ holds *almost surely for labeled structures* in $\mathscr{C}$ when $\mu^{\mathscr{C}}(\varphi) = 1$, and *almost surely for unlabeled structures* in $\mathscr{C}$ when $\nu^{\mathscr{C}}(\varphi) = 1$.

Möhring (1985) asked if the class of partial orders has a labeled first-order 0–1 law. We answer this question affirmatively, showing, in fact, that the class of partial orders has both a labeled and unlabeled 0–1 law for

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inductive fixed-point logic, a logic that properly contains first-order logic, and that $\mu_n^{\mathscr{C}}(\varphi)$ and $\nu_n^{\mathscr{C}}(\varphi)$ converge to the same value for all sentences φ in this logic. Moreover, we determine the computational complexity of the set of sentences φ such that $\mu_n^{\mathscr{C}}(\varphi)$ converges to 1. For first-order logic, this set is *PSPACE*-complete; for inductive fixed-point logic, this set is *EXPTIME*-complete.

When $\mathscr{C}$ is the class of all structures of a given relational similarity type, rather than the class of all partial orders, the analogous results hold. Glebskiĭ, Kogan, Liogon'kiĭ, and Talanov (1969) proved that a labeled first-order 0–1 law pertains and Liogon'kiĭ (1970) proved that an unlabeled first-order 0–1 law pertains. Fagin (1976) later gave a different, simpler proof of the labeled first-order 0–1 law. He showed that certain *extension axioms* (this term is explained in Section 3) constituting a complete theory hold almost surely for the class of all labeled structures, and hence that every sentence following from these axioms holds almost surely. Fagin (1976) also proved an unlabeled 0–1 law using a result in Fagin (1977) stating that if a_n is the number of labeled structures of size n and b_n is the number of unlabeled structures of size n (for some fixed similarity type), then $b_n \sim a_n/n!$ (see also Oberschelp (1968)); in particular, almost every finite relational structure is rigid (i.e., has no automorphisms other than the trivial one). Our proof uses a similar result, due to Prömel (1987), which asymptotically relates the number of labeled and unlabeled structures in certain classes, including the class of partial orders.

Grandjean (1983) showed that when $\mathscr{C}$ is the class of all relational structures of a given similarity type, the set of first-order sentences φ such that $\mu_n^{\mathscr{C}}(\varphi)$ converges to 1 is *PSPACE*-complete. He also derived upper and lower bounds for the complexity of this theory in terms of the maximum arity in the similarity type. His bounds for arity 2 can easily be shown to hold for the class of partial orders if one uses some facts from this paper, but we will present only the proof of *PSPACE*-completeness.

Blass, Gurevich, and Kozen (1985) proved both a labeled and unlabeled 0–1 law for inductive fixed-point logic where $\mathscr{C}$ is the class of all relational structures of a given similarity type, and also showed the set of inductive fixed-point sentences φ such that $\mu_n(\varphi) = 1$ is *EXPTIME*-complete. Both these results and Grandjean's results depend on considerations concerning the extension axioms used by Fagin.

Our work was motivated by a paper of Kleitman and Rothshild (1975) giving an asymptotic estimate for the number of labeled partial orders. Their analysis gives a great deal of information about the structure of random partial orders. In particular, we show that it implies that certain extension axioms for partial orders, similar to those used by Fagin in the case of relational structures, hold almost surely. Our results then follow in the same way as the analogous results for relational structures.

We will use a broader definition of inductive fixed-point logic than was used by Blass, Gurevich, and Kozen (1985); our definition, which is based on the notation for inductive definitions of Moschovakis (1974a), does not increase expressiveness of the logic, but does allow for more succinct expressions. Since our proof works for relational structures, we slightly improve the upper bound result of Blass, Gurevich, and Kozen. We also show how to correct an error in their proof of the upper bound result.

2. Logics with Fixed-Point Operators

In this section we define two logics with fixed-point operators: least fixed-point logic and inductive fixed-point logic. These logics incorporate inductive definitions into first-order logic; here inductively defined relations are fixed-points of monotone or inflationary operators in the logics (see below).

The study of inductive definitions has a long history in recursion theory (see the survey article of Aczel (1977)). Moschovakis (1974a) was the first to study inductive definitions in a general setting. He regarded inductively defined sets as least fixed-points of first-order definable monotone operators. Aho and Ullman (1979), considering database languages, proposed adding a least fixed-point opeator to the syntax of first-order logic. Immerman (1986) and Vardi (1982) showed that least fixed-point logic "captures" polynomial time computability on structures with an underlying linear order. That is, C is the class of structures satisfying a least fixed-point sentence if and only if C is polynomial time computable.

Inductive definitions using inflationary operators (sometimes called non-monotone inductive definitions) also originated in recursion theory (see Aczel (1977)) and again Moschovakis (1974b) was the first to study them in a general setting. Gurevich (1984) proposed adding inflationary operators to the syntax of first-order logic and suggested the name *inductive fixed-point logic*. Gurevich and Shelah (1985) showed that least fixed-point logic and inductive fixed-point logic have the same expressive power on finite structures. (This is not the case for infinite structures. In particular, extended ordinal notations, which first motivated recursion theorists to use inflationary rather than monotone operators, cannot be expressed in least fixed-point logic. See Aczel (1977).)

We describe how to build formulas of a fixed-point logic for a given language of constant, relation, and function symbols. Our notation is based on the notation used by Moschovakis (1974a) for inductive definitions. We believe this notation is a little clearer than has been used in other places (say in Immerman (1986) or Gurevich (1984)) and in some cases it may be more concise.

Formulas for a fixed-point logic over a language may contain any of the symbols occurring in first-order formulas over the language and possibly some *relation variables* P_i^j, where $i, j \geqslant 0$. In each case the arity of P_i^j is j and the subscript and superscript of P_i^j are expressed in binary notation. We define formulas φ inductively, and at the same time define free(φ), the set of free variables in φ. An *atomic formula* φ is either a first-order atomic formula or a formula $P_i^j(x_1, ..., x_j)$; in the former free(φ) is the same as in first-order logic; in the latter, $\text{free}(\varphi) = \{P_i^j, x_1, ..., x_j\}$. More complex formulas φ may be constructed using the standard logical connectives and first-order quantifiers. In these cases free(φ) is defined just as in first-order logic. The only other way to construct more complex formulas is to make an *implicit definition.* Let ψ and θ be formulas, P_i^j a relation variable, and $x_1, ..., x_j$ a sequence of distinct element variables. Then φ given by

$$[P_i^j(x_1, ..., x_j) \equiv \theta]\ \psi$$

is also a formula, where we require in *least fixed-point logic* that P_i^j occurs only positively in θ (i.e., every occurrence is within the scope of an even number of negations) and in *inductive fixed-point logic* that θ is of the form $P_i^j(x_1, ..., x_j) \vee \theta'$. In both cases

$$\text{free}(\varphi) = ((\text{free}(\theta) - \{x_1, ..., x_j\}) \cup \text{free}(\psi)) - \{P_i^j\}.$$

The part of φ within brackets *implicitly defines* the interpration of P_i^j in ψ. Let us make this idea precise.

Fix a structure M and assign values in M to the symbols in free(φ)–i.e., assign elements of M to element variables and relations on M of the appropriate arity to relation variables. For every such assignment, $\theta = \theta(P_i^j, x_1, ..., x_j)$ defines an operator F on the set of j-ary relations on M: $F(R) = \{(a_1, ..., a_j) \mid M \models \theta(R, a_1, ..., a_j)\}$. If P_i^j occurs only positively in θ, F is *monotone*; i.e., $F(R) \subseteq F(R')$ whenever $R \subseteq R'$. Let $F^0(R) = R$, $F^{\alpha+1}(R) = F(F^\alpha(R))$ and if α is a limit ordinal, $F^\alpha(R) = \bigcup_{\beta<\alpha} F^\beta(R)$. By induction $F^\alpha(\varnothing) \subseteq F^\beta(\varnothing)$ whenever $\alpha < \beta$. There must be an ordinal κ such that $F^\alpha(\varnothing) = F^\kappa(\varnothing)$ whenever $\alpha \geqslant \kappa$. Thus, $F^\kappa(\varnothing)$ is a fixed-point for F; it is easy to prove that it is the least fixed-point. Then φ is true in M (at the given assignment) just in case ψ is true in M when $F^\kappa(\varnothing)$ interprets P_i^j (all other free variables interpreted as in the assignment). This describes the semantics for φ in least fixed-point logic.

If θ is of the form $P_i^j(x_1, ..., x_j) \vee \theta'$, F may no longer be monotone, but it is, in the terminology of Gurevich and Shelah (1985) *inflationary*; i.e., it is always the case that $R \subseteq F(R)$. Therefore, it is still true that $F^\alpha(\varnothing) \subseteq F^\beta(\varnothing)$ whenever $\alpha < \beta$ and that there is an ordinal κ such that $F^\alpha(\varnothing) = F^\kappa(\varnothing)$ whenever $\alpha \geqslant \kappa$. Again, $F^\kappa(\varnothing)$ is a fixed-point for F (although not necessarily the least fixed-point) so we use it to interpret P_i^j

in ψ as in the preceding paragraph. This describes the semantics for φ in inductive fixed-point logic.

A *sentence* (in either logic) is a formula φ with free$(\varphi) = \varnothing$. We also use the notation

$$[P_i^j(x_1, ..., x_j) \equiv \theta]_m \psi,$$

where m is a non-negative integer, to indicate that $F^m(\varnothing)$ interprets P_i^j in ψ. As m increases we obtain better approximations to the fixed-point interpretation. Notice that a sentence which is formed using only connectives, quantifiers, and these approximations is equivalent to a first-order sentence since $F^m(\varnothing)$ is first-order definable.

As an example, consider the language containing just a binary relation symbol E which we take to denote the edge relation on the class of graphs. The least fixed-point sentence

$$[P_0^2(x, y) \equiv (x = y \ \vee \exists z(P_0^2(x, z) \wedge E(z, y))] \, \forall x, y P_0^2(x, y)$$

asserts that a graph is connected. The relation variable P_0^2 interprets the path relation in each graph: it holds between two points precisely when there is a path between them. The sentence

$$[P_0^2(x, y) \equiv (x = y \vee \exists z(P_0^2(x, z) \wedge E(z, y))]_m \, \forall x, y P_0^2(x, y)$$

asserts that the graph has diameter at most $m - 1$; i.e., the distance between every pair of points is at most $m - 1$.

Since least fixed-point logic and inductive fixed-point logic have the same expressiveness on finite structures, proof of a 0–1 law for one establishes it for the other. We must be more careful when considering computational complexity. Every sentence φ in least fixed-point logic is equivalent to a sentence in inductive fixed-point logic whose length is less than twice the length of φ and which can be computed from φ in linear time. Simply replace every implicit definition $[P_i^j(x_1, ..., x_j) \equiv \theta]$ in φ with $[P_i^j(x_1, ..., x_j) \equiv P_i^j(x_1, ..., x_j) \vee \theta]$. The resulting sentence has the same truth value as φ. The other direction is much difficult and not so easily analyzed. However, the proofs we give work equally well for both logics. For simplicity, we state all our results for inductive fixed-point logic.

3. The 0–1 Laws for Partial Orders

We begin by specifying a first-order theory T_{as} in the language of partial orders. (The *as* subscript stands for *almost sure.*) First, T_{as} contains the usual axioms for partial orders. Next, it contains an axiom asserting that there are no clains of length greater than three:

$$\forall x_0, x_1, x_2, x_3 \left(\bigwedge_{0 \leqslant i \leqslant 2} x_i \leqslant x_{i+1} \rightarrow \bigvee_{0 \leqslant i \leqslant 2} x_i = x_{i+1} \right).$$

Every partial order satisfying this axiom can be partitioned into three levels L_0, L_1, and L_2. The least elements in the partial order constitute L_0; those elements lying above elements in L_0, but not above elements in any other level, constitute L_1; those elements lying above elements in L_1 constitute L_2. Clearly, L_0, L_1, and L_2 are first-order definable. We now describe the extension axioms. For every j, k, $l \geqslant 0$ there is an axiom saying that for all distinct $x_0, ..., x_{j-1}$ and $y_0, ..., y_{k-1}$ in L_1 and all distinct $z_0, ..., z_{l-1}$ in L_0, there is an element z in L_0 not equal to $z_0, ..., z_{l-1}$, such that

$$\bigwedge_{i<j} z \leqslant x_i \wedge \bigwedge_{i<k} z \nleqslant y_i.$$

For every j, k, $l \geqslant 0$ there is an axiom saying that for all distinct $x_0, ..., x_{j-1}$ and $y_0, ..., y_{k-1}$ in L_1 and all distinct $z_0, ..., z_{l-1}$ in L_2, there is an element z in L_2 not equal to $z_0, ..., z_{l-1}$, such that

$$\bigwedge_{i<j} x_i \leqslant z \wedge \bigwedge_{i<k} y_i \nleqslant z.$$

For every j, j', k, k', $l \geqslant 0$ there is an axiom saying that for all distinct $x_0, ..., x_{j-1}$ and $y_0, ..., y_{k-1}$ in L_0, all distinct $x'_0, .., x'_{j'-1}$ and $y'_0, ..., y'_{k'-1}$ in L_2 , and all distinct $z_0, ..., z_{l-1}$ in L_1, there is an element z in L_1 not equal to $z_0, ..., z_{l-1}$, such that

$$\bigwedge_{i<j} x_i \leqslant z \wedge \bigwedge_{i<k} y_i \nleqslant z \wedge \bigwedge_{i<j'} z \leqslant x'_i \wedge \bigwedge_{i<k'} z \nleqslant y'_i.$$

This concludes the description of T_{as}.

Note that in every partial order satisfying T_{as}, if x is an element at level L_0 and y is an element at level L_2, then $x \leqslant y$. Also, T_{as} has no finite models.

The extension axioms of T_{as} resemble the axioms for the theory of almost all relational structures given in Fagin (1976). In particular, $\aleph_0$-categoricity can be demonstrated for T_{as} by a simple back and forth argument, just as it can for the theory of almost all relational structures. $\aleph_0$-categoricity of the theory of almost all relational structures was known before its connection with finite models was known; see Gaifman (1964), and for a history of this theory, Lynch (1980).

PROPOSITION 3.1. *Theory T_{as} is $\aleph_0$-categorical (and hence complete since it has no finite models).*

Proof. To show that two arbitrary countable models of T_{as} are isomorphic, we show that a finite partial isomorphism between them can be extended to include a new element. (This is the origin of the term *extension axiom*). We will suppose that our partial isomorphisms respect the levels of elements (that is, membership in each L_i) as well as their relationships between elements in the domain of the partial isomorphism. Suppose that the partial isomorphism maps elements $a_0, a_1, ..., a_{k-1}$ in the first model to elements $b_0, b_1, ..., b_{k-1}$ in the second model and we wish to extend to include another element a_k at level L_0 in, say, the first model. We must find an element b_k at level L_0 in the second model so that for each element a_i at level L_1, $a_k \leqslant a_i$ holds precisely when $b_k \leqslant b_i$ holds. The extension axioms of T_{as} ensure that such a b_k exists and is not equal to any of the elements $b_0, b_1, ..., b_{k-1}$. The other cases for extending partial isomorphisms follow similarly from the extension axioms. ∎

Now by a well-known theorem of Kleitman and Rothshild (1975) on the asymptotic enumeration of labeled partial orders, we obtain our first-order 0–1 law.

THEOREM 3.2. *The class of partial orders has a labeled first-order* 0–1 *law.*

Proof. We will show that each of the sentences in T_{as} has labeled asymptotic probability 1 in the class of partial orders. By Proposition 3.1, T_{as} is complete, so the theorem then follows by compactness just as in Fagin (1976).

Recall that $\mathscr{C}_n$ is the set of partial orders on the set $n = \{0, 1, ..., n-1\}$. Kleitman and Rothshild (1975) showed that there is a function $g(n) = o(n)$ such that if $\mathscr{D}_n$ is the subset of $\mathscr{C}_n$ consisting of partial orders with no chains of length greater than three with the levels L_0, L_1, and L_2 (defined as in the definition of T_{as}), satisfying

$$\frac{n}{4} - g(n) \leqslant |L_0|, |L_2| \leqslant \frac{n}{4} + g(n)$$

and

$$\frac{n}{2} - g(n) \leqslant |L_1| \leqslant \frac{n}{2} + g(n)$$

then $|\mathscr{C}_n| / |\mathscr{D}_n| = 1 + O(1/n)$. Thus, it suffices to show that for every sentence φ in T_{as}, $\mu_n^{\mathscr{D}}(\varphi)$ approaches 1 as n goes to ∞.

The proof of the 0–1 law for relational structures in Fagin (1976) used the probabilistic independence of relations holding of distinct tuples of elements. We show that $\mathscr{D}_n$ is closely associated with a class of structures with a similar independence property. To do this we first transform the problem from one about partial orders to one about directed graphs.

The *Hasse diagram* of a partial order $(n, \leqslant)$ is a directed graph (n, E), where xEy holds of distinct elements x and y in (n, E) precisely when $x \leqslant y$ and there is no z distinct from x and y such that $x \leqslant z \leqslant y$. For each partial order $(n, \leqslant)$ in $\mathscr{D}_n$, form a structure (n, E, L_0, L_1, L_2) by taking the Hasse diagram (n, E) of $(n, \leqslant)$ together with unary relations L_0, L_1, L_2 for the three levels. Let $\mathscr{D}'_n$ be the set of all such structures. Everty first-order sentence φ in the language of $\mathscr{D}_n$ can be translated into a first-order sentence φ' in the language of $\mathscr{D}'_n$ in such a way that φ holds in a partial order in $\mathscr{D}_n$ precisely when φ' holds in the coresponding structure in $\mathscr{D}'_n$. Our problem is equivalent to showing that every sentence φ in T_{as}, $\mu_n^{\mathscr{D}'}(\varphi')$ approaches 1.

Now consider $\mathscr{E}_n$, the class of all structures (n, E, L_0, L_1, L_2) such that L_0, L_1, and L_2 partition n with

$$\frac{n}{4} - g(n) \leqslant |L_0|, |L_2| \leqslant \frac{n}{4} + g(n)$$

and

$$\frac{n}{2} - g(n) \leqslant |L_1| \leqslant \frac{n}{2} + g(n),$$

where edges in E can go only from L_i to L_{i+1} for $i = 0, 1$. Clearly, $\mathscr{D}'_n \subseteq \mathscr{E}_n$. Given p, q, and r with $p + q + r = n$, there are

$$\binom{n}{p}\binom{n-p}{q}(2^p - 1)^q (2^q - 1)^r$$

structures in $\mathscr{D}'_n$. To see this, note that we choose p elements for L_0, then q elements for L_1; each of the q elements at level L_1 lies above some nonempty subset of L_0, and each of the r elements at level L_2 lies above some nonempty subset of L_1. Similarly, there are

$$\binom{n}{p}\binom{n-p}{q} 2^{pq} 2^{qr}$$

structures in $\mathscr{E}_n$ with $|L_0| = p$, $|L_1| = q$, and $|L_2| = r$. The ratio of these two quantities is $(1 - 2^{-p})^q\ (1 - 2^{-q})^r$, which approaches 1 uniformly for $n/4 - g(n) \leqslant p, r \leqslant n/4 + g(n)$, and $n/2 - g(n) \leqslant q \leqslant n/2 + g(n)$. Hence, $|\mathscr{D}'_n| / |\mathscr{E}_n|$ approaches 1 as n goes to ∞.

Thus, it suffices to show that for every sentence φ in T_{as}, $\mu_n^{\mathscr{E}}(\varphi')$ approaches 1. This is clearly true if φ is one of the axioms for the theory of partial orders or the sentence asserting there are no chains of length greater than three.

For p, q, and r with $p+q+r=n$ consider the structures in $\mathscr{E}_n$ with $|L_0|=p$, $|L_1|=q$, and $|L_2|=r$. Fix nonnegative integers j, k, l. In one of the structures choose a sequence of distinct elements $a_0, a_1, ..., a_{j-1}$, b_0, $b_1, ..., b_{k-1}$ from L_1 and a sequence of distinct elements $c_0, c_1, ..., c_{l-1}$ from L_0. The probability that a particular element in L_0 has an edge to each of the elements $a_0, ..., a_{j-1}$ but to none of the elements $b_0, ..., b_{k-1}$ is 2^{-j-k}. The probability that no element in L_0 distinct from $c_0, ..., c_{l-1}$ has this property, then, is $(1-2^{-j-k})^{p-l}$. The probability that for some sequence $x_0, ..., x_{j-1}$, $y_0, ..., y_{k-1}$ of distinct elements in L_1 and some sequence $z_0, ..., z_{l-1}$ of distinct elements in L_0 there is no element z in L_0 distinct from $z_0, ..., z_{l-1}$ with an edge to each of the elements $x_0, ..., x_{j-1}$ but to none of the elements $y_0, ..., y_{k-1}$ is at most

$$\binom{q}{j+k}\binom{p}{l}(1-2^{-j-k})^{p-l}$$

which approaches 0 as n goes to ∞. Thus, if φ is the first kind of extension axiom listed at the beginning of this section, then $\mu_n^{\mathscr{E}}(\varphi')$ approaches 1 as n goes to ∞. The other two kinds of extension axioms are handled in the same manner.

Therefore, the class of partial orders has a labeled first-order 0–1 law. ∎

COROLLARY 3.3. *The class of partial orders has an unlabeled first-order 0–1 law. In fact, $\mu^{\mathscr{C}}(\varphi)=\nu^{\mathscr{C}}(\varphi)$ for all first-order sentences φ.*

Proof. Prömel (1987) shows that when a_n is the number of labeled partial orders of size n and b_n is the number of unlabeled partial orders of size n, $b_n \sim a_n/n!$. The proof then follows as in Fagin (1976). If $\mu^{\mathscr{C}}(\varphi)=1$ for a first-order sentence φ then

$$n!\, b_n \nu_n^{\mathscr{C}}(\varphi) \geqslant a_n \mu_n^{\mathscr{C}}(\varphi)$$

so

$$1 \geqslant \nu_n(\varphi) \geqslant \frac{a_n/n!}{b_n}\, \mu_n^{\mathscr{C}}(\varphi)$$

and therefore $\nu_n^{\mathscr{C}}(\varphi)$ approaches 1. We conclude that $\mu^{\mathscr{C}}(\varphi)=\nu^{\mathscr{C}}(\varphi)$ for all first-order φ. ∎

We now extend these results to inductive fixed-point logic.

THEOREM 3.4. *The class of partial orders has a labeled inductive fixed-point 0–1 law.*

Proof. Blass, Gurevich, and Kozen (1985) show that if a class $\mathscr{C}$ of structures has a first-order 0–1 law and the set of first-order sentences with probability 1 is an $\aleph_0$-categorical theory, then that class has an inductive fixed-point 0–1 law. ▮

For later reference we remark that Blass, Gurevich, and Kozen show more: For every first-order $\aleph_0$-categorical theory T and every inductive fixed-point formula φ there is a finite subtheory T' of T and a first-order formula φ' such that $T' \models \varphi \leftrightarrow \varphi'$. From this it follows that every inductive fixed-point sentence φ in the language of partial orders is equivalent to some first-order sentence φ' on almost every finite partial order, so $\mu^{\mathscr{C}}(\varphi)$ is either 0 or 1. The sentence φ' is formed by replacing every implicit definition $[P_i^j(x_1, ..., x_j) \equiv \theta]$ with the approximation $[P_i^j(x_1, ..., x_j) \equiv \theta]_m$, where m is at least the number of complete j-types over T. By the Ryll–Nardzewski theorem (Theorem 2.3.12(e) of Chang and Keisler, 1973) there are only finitely many complete j-types over an $\aleph_0$-categorical theory, so this replacement is always well defined.

The following corollary is immediate.

COROLLARY 3.5. *The class of partial orders has an unlabeled inductive fixed-point* 0–1 *law. In fact,* $\mu^{\mathscr{C}}(\varphi) = \nu^{\mathscr{C}}(\varphi)$ *for all inductive fixed-point sentences* φ.

4. COMPLEXITY OF THE ALMOST SURE THEORIES OF PARTIAL ORDERS

We now determine the computational complexities of the first-order and inductive fixed-point theories of almost all partial orders. The first-order theory, as we saw in Theorem 3.2, is just the deductive closure of T_{as}. We show that this theory is *PSPACE*-complete by an argument similar to the one used by Grandjean (1983) to show that the theory of almost all finite relational structures is *PSPACE*-complete. The inductive fixed-point theory of almost all partial orders, as we saw in Theorem 3.4, is the set of inductive fixed-point sentences true in models of T_{as}. We show that this theory is *EXPTIME*-complete by an argument similar to the one used by Blass, Gurevich, and Kozen (1985) to show that the inductive fixed-point theory of almost all finite relational structures is *EXPTIME*-complete.

Following the approach of Grandjean we define a **v**-*description* for partial orders (Grandjean's definition is for relational structures). This is a formula with free variables **v** formed by taking a conjunction of the following:

(a) for each pair of variables v_i and v_j in **v** with $i \neq j$, the formula $v_i \neq v_j$;

(b) for each variable v_i in **v**, precisely one of the formulas $L_0(v_i)$, $L_1(v_i)$, $L_2(v_i)$;

(c) for each pair of variables x and y such that $L_i(x)$ and $L_{i+1}(y)$ are conjuncts, precisely one of the formulas $x \leqslant y$, $x \nleqslant y$; and

(d) for each pair of variables x and y such that $L_0(x)$ and $L_2(y)$ are conjuncts, the formula $x \leqslant y$.

In the case where $\mathbf{v}$ is of length 0, we will make a convention that there is exactly one $\mathbf{v}$-description, a tautology τ. Observe that for $\mathbf{v}$ of length n there are

$$\sum_{i+j+k=n} \binom{n}{i}\binom{n-i}{j} 2^{ij} 2^{jk}$$

$\mathbf{v}$-descriptions. This quantity is $O(2^{(n^2+3n)/4})$. (See Kim (1982, p. 148) for a precise asymptotic estimate of this sum.)

When all the conjuncts of a $\mathbf{v}$-description $\delta(\mathbf{v})$ occur as conjuncts of a $(\mathbf{v}, w)$-description $\delta'(\mathbf{v}, w)$, where w is a variable not occurring in $\mathbf{v}$, we say $\delta'(\mathbf{v}, w)$ is an *extension* of $\delta(\mathbf{v})$.

THEOREM 4.1. *Let $\mathscr{C}$ be the class of partial orders. The problem of determining if $\mu^{\mathscr{C}}(\varphi) = 1$ for first-order sentences φ is* PSPACE*-complete.*

Proof. Another way of stating the theorem is to say that T_{as} is *PSPACE*-complete.

Stockmeyer (1977) showed *PSPACE*-hardness for every complete first-order theory T for which there is a formula $\varphi(\mathbf{x})$ with both $\varphi(\mathbf{x})$ and $\neg\varphi(\mathbf{x})$ satisfied in some model of T (this is true even if T is not complete). Since our languages have equality this condition is trivially met when T has a model of power greater than 1. This is certainly the case for T_{as} so *PSPACE*-hardness is immediate.

Now we must show that T_{as} is in *PSPACE*. Observe that if $\mathbf{a}$ is a sequence of n distinct elements in a model of T_{as}, the $\mathbf{v}$-description satisfied by $\mathbf{a}$ in this model completely determines the complete n-type of $\mathbf{a}$. For if $\mathbf{b}$ is a sequence of distinct elements from some other model of T_{as} satisfying the same $\mathbf{v}$-description, the mapping from elements in $\mathbf{a}$ to corresponding elements in $\mathbf{b}$ is a partial isomorphism which respects levels. We may assume by the Löwenheim–Skolem theorem that the models are countable. As we saw in the proof of Proposition 3.1, the partial isomorphism can be extended to an isomorphism. Thus, $\mathbf{a}$ and $\mathbf{b}$ satisfy the same formulas.

This observation implies the following for all $\mathbf{v}$- descriptions $\delta(\mathbf{v})$:

(i) If φ is atomic, then $T_{\text{as}} \models \delta(\mathbf{v}) \to \varphi$ if and only if φ is a conjunct of $\delta(\mathbf{v})$.

(ii) If φ is of the form $\neg\psi$, then $T_{\text{as}} \models \delta(\mathbf{v}) \to \varphi$ if and only if $T_{\text{as}} \not\models \delta(\mathbf{v}) \to \psi$.

(iii) If φ is of the form $\psi_0 \vee \psi_1$, then $T_{as} \models \delta(\mathbf{v}) \rightarrow \varphi$ if and only if $T_{as} \models \delta(\mathbf{v}) \rightarrow \psi_0$ or $T_{as} \models \delta(\mathbf{v}) \rightarrow \psi_1$.

(iv) If φ is of the form $\exists w \psi(w)$ (there may be free variables in ψ besides w), then $T_{as} \models \delta(\mathbf{v}) \rightarrow \varphi$ if and only if either for some v_i in $\mathbf{v}$ $T_{as} \models \delta(\mathbf{v}) \rightarrow \psi(v_i)$ or for some w not in $\mathbf{v}$ and some extension $\delta'(\mathbf{v}, w)$ of $\delta(\mathbf{v})$, $T_{as} \models \delta'(\mathbf{v}, w) \rightarrow \psi(w)$.

Now it is easy to translate this list of equivalences, as Grandjean does, into a polynomial time program for an alternating Turing machine. This program takes as input a $\mathbf{v}$-description $\delta(\mathbf{v})$ and formula φ, and returns a value of *true* or *false* depending on whether or not $T_{as} \models \delta(\mathbf{v}) \rightarrow \varphi$. We may regard the program as consisting of a recursive procedure with formal parameters $\delta(\mathbf{v})$ and φ. We assume that conjunction and universal quantification are defined in terms of negation, disjunction, and existential quantification, so φ will be in one of the forms given above. If φ is atomic, the procedure computes the return value directly according to (i). In all ther cases it computes the return value according to recursive procedure calls according to the equivalences listed.

To see that this program operates in alternating polynomial time observe that all the $(\mathbf{v}, w)$-descriptions extending a particular $\mathbf{v}$-description can be generated in alternating time uniformly polynomial in the length of the $\mathbf{v}$-description. Now a sentence φ is true in all models of T_{as} precisely when $T_{as} \models \tau \rightarrow \varphi$ (recall that τ is a tautology) so we simply apply the program to τ and φ. A well-known result of Chandra, Kozen, and Stockmeyer (1981) states that alternating-*PTIME* is *PSPACE* so T_{as} is in *PSPACE*. ▌

It is worth noting that Grandjean's careful analysis of upper and lower bounds for the theory of almost all binary relations carries through here. T_{as} has the same upper and lower bounds as the theory of almost all binary relations: T_{as} is in

$$DSPACE(n^2/\log n) - NSPACE(o(n/(\log n \log\log n)^{1/2})).$$

We now turn to the inductive fixed-point theory of almost all partial orders.

THEOREM 4.2. *Let $\mathscr{C}$ be the class of partial orders. The problem of determining if $\mu^{\mathscr{C}}(\varphi) = 1$ for inductive fixed-point sentences φ is* EXPTIME-*complete.*

Proof. Blass, Gurevich, and Kozen (1985) showed that every inductive fixed-point theory T for which there is a formula $\varphi(\mathbf{x})$, where both $\varphi(\mathbf{x})$ and $\neg\varphi(\mathbf{x})$ are satisfied in some model of T, is *EXPTIME*-hard. This condition is trivially met when T has a model of power greater than 1, so *EXPTIME*-hardness for T_{as} is immediate.

The proof of Blass, Gurevich, and Kozen that the almost sure inductive fixed-point theory of relational structures is in *EXPTIME* is not quite correct. We outline the repaired proof, which is the same for relational structures and partial orders.

As we noted at the end of the previous section, an inductive least fixed-point sentence φ has labeled asymptotic probability 1 precisely when an associated first-order sentence φ' has labeled asymptotic probability 1. Recall that φ' is formed by replacing every implicit definition $[P_i^j(x_1, ..., x_i) \equiv \theta]$ with the approximation $[P_i^j(x_1, ..., x_j) \equiv \theta]_m$, where m is at least the number of complete j-types over T_{as}. We saw in the previous theorem that the number of complete j-types over T_{as} is given by the number of **v**-description, which is $O(2^{j^2/4})$, so φ' is easily obtained from φ.

Blass, Gurevich, and Kozen proceed essentially as follows. To the list of equivalences for determining whether or not $T_{as} \models \delta(\mathbf{v}) \to \varphi$ in Theorem 4.1 add two more equivalences:

(v) If φ is of the form $[P_i^j(x_1, ..., x_j) \equiv \theta]_0\ \psi$, then $T_{as} \models \delta(\mathbf{v}) \to \varphi$ if and only if $T_{as} \models \delta(\mathbf{v}) \to \psi'$, where ψ' is formed by replacing every free subformula $P_i^j(y_1, ..., y_j)$ of ψ (i.e., every subformula where P_i^j occurs freely in ψ) with $\neg\tau$, the negation of a tautology.

(vi) If φ is of the form $[P_i^j(x_1, ..., x_j) \equiv \theta]_{m+1}\ \psi$, then $T_{as} \models \delta(\mathbf{v}) \to \varphi$ if and only if $T_{as} \models \delta(\mathbf{v}) \to \psi'$, where ψ' is formed by replacing every free subformula $P_i^j(y_1, ..., y_j)$ of ψ with the formula θ' which is formed by replacing every free subformula $P_i^j(y_1, ..., y_j)$ of θ by $[P_i^j(x_1, ..., x_j) \equiv \theta]_m\ \psi$.

Blass, Gurevich, and Kozen add these two cases to the recursive program described in Theorem 4.1; they claim the result is an alternating polynomial space program. Since *EXPTIME* is alternating-*PSPACE* (see Chandra, Kozen, and Stockmeyer (1981)) it is possible to determine if $\mu^{\mathscr{C}}(\varphi) = 1$ by replacing implicit definitions in φ with the appropriate approximations to form φ' as described above, and applying the algorithm to the pair (τ, φ'). The algorithm they give appears somewhat simpler because they use a more restrictive syntax: they allow formulas of the form $[P_i^j(x_1, ..., c_j) \equiv \theta]_m\ \psi$ only when ψ is a formula $P_i^j(y_1, ..., y_j)$.

Unfortunately, the algorithm, as stated, can use more than alternating polynomial space, even for their restricted syntax. It is true that the **v**-descriptions occurring during execution of the algorithm require only polynomial space, but substitution of formulas in case (vi) may result in exponential growth in the second argument. For example, suppose φ is of the form

$$[P_1^1(x) \equiv [P_2^1(x) \equiv \cdots [P_k^1(x) \equiv \theta(P_1^1, ..., P_k^1)]_1\ P_k^1(x) \cdots]_1\ P_2^1(x)]_1\ P_1^1(x),$$

where the relation variables $P_1^1, ..., P_k^1$ each occur twice in θ. Following the algorithm, we apply the substitution in (vi) k times, more than doubling the size of the formula each time. Even if θ were modified so that each relation variable occurs just once, we would still have exponential growth.

The problem is that reducing the outermost definition and substituting in case (vi) can result in a much longer sentence. To overcome this difficulty we formulate another algorithm which reduces inner definitions. It is based on the following set of equivalences. As before, $\delta(\mathbf{v})$ is a $\mathbf{v}$-description. We will assume that φ is of the form

$$[Q_1(\mathbf{x}_1) \equiv \theta_1]_{m_1} \cdots [Q_k(\mathbf{x}_k) \equiv \theta_k]_{m_k} \varphi',$$

where symbols $Q_1, ..., Q_k$ denote distinct relation variables (duplications can be easily eliminated), φ does not have free relation variables, and φ' does not begin with an implicit definition:

(i′) If φ' is atomic but does not contain a relation variable, then $T_{\mathrm{as}} \models \delta(\mathbf{v}) \rightarrow \varphi$ if and only if φ' is a conjunct of $\delta(\mathbf{v})$.

(ii′) If φ' is of the form $\neg\psi$, then $T_{\mathrm{as}} \models \delta(\mathbf{v}) \rightarrow \varphi$ if and only if

$$T_{\mathrm{as}} \not\models \delta(\mathbf{v}) \rightarrow [Q_1(\mathbf{x}_1) \equiv \theta_1]_{m_1} \cdots [Q_k(\mathbf{x}_k) \equiv \theta_k]_{m_k} \psi.$$

(iii′) If φ' is of the form $\psi_0 \vee \psi_1$, then $T_{\mathrm{as}} \models \delta(\mathbf{v}) \rightarrow \varphi$ if and only if

$$T_{\mathrm{as}} \models \delta(\mathbf{v}) \rightarrow [Q_1(\mathbf{x}_1) \equiv \theta_1]_{m_1} \cdots [Q_k(\mathbf{x}_k) \equiv \theta_k]_{m_k} \psi_0$$

or

$$T_{\mathrm{as}} \models \delta(\mathbf{v}) \rightarrow [Q_1(\mathbf{x}_1) \equiv \theta_1]_{m_1} \cdots [Q_k(\mathbf{x}_k) \equiv \theta_k]_{m_k} \psi_1.$$

(iv′) If φ' is of the form $\exists w \psi(w)$ (there may be free variables in ψ besides w), then $T_{\mathrm{as}} \models \delta(\mathbf{v}) \rightarrow \varphi$ if and only if either for some v_i in $\mathbf{v}$

$$T_{\mathrm{as}} \models \delta(\mathbf{v}) \rightarrow [Q_1(\mathbf{x}_1) \equiv \theta_1]_{m_1} \cdots [Q_k(\mathbf{x}_k) \equiv \theta_k]_{m_k} \psi(v_i)$$

or for some w not in $\mathbf{v}$ and some extension $\delta'(\mathbf{v}, w)$ of $\delta(\mathbf{v})$

$$T_{\mathrm{as}} \models \delta'(\mathbf{v}, w) \rightarrow [Q_1(\mathbf{x}_1) \equiv \theta_1]_{m_1} \cdots [Q_k(\mathbf{x}_k) \equiv \theta_k]_{m_k} \psi(w).$$

(It may be necessary to rename variables in ψ to avoid conflicts when substituting.)

(v′) If φ' is of the form $Q_i(\mathbf{y})$ for some i with $1 \leqslant i \leqslant k$, and $m_i = 0$, then $T_{\mathrm{as}} \not\models \delta(\mathbf{v}) \rightarrow \varphi$.

(vi′) If φ' is of the form $Q_i(\mathbf{y})$ for some i with $1 \leqslant i \leqslant k$, and $m_i > 0$, then $T_{\mathrm{as}} \models \delta(\mathbf{v}) \rightarrow \varphi$ if and only if $T_{\mathrm{as}} \models \delta(\mathbf{v}) \rightarrow \psi$, where ψ is the formula

$$[Q_1(\mathbf{x}_1) \equiv \theta_1]_{m_1} \cdots [Q_{i-1}(\mathbf{x}_{i-1}) \equiv \theta_{i-1}]_{m_{i-1}} [Q_i(\mathbf{x}_i) \equiv \theta_i]_{m_i - 1} \theta'$$

and θ' is formed by replacing each of the variables in $\mathbf{x}_i$ free in θ with the corresponding variables in $\mathbf{y}$ (also, perhaps some of the bound variables in θ must be renamed to avoid conflicts).

Now it is easy to see that if we translate this list of equivalences into a recursive program for an alternating Turing machine, the algorithm terminates in cases (i′) and (v′) and in cases (ii′)–(iv′) the arguments for the recursive call are shorter. To see that case (vi′) does not result in overly fast growth in argument length, observe that in none of the cases (i′)–(vi′) is a subformula of an implicit definition replaced (this is where we differ from Blass, Gurevich, and Kozen). Thus, every implicit definition that occurs during a recursion was present in the original formula, so the argument

$$[Q_1(\mathbf{x}_1) \equiv \theta_1]_{m_1} \cdots [Q_{i-1}(\mathbf{x}_{i-1}) \equiv \theta_{i-1}]_{m_{i-1}} [Q_i(\mathbf{x}_i) \equiv \theta_i]_{m_i - 1}\, \theta'$$

of the recursive call in (vi′) is bounded throughout the execution of the algorithm by twice the length of the original formula. It follows that this algorithm requires only alternating polynomial space and, therefore, that the set of inductive fixed-point formulas φ such that $\mu^{\mathscr{C}}(\varphi) = 1$ is in *EXPTIME*. ▌

5. Final Remarks

Our results do not extend to monadic second-order logic. Kaufmann and Shelah (1983) showed that when $\mathscr{C}$ is the class of all structures for a similarity type with a nonunary relation, there are monadic second-order sentences φ such that $\mu_n^{\mathscr{C}}(\varphi)$ does not converge to any value. Their argument can be modified to work for partial orders. Grandjean (1983) shows that for the same class $\mathscr{C}$ considered by Kaufmann and Shelah, the set of monadic second-order sentences φ such that $\mu_n(\varphi) = 1$ is undecidable. Again, the argument can be modified to work for partial orders. Is there another logic more appropriate than monadic second-order logic?

One application where random structures occur is the determination of expected times for algorithms. The results here do not seem well suited for determining expected times of algorithms for partial orders. The result of Kleitman and Rothshild used in Theorem 3.2 is evidence that the uniform measure on labeled partial orders is not realistic in general. Our analysis in Theorem 3.2 is more evidence for this assessment. The measures on finite partial orders investigated by Winkler (1985) may prove more useful. Is it the case that the probabilities of first-order (or inductive fixed-point) sentences always converge for these measures? If so, can the limits be computed and how difficult are they to compute? Answers to these questions may help in determining realistic expected times for partial order algorithms.

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