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Extinction and Positivity for The Evolution P-Laplacian Equation

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The aim of this paper is to discuss the extinction and positivity for the evolution P-Laplacian equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right)$$

with $p > 1$. In particular, the necessary and sufficient condition for extinction is obtained. © 1995 Academic Press, Inc.

1. INTRODUCTION

In this paper we consider the evolution P-Laplacian equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) \tag{1.1}$$

in Q with initial-boundary conditions

$$u(x,0) = u_0(x), \forall x \in [-1, 1]; \qquad u(\pm 1, t) = 0, \forall t \in \mathbf{R}, \tag{1.2}$$

where $p > 1$ is a given real number, and u_0 is a nonzero nonnegative continuous function in $\mathbf{I}$ with $u_0(\pm 1) = 0$, and $Q = \mathbf{I} \times \mathbf{R}$, $\mathbf{I} = [-1, 1]$, $\mathbf{R} = (0, +\infty)$.

Equation (1.1) appears in a number of applications to describe the evolu-

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tion of diffusion processes, in particular non-Newtonian flow in a porous medium; cf. [1–3].

The quasilinear equation (1.1) is degenerate if $p > 2$ or singular if $1 < p < 2$, since the modulus of ellipticity degenerates ($p > 2$) or blows up ($1 < p < 2$) at points where $\partial u/\partial x = 0$, and therefore has no classical solution in general. We consider therefore its weak solutions.

DEFINITION 1.1. A nonnegative function u is said to be a weak solution of (1.1)–(1.2), if u satisfies the following conditions:

$$u \in L^{\infty}(Q) \cap C(Q), \qquad \frac{\partial u}{\partial x} \in L^{p}_{\text{loc}}(Q) \tag{1.3}$$

$$\int_{-1}^{1} u(x,T)\phi(x,T)\,dx - \int_{-1}^{1} u_0(x)\phi(x,0)\,dx = \int_0^T \int_{-1}^{1} \left(u\frac{\partial \phi}{\partial t} - \left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{\partial u}{\partial x}\frac{\partial \phi}{\partial x}\right) dx\,dt \tag{1.4}$$

for all $T \in (0, \infty)$ and all $\phi \in C(0, T; W_0^{1,p}(I))$ with $\partial\phi/\partial t \in L^p(Q)$.

The existence, uniqueness, and regularity of solutions of the Cauchy problem (1.1)–(1.2) have been obtained by a number of authors; cf. [4–13].

In this paper our interest is to investigate the extinction and positivity of solutions. Our main results are the following theorems.

THEOREM 1.1. *Let u be a weak solution of* (1.1)–(1.2). *If* $1 < p < 2$, *then there exists a time T such that*

$$u(x, t) = 0$$

for all $(x, t) \in \mathbf{I} \times (T, \infty)$.

THEOREM 1.2. *Let u be a weak solution of* (1.1)–(1.2). *If* $p \geq 2$, *then there exists a time T such that*

$$u(x, t) > 0$$

for all $(x, t) \in (-1, 1) \times (T, +\infty)$.

Remark 1.1. Theorems 1.1 and 1.2 imply that Eq. (1.1) has the extinctive property if and only if $1 < p < 2$.

Such results have been obtained for the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2}$$

with $0 < m < 1$ by L. C. Evans in [14], in which the proofs cannot be extended to our case. Our approach is different from [14]; it is based on a comparison principle.

The proofs of our main results are completed in Sections 3 and 4. We first prove some fundamental lemmas in Section 2.

2. Fundamental Lemmas

Lemma 2.1. *Let u be a weak solution of* (1.1)–(1.2). *If v satisfies*

$$\frac{\partial v}{\partial t} \geq \frac{\partial}{\partial x}\left(\left|\frac{\partial v}{\partial x}\right|^{p-2} \frac{\partial v}{\partial x}\right)$$

$$v(x, 0) \geq u(x, 0),\ \forall x \in \mathbf{I}; \qquad v(\pm 1, t) \geq u(\pm 1, t),\ \forall t \in \mathbf{R},$$

then we have

$$v(x, t) \geq u(x, t)$$

for all $(x, t) \in Q$.

The proof is similar to that given in [8].

Lemma 2.2. *Let u be a weak solution. Then we have*

(i) *If $p > 2$, then*

$$\frac{\partial u}{\partial t} \geq -\frac{u}{(p-2)t}$$

in the sense of distributions.

(ii) *If $2 > p > 1$, then*

$$\frac{\partial u}{\partial t} \leq \frac{u}{(2-p)t}$$

in the sense of distributions.

Proof. Denote

$$u_r(x, t) = r u(x, r^{p-2} t)$$

for all $(x, t) \in Q$ and all $r \in (\frac{1}{2}, 1)$. Clearly, u_r is a weak solution Eq. (1.1) with the following initial-boundary condition

$$u_r(x, 0) = ru_0(x), \forall x \in \mathbf{I}; \qquad u_r(\pm 1, t) = 0, \forall t \in \mathbf{R}. \tag{2.1}$$

Noting $r \in (\frac{1}{2}, 1)$ and using (1.2) and (2.1) we get

$$u_r(x, 0) \leq u_0(x), \forall x \in \mathbf{I}; \qquad u_r(\pm 1, t) = u(\pm 1, t), \forall t \in \mathbf{R}. \tag{2.2}$$

Applying the comparison principle we have

$$u_r(x, t) \leq u(x, t) \tag{2.3}$$

for all $(x, t) \in Q$.

We consider the cases: (1) $p > 2$, (2) $1 < p < 2$, respectively.

(1) For $p > 2$, by (2.3), we get

$$\frac{[u(x, \lambda t)]^{p-2} - [u(x, t)]^{p-2}}{\lambda t - t} \geq \frac{(1/\lambda - 1)\,[u(x, t)]^{p-2}}{\lambda t - t},$$

where $\lambda = r^{p-2}$. Letting $\lambda \to 1-$, we get

$$\frac{\partial}{\partial t}[u(x, t)]^{p-2} \geq -\frac{1}{t}[u(x, t)]^{p-2}$$

in the distribution, which implies that (i) holds.

(2) For $1 < p < 2$, by (2.3), we have

$$\frac{[u(x, \lambda t)]^{2-p} - [u(x, t)]^{2-p}}{\lambda t - t} \leq \frac{(\lambda - 1)\,[u(x, t)]^{2-p}}{\lambda t - t},$$

where $\lambda = r^{p-2}$. Letting $\lambda \to 1+$, we get

$$\frac{\partial}{\partial t}[u(x, t)]^{2-p} \leq \frac{1}{t}[u(x, t)]^{2-p}$$

in the distribution, which implies that (ii) holds. Thus the proof is completed.

LEMMA 2.3. *Let u be a weak solution of* (1.1)–(1.2). *If $p > 2$, then*

$$\operatorname{supp} u(\cdot, s) \subset \operatorname{supp} u(\cdot, t)$$

for all s, t with $0 < s < t$.

The proof follows from Lemma 2.2.

3. Extinction of Solutions

In this section we shall prove Theorem 1.1 and assume that $1 < p < 2$. Denote

$$s_+ = \max\{s, 0\}$$

for all $s \in (-\infty, +\infty)$.

Define an auxiliary function

$$v(x,t) = k(T-t)_+^{1/(2-p)} \ln(3+x), \tag{3.1}$$

where

$$k = \left[\frac{1}{3^p \ln 3}\right]^{1/(2-p)}, \qquad T = \left(\frac{\|u_0\|_\infty}{k \ln 2}\right)^{2-p}. \tag{3.2}$$

Let us compute

$$\frac{\partial v}{\partial t} = -\frac{k}{2-p}(T-t)_+^{(p-1)/(2-p)} \ln(3+x) \tag{3.3}$$

$$\begin{aligned} \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) &= \frac{\partial}{\partial x}\left[\left(\frac{k(T-t)_+^{1/(2-p)}}{3+x}\right)^{p-2}\left(\frac{k(T-t)_+^{1/(2-p)}}{3+x}\right)\right] \\ &= \frac{\partial}{\partial x}\left(\frac{k(T-t)_+^{1/(2-p)}}{3+x}\right)^{p-1} = -(p-1)\,k^{p-1}\,(T-t)_+^{(p-1)/(2-p)}\,(3+x)^{-p}. \end{aligned} \tag{3.4}$$

Using (3.1)–(3.4) we get

$$\frac{\partial v}{\partial t} \geq \frac{\partial}{\partial x}\left(\left|\frac{\partial v}{\partial x}\right|^{p-2}\frac{\partial v}{\partial x}\right), \tag{3.5}$$

$$v(x,0) \geq u_0(x), \forall x \in [-1,1]; \qquad v(\pm 1, t) \geq 0, \forall t \in \mathbf{R}. \tag{3.6}$$

Applying Lemma 2.1 and (3.5)–(3.6) we get

$$u(x,t) \leq v(x,t)$$

for all $(x,t) \in \mathbf{Q}$. By the definition of $v(x,t)$ we have

$$u(x,t) \le v(x,t) = 0$$

for all $(x, t) \in \mathbf{I} \times (T, +\infty)$. Thus the proof of Theorem 1.1 is completed.

4. Positivity of Solutions

In this section we shall prove Theorem 1.2 and assume that $p \geq 2$. In the case $p = 2$, the conclusion of Theorem 1.2 follows from the theory of the linear uniformly parabolic equations. Here we consider only the case $p > 2$.

In order to prove Theorem 1.2 we need the following lemmas.

Lemma 4.1. *Let u be a weak solution of* (1.1)–(1.2) *and $p > 2$. If*

$$u_0(0) > 0,$$

the there exists a time T_0 such that

$$u(x,t) > 0$$

for all $(x, t) \in (-1, 1) \times (T_0, +\infty)$.

Proof. Let us consider the fundamental solution of Eq. (1.1)

$$\Phi_{k,R}(x,t;x_0) = kRS(t)^{-1/\lambda}\left\{1 - \left[\frac{|x-x_0|}{S^{1/\lambda}(t)}\right]^{p/(p-1)}\right\}_+^{(p-1)/(p-2)} \tag{4.1}$$

for all positive constants k and R and all $x_0 \in (-1, 1)$, where

$$S(t) = R^\lambda + \lambda\left(\frac{p}{p-2}\right)^{p-1} k^{p-2}t, \qquad \lambda = 2(p-1). \tag{4.2}$$

From

$$u_0(0) > 0,$$

it follows that

$$u_0(x) > k, \qquad \forall x \in (-R, R) \subset \mathbf{I}$$

for some positive constant R. Clearly, we have

$$\frac{\partial\Phi_{k,R}(x,t;0)}{\partial t}=\frac{\partial}{\partial x}\left(\left|\frac{\partial\Phi_{k,R}(x,t;0)}{\partial x}\right|^{p-2}\frac{\partial\Phi_{k,R}(x,t;0)}{\partial x}\right) \tag{4.3}$$

$$\begin{aligned}\Phi_{k,R}(x,0;0)\le u_0(x),\ \forall x\in[-1,1];\\ \Phi_{k,R}(\pm 1,t;0)=0,\ \forall t\in(0,T^*),\end{aligned} \tag{4.4}$$

where

$$T^*=\frac{1-R^\lambda}{\lambda k^{p-2}(p/p-2)^{p-1}}. \tag{4.5}$$

By Lemma 2.1, using (1.2), (4.3), and (4.4), we get

$$u(x,t)\ge\Phi_{k,R}(x,t;0) \tag{4.6}$$

for all $(x, t) \in [-1, 1] \times (0, T^*)$, which implies that

$$u(x,T^*)>0 \tag{4.7}$$

for all $x \in (-1, 1)$. By (4.7) and Lemma 2.3 we get

$$u(x,t)>0$$

for all $(x, t) \in (-1, 1) \times (T^*, +\infty)$. Thus the proof is completed.

LEMMA 4.2. *Let u be a weak solution of* (1.1)–(1.2) *and $p > 2$. If*

$$u_0(x_0)>0 \qquad x_0\ne 0, \tag{4.8}$$

then there exists a time T such that

$$u(0,T)>0. \tag{4.9}$$

Proof. Without loss of generality we may assume that

$$0<x_0<1. \tag{4.10}$$

From (4.8), it follows that

$$u_0(x)>k_0=\frac{1}{2}u_0(x_0)>0$$

for all $x \in (x_0 - R_0, x_0 + R_0)$ with some small positive number R_0.

Similar to (4.6), we get

$$u(x, t) \geq \Phi_{k_0, R_0}(x, t; x_0) \tag{4.11}$$

for all $(x, t) \in \mathbf{I} \times (0, T_0)$, where

$$T_0 = \min\{T_0^+, T_0^-\} \qquad T_0^- = \frac{|x_0 + 1|^1 - R_0^\lambda}{\lambda k^{p-2}(p/p-2)^{p-1}}, \qquad T_0^+ = \frac{|x_0 - 1|^\lambda - R_0^\lambda}{\lambda k^{p-2}(p/p-2)^{p-1}} \tag{4.12}$$

which implies

$$u(x, T_0) > 0 \tag{4.13}$$

for all $(x, t) \in (2x_0 - 1, 1)$.

If $x_0 < \frac{1}{2}$, then by (4.11) we have (4.9).

If $x_0 \geq \frac{1}{2}$, we define

$$x_1 = x_0 - d_k,$$

where

$$x_0 > d_k = \frac{1}{k}(1 - x_0) > 0$$

and $k > 2$ is a positive integer.

Similar to (4.13), there exists a time T_1 such that

$$u(x, T_0 + T_1) > 0 \tag{4.14}$$

for all $x \in (2x_1 - 1, 1)$.

If $x_1 < \frac{1}{2}$, then by (4.14) we have (4.9).

If $x_1 \geq \frac{1}{2}$, we define

$$x_2 = x_1 - d_k = x_0 - 2d_k.$$

Similar to (4.13), there exists a time T_2 such that

$$u(x, T_0 + T_1 + T_2) > 0 \tag{4.14}$$

for all $x \in (2x_2 - 1, 1)$.

Repeating the above process we can find two positive integer n^* and k^* such that

$$x_{n^*} = x_0 - n^* d_{k^*}, \qquad -1 < 2x_{n^*} - 1 < 0$$

and

$$u(x, T_1 + T_2 + \cdots + T_{n^*}) > 0$$

for all $x \in (2x_{n^*} - 1, 1)$. Thus the proof is completed.

Proof of Theorem 1.2. From Lemma 4.1 and Lemma 4.2 there exists a time T such that

$$u(x, T) > 0$$

for all $x \in (-1, 1)$. By Lemma 2.3, we have the conclusion of Theorem 1.2. Thus the proof is completed.

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