Behaviors of solutions for a singular diffusion equation

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Abstract

This paper is devoted to some behaviors of solutions of the initial-boundary problem for a singular diffusion equation, namely, localization and large time behavior. After given some special explicit solutions it is proved that solutions of the problem possess the localization property. Next, $L^2$ decay estimate as $t \to \infty$ is proved by a rather standard energy method. Finally, by comparison with a special solution the expected $L^\infty$ decay estimate is derived.

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1. Introduction

In this paper, we consider a singular diffusion equation with nondivergence form

$$u_t = u \text{ div}(|\nabla u|^{p-2} \nabla u) - \gamma |\nabla u|^p \quad \text{in } \Omega_\infty = \Omega \times (0, \infty),$$

(1.1)

with the Dirichlet boundary value condition

$$u(x, t) = 0 \quad \text{on } \partial \Omega \times (0, \infty)$$

(1.2)

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and the initial value condition
\[ u(x, 0) = u_0(x) \quad \text{in } \Omega, \quad (1.3) \]
where \( \Omega \subset \mathbb{R}^N \) is a bounded domain with appropriately smooth boundary \( \partial \Omega \), \( p > 1 \), \( \gamma \geq 0 \) and \( u_0 \) is a nonnegative function.

(1.1) arises in some models describing physical phenomenon. For example, (1.1) with \( p = 2 \) can be derived as a model of groundwater flow in a water-absorbing, fissurized porous rock bed (see [1,2]). Note that (1.1) is closely related to the filtration equation (see [3–5])
\[ v_t = \text{div}\{|\nabla v|^p - 2 \nabla v\} \quad (m \neq 0). \quad (1.4) \]
Indeed, if \( m(p - 1) - 1 \neq 0 \) and let
\[ \gamma = (p - 1)/(1 - m(p - 1)), \quad u = m|\gamma|^{(p-2)/(p-1)}v^{-1/\gamma}, \]
then (1.1) can be transformed formally into (1.4) (see [6]).

Since (1.1) may be degenerate or singular at points where \( u = 0 \) or \( |\nabla u| = 0 \), the problem does not admit classical solutions in general. Therefore we need to consider weak solutions. Moreover, only nonnegative solutions are considered. Generally speaking, weak solutions of problem (1.1)–(1.3) are not uniquely determined by the initial value. Indeed, many weak solutions were constructed in [7–9] for the case \( p = 2 \) and [10,11] for the case \( p \neq 2 \) where the general existence results were also obtained. In addition, the weak solutions for (1.1) may not be continuous (see [12]). Some other results can be referred to [13–16] and references therein. In the present paper, we are interested in some behaviors of solutions, namely, localization property and large time behavior. We point out that in the case \( p = 2 \) and \( \gamma = 0 \), the localization property of solutions was discovered independently by Dal Passo and Luckhaus [7] and by Ughi [8]. By means of a similar idea used in [7], we extend and generalize the result to the case \( p \neq 2 \) and \( \gamma > 0 \). Another interest of this paper is to study large time behavior of solutions for (1.1). More specifically, \( L^2 \) decay estimate as \( t \to \infty \) is proved by a rather standard energy method, and the expected \( L^\infty \) decay estimate as \( t \to \infty \) is derived by comparison with a special solution.

For \( T > 0 \), denote \( \Omega_T = \Omega \times (0, T) \).

**Definition 1.1.** A nonnegative function \( u \) is called a weak sup-solution (sub-solution) for (1.1) on \( \Omega_\infty \), if for any \( T > 0 \), the following two conditions are satisfied:

(a) \( u \in L^\infty(\Omega_T) \cap L^p(0, T; W^{1,p}(\Omega)) \) with \( u_t \in L^2(\Omega_T) \);
(b) for any \( \varphi \in L^\infty(\Omega_T) \cap L^p(0, T; W^{1,p}_0(\Omega)) \) with \( \varphi \geq 0 \), there holds
\[ \intint_{\Omega_T} (u_t \varphi + u|\nabla u|^{p-2} \nabla u \nabla \varphi + (1 + \gamma)|\nabla u|^p \varphi) \, dx \, dt \geq 0 \quad (\leq 0). \]

A nonnegative function \( u \) is called a weak solution for (1.1) on \( \Omega_\infty \), if \( u \) is both a weak sup-solution and a weak sub-solution for (1.1) on \( \Omega_\infty \).

**Definition 1.2.** A nonnegative function \( u \) is called a weak solution of problem (1.1)–(1.3), if \( u \) is a weak solution for (1.1) on \( \Omega_\infty \), and satisfies the following two conditions:

(a) for any \( t > 0 \), \( u(t) \in W^{1,p}_0(\Omega) \);
(b) \( u(t) \to u_0 \) in \( L^1(\Omega) \) as \( t \to 0 \).
The paper is organized as follows. In Section 2 we first construct some special explicit solutions, and then the localization property is proved. In Section 3 some decay estimates as \( t \to \infty \) are obtained.

2. Localization of solutions

First, we give some special explicit solutions for (1.1) on \( \Omega_{\infty} \).

**Proposition 2.1.** The following functions are weak solutions for (1.1) on \( \Omega_{\infty} \):

\[
\begin{align*}
F_0(x, t) &= [A - \kappa \psi_0(x)]_+ \quad (\gamma = 0), \\
F_1(x, t) &= \frac{[A(t + t_0)^{-\alpha} - \kappa \psi_0(x)]_+}{(t + t_0)^{1/(p-1)}} \quad (\gamma < \gamma_c), \\
F_2(x, t) &= (Ae^{-Nt} - \psi_0(x))_+ \quad (\gamma = \gamma_c), \\
F_3(x, t) &= \begin{cases} \\
[A(t_0 - t)\alpha - \kappa \psi_0(x)]_+ & t \in (0, t_0), \\
0, & t \geq t_0, \\
\end{cases} \quad (\gamma > \gamma_c),
\end{align*}
\]

where \( t_0, A > 0, x_0 \in \mathbb{R}^N \) and

\[
\begin{align*}
\gamma_c &= \frac{p - 1}{p} N, \\
\alpha &= \frac{\gamma}{(p - 1)|\gamma_c - \gamma|}, \\
\kappa &= \frac{1}{p|\gamma_c - \gamma|} \left| \frac{1}{(p-1)} \right|, \\
\psi_0(x) &= \frac{p - 1}{p} |x - x_0|^{p/(p-1)}.
\end{align*}
\]

Moreover, for any \( t_1 \geq 0 \) we have

\[
\text{supp } F_i(t_1) = \text{supp } F_i(0),
\]

\[
\text{supp } F_i(t_2) \subseteq \text{supp } F_i(t_1) \subseteq \text{supp } F_i(0) \quad \text{if } t_2 > t_1 \ (i = 1, 2, 3),
\]

i.e., \( F_i \ (i = 0, 1, 2, 3) \) possess the localization property.

**Proof.** By simple calculation we have

\[
\nabla \psi(x_0) = 0, \quad \nabla \psi(x) = (x - x_0)|x - x_0|^{(2-p)/(p-1)}, \quad x \neq x_0,
\]

which imply that

\[
|\nabla \psi(x)|^{p-2} \nabla \psi(x) = x - x_0, \quad x \in \mathbb{R}^N,
\]

and hence

\[
\text{div } |\nabla \psi|^{p-2} \nabla \psi = N \quad \text{in } \mathbb{R}^N.
\]

By means of the fact, it is not difficult to check that \( F_i \ (i = 0, 1, 2, 3) \) are weak solutions for (1.1) on \( \Omega_{\infty} \).

The other conclusions of Proposition 2.1 are obvious, and thus the proposition is proved.

In order to show a general result on localization of weak solutions, we need to define the support of a nonnegative function \( w : \Omega \to \mathbb{R}^+ \cup \{0\} \) that is not necessarily continuous:

\[
\text{supp } w = \left\{ x \in G; \lim_{\rho \to 0^+} \frac{\mu(G \cap B_{\rho}(x))}{\mu(B_{\rho}(x))} > 0 \right\},
\]
where $G = \{x \in \Omega; w(x) > 0\}$, $B_{\rho}(x) = \{y \in \Omega; |x - y| < \rho\}$, and $\mu(E)$ is the Lebesgue measure of set $E$ in $\mathbb{R}^N$. It is easy to see that if $w \in C(\Omega)$, then $\text{supp} \ w = \bar{G}$.

**Theorem 2.2.** (1) Assume $\gamma \geq 0$ and $0 \leq u_0 \in L^1(\Omega)$. If $u$ is a weak solution of problem (1.1)–(1.3), and $u(t_1) \neq 0$ and supp $u(t_1) \subseteq$ but $\neq \bar{\Omega}$ for some $t_1 > 0$, then

$$\text{supp} \ u(t) \subseteq \text{supp} \ u(t_1) \quad \text{a.e. in } (t_1, \infty).$$

(2) Besides the assumptions of (1), let $u_0 \neq 0$ and supp $u_0 \subseteq$ but $\neq \bar{\Omega}$. If $u$ is a weak solution of problem (1.1)–(1.3), then

$$\text{supp} \ u(t) \subseteq \text{supp} \ u_0 \quad \text{a.e. in } (0, \infty).$$

**Proof.** For any $t > t_1$ and any $\varphi \in L^\infty(\Omega_{1,t}) \cap L^p(t_1, t; W^{1,p}_0(\Omega))$ with $\varphi \geq 0$, we have

$$\int_{\Omega_{1,t}} (u \varphi + u |\nabla u|^{p-2} \nabla u \nabla \varphi + (1 + \gamma) |\nabla u|^p \varphi) \, dx \, d\tau = 0, \quad (2.1)$$

where $\Omega_{1,t} = \Omega \times (t_1, t)$.

For $\sigma \in (0, 1)$, let $\psi = \psi_\sigma = \inf \{d(x) \sigma, 1\}$, where $d(x) = \text{dist}(x, \text{supp} \ u(t_1) \cup \partial \Omega)$. Since the distance function $d(x)$ is Lipschitz with constant 1, Rademacher’s theorem implies that it is differentiable almost everywhere (see [17, pp. 49–51]). Hence, for any $\epsilon > 0$, $\varphi = \frac{\psi}{u + \epsilon}$ can be chosen as a test function. Substituting it into (2.1), we derive

$$\int_{\Omega_{1,t}} \left( \frac{u \varphi}{u + \epsilon} + u |\nabla u|^{p-2} \nabla u \nabla \frac{\psi}{u + \epsilon} + (1 + \gamma) |\nabla u|^p \frac{\psi}{u + \epsilon} \right) \, dx \, d\tau = 0$$

and hence

$$\int_{\Omega_{1,t}} \frac{u \varphi}{u + \epsilon} \, dx \, d\tau + \int_{\Omega_{1,t}} \frac{u}{u + \epsilon} |\nabla u|^{p-2} \nabla u \nabla \psi \, dx \, d\tau$$

$$+ \epsilon \int_{\Omega_{1,t}} \frac{|\nabla u|^p}{(u + \epsilon)^2} \psi \, dx \, d\tau + \gamma \int_{\Omega_{1,t}} |\nabla u|^p \frac{\psi}{u + \epsilon} \, dx \, d\tau = 0.$$

Since $\gamma \geq 0$, we obtain

$$\int_{\Omega} \left[ \ln(u(t) + \epsilon) - \ln(u(t_1) + \epsilon) \right] \psi \, dx \leq - \int_{\Omega_{1,t}} \frac{u}{u + \epsilon} |\nabla u|^{p-2} \nabla u \nabla \psi \, dx \, d\tau \leq C,$$

where $C$ is a positive constant independent of $\epsilon$. Therefore, by noticing $u(t_1) \cdot \psi = 0$, we have

$$\int_{\Omega} \chi_{(\text{supp} \ \psi)} \left[ \ln(u(t) + \epsilon) - \ln \epsilon \right] \psi \, dx \leq C,$$

where $C$ is a positive constant independent of $\epsilon$. Then, for any $\delta > 0$ and for a.e. $t > t_1$ we obtain

$$\int_{\Omega} \chi_{((x \in \Omega | u(t) > \delta) \cap \{x \in \Omega; \psi = 1\})} \left[ \ln(u(t) + \epsilon) - \ln \epsilon \right] \, dx \leq C. \quad (2.2)$$
Suppose that there exists \( t_2 > t_1 \) such that \( \mu(\{x \in \Omega; \psi = 1\} | u(t_2) > \delta) > 0 \). Then it follows from (2.2) that
\[
\mu(\{x \in \{x \in \Omega; \psi = 1\} | u(t_2) > \delta\}) \leq (\ln(\delta + \epsilon) - \ln \epsilon) \leq C,
\]
then letting \( \epsilon \to 0 \) to yield a contradiction. Hence
\[
\mu(\{x \in \{x \in \Omega; \psi = 1\} | u(t) > \delta\}) = 0 \quad \text{a.e. in} \quad (t_1, \infty).
\]
This implies that
\[
\mu(\{x \in \{x \in \Omega; \psi = 1\} | u(t) > 0\}) = 0 \quad \text{a.e. in} \quad (t_1, \infty).
\]
The conclusion (1) follows from this and the arbitrariness of \( \sigma \in (0, 1) \).

By a similar proof as in that of (1), the conclusion (2) can be proved. The proof of Theorem 2.2 is complete. \( \square \)

3. Large time behavior of solutions

Our first main result, namely, \( L^2 \) decay estimate as \( t \to \infty \), is the following:

**Theorem 3.1.** Let \( \gamma \geq 0 \) and \( 0 \leq u_0 \) and \( 0 < A \equiv \int \Omega u_0^2 \, dx < \infty \). If \( u \) is a weak solution of problem (1.1)–(1.3), then there exists a positive constant \( C \) depending only on \( p \) and \( \Omega \) such that
\[
\int_{\Omega} u^2(t) \, dx \leq \left[ \frac{1}{Ct + A^{(1-p)/2}} \right]^{2/(p-1)}, \quad t > 0.
\]

**Proof.** Taking \( \varphi = u \) in the integral equality satisfied by \( u \), we derive
\[
\int_{0}^{t} \int_{\Omega} (u_{\tau} u + (2 + \gamma) u |\nabla u|^p) \, dx \, d\tau = 0
\]
and hence
\[
\int_{\Omega} u^2(t) \, dx - \int_{\Omega} u^2_0 \, dx = -(4 + 2\gamma) \int_{0}^{t} \int_{\Omega} u |\nabla u|^p \, dx \, d\tau. \tag{3.1}
\]
Let
\[
\Phi(t) = \int_{\Omega} u^2(t) \, dx.
\]
Then it follows from (3.1) and \( \gamma \geq 0 \) that
\[
\Phi'(t) = -(4 + 2\gamma) \int_{\Omega} u(t) |\nabla u(t)|^p \, dx \leq -4 \int_{\Omega} u(t) |\nabla u(t)|^p \, dx
\]
\[
\leq -\frac{4}{p+1} \int_{\Omega} |\nabla u(t)|^{(p+1)/p} \, dx. \tag{3.2}
\]
By Poincaré’s inequality, we derive that there exists a positive constant $C$ depending only $p$ and $\Omega$ such that
\[ \int_{\Omega} u(t)^{p+1} \, dx \leq C \int_{\Omega} \left| \nabla u(t) \right|^{(p+1)/p} \, dx, \]
which and (3.2) imply that
\[ \Phi'(t) \leq -C \int_{\Omega} u(t)^{p+1} \, dx. \]
On the other hand, it follows from Hölder’s inequality that
\[ \Phi(t) = \int_{\Omega} u^2(t) \, dx \leq C \left( \int_{\Omega} u(t)^{p+1} \, dx \right)^{2/(p+1)}, \]
where $C$ is a positive constant depending only $p$ and $\Omega$, and hence
\[ \Phi'(t) \leq -C \Phi(t)^{(1+p)/2}, \]
which gives
\[ \Phi(t) \leq \left[ \frac{1}{Ct + A^{(1-p)/2}} \right]^{2/(p-1)}. \]
This ends the proof of Theorem 3.1. □

Furthermore, we obtain the following $L^\infty$ decay estimate.

**Theorem 3.2.** Let $\gamma \geq 0$ and $0 \leq u_0 \in L^\infty(\Omega)$. If $u$ is a weak solution of problem (1.1)–(1.3), then there exists a positive constants $L$ depending only on $p$, $N$ and $\Omega$, such that
\[ u(t) \leq \frac{L}{(t+1)^{1/(p-1)}} \quad \text{a.e. in } \Omega. \]

To prove Theorem 3.2, we need to establish the following comparison theorem.

**Theorem 3.3.** Let $\gamma \geq 0$. Assume that $u_2$ and $u_1$ are weak sup-solution and sub-solution for (1.1) on $\Omega_{\infty}$, respectively, and $u_2 \geq c$ a.e. in $\Omega_T$ for some $c > 0$ and $T > 0$. If $u_2(x, 0) \geq u_1(x, 0)$ a.e. in $\Omega$, and $u_2(x, t) \geq u_1(x, t)$ a.e. on $\partial \Omega \times (0, T)$, then $u_2 \geq u_1$ a.e. in $\Omega_T$.

**Proof.** First, for any $\varphi \in L^\infty(\Omega_t) \cap L^p(0, t; W^{1,p}_0(\Omega))$ with $\varphi \geq 0$ we have
\[ \iint_{\Omega_t} \left( \frac{\partial u_2}{\partial \tau} \varphi + u_2 |\nabla u_2|^{p-2} \nabla u_2 \nabla \varphi + (1 + \gamma)|\nabla u_2|^p \varphi \right) \, dx \, d\tau \geq 0, \]
\[ \iint_{\Omega_t} \left( \frac{\partial u_1}{\partial \tau} \varphi + u_1 |\nabla u_1|^{p-2} \nabla u_1 \nabla \varphi + (1 + \gamma)|\nabla u_1|^p \varphi \right) \, dx \, d\tau \leq 0. \quad (3.3) \]
Let $g(s)$ and $\text{sgn}_\delta(z)$ be defined by
\[ g(s) = s^{1-\gamma/(p-1)} \left[ 1 - \gamma/(p-1) \right]^{-1}. \]
and

\[ \text{sgn}_\delta(z) = \begin{cases} 
1, & z > \delta, \\
\frac{z}{\delta}, & |z| \leq \delta, \\
-1, & z < -\delta.
\end{cases} \]

Since \( u_1, u_2 \in L^\infty(\Omega_T) \cap L^p(0, T; W^{1,p}(\Omega)) \), \( g(s) \) is increasing and \( u_2 \geq u_1 \) a.e. on \( \partial\Omega \times (0, T) \), we have \((g(u_1) - g(u_2))_+ \in L^\infty(\Omega_T) \cap L^p(0, T; W^{1,p}_0(\Omega)) \) and hence \( \text{sgn}_\delta((g(u_1) - g(u_2))_+) \in L^\infty(\Omega_T) \cap L^p(0, T; W^{1,p}_0(\Omega)) \). This and \( u_2 \geq c > 0 \) a.e. in \( \Omega_T \) imply \( \varphi_{u_2} = u_2^{-1-\gamma} \text{sgn}_\delta((g(u_1) - g(u_2))_+) \in L^\infty(\Omega_T) \cap L^p(0, T; W^{1,p}_0(\Omega)) \). Define \( \varphi_{u_1} = u_1^{-1-\gamma} \times \text{sgn}_\delta((g(u_1) - g(u_2))_+) \) whenever \( u_1 > c, \varphi_{u_1} = 0 \) whenever \( u_1 \leq c \). Then also we have \( \varphi_{u_1} \in L^\infty(\Omega_T) \cap L^p(0, T; W^{1,p}_0(\Omega)) \). So \( \varphi_{u_2} \) and \( \varphi_{u_1} \) can be chosen in (3.3) as test functions.

Therefore

\[
\begin{align*}
\int_{\Omega_t} \left[ \frac{\partial u_2}{\partial \tau} u_2^{-1-\gamma} \text{sgn}_\delta((g(u_1) - g(u_2))_+) \\
+ u_2^{-\gamma} |\nabla u_2|^{p-2} \nabla u_2 \nabla (g(u_1) - g(u_2))_+ \text{sgn}_\delta'(((g(u_1) - g(u_2))_+) \right] dx \ d\tau & \geq 0, \\
\int_{\Omega_t} \left[ \frac{\partial u_1}{\partial \tau} u_1^{-1-\gamma} \text{sgn}_\delta((g(u_1) - g(u_2))_+) \\
+ u_1^{-\gamma} |\nabla u_1|^{p-2} \nabla u_1 \nabla (g(u_1) - g(u_2))_+ \text{sgn}_\delta'(((g(u_1) - g(u_2))_+) \right] dx \ d\tau & \leq 0
\end{align*}
\]

and hence

\[
\begin{align*}
\int_{\Omega_t} (f(u_1) - f(u_2))_+ \text{sgn}_\delta((g(u_1) - g(u_2))_+) dx \ d\tau \\
+ \int_{\Omega_t} \left[ |\nabla g(u_1)|^{p-2} \nabla g(u_1) - |\nabla g(u_2)|^{p-2} \nabla g(u_2) \right] \\
\times \nabla (g(u_1) - g(u_2))_+ \text{sgn}_\delta'(((g(u_1) - g(u_2))_+) dx \ d\tau & \leq 0, \tag{3.4}
\end{align*}
\]

where \( f : (0, \infty) \to R \) is defined by

\[ f(s) = \begin{cases} 
\ln s & (\gamma = 0), \\
-\frac{s^{-\gamma}}{\gamma} & (\gamma > 0).
\end{cases} \]

By means of the following fact:

\[
\begin{align*}
\int_{\Omega_t} (|\nabla g(u_1)|^{p-2} \nabla g(u_1) - |\nabla g(u_2)|^{p-2} \nabla g(u_2) ) \\
\times \nabla (g(u_1) - g(u_2))_+ \text{sgn}_\delta'(((g(u_1) - g(u_2))_+) dx \ d\tau & \geq 0,
\end{align*}
\]

we obtain from (3.4)

\[
\int_{\Omega_t} (f(u_1) - f(u_2))_+ \text{sgn}_\delta((g(u_1) - g(u_2))_+) dx \ d\tau \leq 0.
\]
Letting $\delta \to 0$ shows
\[ \int\int_\Omega (f(u_1) - f(u_2)) \frac{\tau}{\tau} \text{sgn}((g(u_1) - g(u_2))_+) \, dx \, d\tau \leq 0. \]

Using $\text{sgn}((g(u_1) - g(u_2))_+) = \text{sgn}((f(u_1) - f(u_2))_+)$ we derive
\[ \int\int_\Omega (f(u_1) - f(u_2)) \frac{\tau}{\tau} \text{sgn}((f(u_1) - f(u_2))_+) \, dx \, d\tau \leq 0. \]

It follows from $u_1(x, 0) \leq u_2(x, 0)$ a.e. in $\Omega$ that
\[ \int_\Omega (f(u_1) - f(u_2))_+(x, t) \, dx \leq 0, \]

which implies that $(f(u_1) - f(u_2))_+(x, t) = 0$ a.e. in $\Omega_T$, i.e., $u_1 \leq u_2$ a.e. in $\Omega_T$. This completes the proof of Theorem 3.3.

**Proof of Theorem 3.2.** From Proposition 2.1, we see that $F_0$ is a weak solution for (1.1) with $\gamma = 0$ on $\Omega_\infty$, and thus it is a weak sup-solution for (1.1).

For fixed $x_0 \in R^N$ and $t_0 = 1$, choosing a sufficiently large $A > 0$ such that
\[ A - \kappa \psi_0(x) \geq |u_0|_\infty + 1 \text{ on } \tilde{\Omega}, \]

we have
\[ F_0(x, 0) \geq u_0(x) \text{ a.e. in } \Omega \]

and
\[ F_0(x, t) \geq C > 0 \text{ in } \Omega_T \text{ for any } T > 0. \]

From Theorem 3.3 we obtain
\[ F_0(x, t) \geq u(x, t) \text{ a.e. in } \Omega_T \text{ for any } T > 0. \]

This ends the proof of Theorem 3.2.

**Remark 3.4.** It is easy to see from Proposition 2.1 that if $\gamma \neq \gamma_c$, the exponent $1/(p - 1)$ in Theorem 3.2 and the exponent $2/(p - 1)$ in Theorem 3.1 are optimal.

**References**