Global Exponential Asymptotic Stability in Nonlinear Discrete Dynamical Systems

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For the nonlinear discrete dynamical system \( x_{k+1} = T x_k \) on bounded, closed and convex set \( D \subseteq \mathbb{R}^n \), we present several sufficient and necessary conditions under which the unique equilibrium point is globally exponentially asymptotically stable. The infimum of exponential bounds of convergent trajectories is also derived.

Key Words: discrete dynamical systems; difference equations; global exponential asymptotic stability; exponential bound; topologically equivalent metric; contraction map.
1. INTRODUCTION

Consider the following nonlinear discrete dynamical system:

\[ x_{k+1} = Tx_k, \quad k = 0, 1, 2, \ldots, \]

(1)

where \( T : \mathbb{R}^n \to \mathbb{R}^n \) is a continuous differentiable nonlinear operator and \( x^* \in D \subset \mathbb{R}^n \) is an equilibrium point (fixed point). Recall that \( x^* \) is said to be globally exponentially asymptotically stable on \( D \) if there exist constants \( M > 0 \) and \( 0 < \alpha < 1 \) such that for any initial value \( x_0 \in D \) \([1–3]\

\[ \|x_k - x^*\| \leq M \cdot \alpha^k \cdot \|x_0 - x^*\|, \quad k = 0, 1, 2, \ldots \]

where \( \alpha \) is an exponential bound independent of the choice of the equivalent norms and \( M \) is a constant determined by the chosen equivalent norm \( \| \| \). The global exponential asymptotic stability of Eq. (1) is an important research topic in nonlinear numerical analysis and control theory \([1–4]\

However, most existing literature mainly discusses the sufficient conditions under which the equilibrium point \( x^* \) is globally exponentially asymptotically stable, seldom studying the sufficient together with necessary conditions. In this paper we present several sufficient and necessary conditions for this issue. By introducing a new characteristic function, we give a quantitative characterization of global exponential asymptotic stability of Eq. (1) on a bounded, closed, and convex subset. In the following sections, we assume that \( D \) is a bounded, closed, and convex subset, \( T(D) \subset D \), and \( x^* \in D \) is the unique equilibrium point of Eq. (1) in \( D \).

2. DEFINITION AND BASIC PROPERTIES OF CHARACTERISTIC FUNCTION

We first introduce two numerical functions. Let

\[ |T|_{(D, x^*)} = \sup_{x \in D, x \neq x^*} \frac{\|Tx - Tx^*\|}{\|x - x^*\|}; \]

(2)

then \( |T|_{(D, x^*)} \) is a function determined by \( T, D, x^* \), and the chosen equivalent norm \( \| \| \). Since \( |T^{n+m}|_{(D, x^*)} \leq |T^n|_{(D, x^*)} \cdot |T^m|_{(D, x^*)} \) holds for any positive integer \( n \) and \( m \), the limitation \( \lim_{n \to \infty} (|T^n|_{(D, x^*)})^{1/n} \) exists and \( \lim_{n \to \infty} (|T^n|_{(D, x^*)})^{1/n} = \inf_{n \geq 0} (|T^n|_{(D, x^*)})^{1/n} \) \([5]\). Let

\[ Lip(T, D, x^*) = \lim_{n \to \infty} (|T^n|_{(D, x^*)})^{1/n}. \]

(3)
Functions \(\text{Lip}(T, D, x^*)\) and \(|T|_{D, x^*}\) have the following basic properties:

1. The value of \(|T|_{D, x^*}\) is varied under different equivalent norms, but \(\text{Lip}(T, D, x^*)\) is a constant independent of the choice of the equivalent norms. Moreover, \(\text{Lip}(T, D, x^*) \leq |T|_{D, x^*}\) holds under any equivalent norm.

2. If \(T\) is a matrix and \(x^* \in D\) is an inner point, then \(|T|_{D, x^*}\) is the matrix norm of \(T\) and \(\text{Lip}(T, D, x^*)\) is the spectral radius of \(T\).

3. \(|T|_{D, x^*} \geq \|T'(x^*)\|\) and \(\text{Lip}(T, D, x^*) \geq \rho(T'(x^*))\), where matrix \(T'(x^*)\) is the derivative (i.e., Jacobian matrix) of \(T\) at \(x^*\) and \(\rho(T'(x^*))\) denotes the spectral radius of \(T'(x^*)\).

The proof of (3) is as follows:

Since \(T\) is differentiable at \(x^*\), for any vector \(h \in \mathbb{R}^n\) such that \(\|h\| = 1\), we have

\[
\lim_{t \to 0} \frac{\|T(x^* + t \cdot h) - T(x^*) - T'(x^*) \cdot t \cdot h\|}{t} = 0.
\]

This implies that

\[
\lim_{t \to 0} \frac{\|T(x^* + t \cdot h) - T(x^*)\|}{\|t \cdot h\|} = \|T'(x^*) \cdot h\|
\]

and, therefore, \(|T|_{D, x^*} \geq \|T'(x^*)\|\). Thus we have \(|T|_{D, x^*} \geq \|T'(x^*)\|\).

Since \(x^*\) is a fixed point, for any integer \(m > 0\), \(T^m(x^*) = x^*\). Therefore, for any integer \(n > 0\), \((T^n)(x^*) = T'(T^{n-1}x^*) \cdot (T^{n-1})(x^*) = \ldots = T'(x^*)\). It concludes that for any integer \(n > 0\), we have \(|T^n|_{D, x^*} \geq \|T^{n'}(x^*)\| = \|T'(x^*)\|\) and

\[
\text{Lip}(T, D, x^*) = \lim_{n \to \infty} \|T^n|_{D, x^*}\|^{1/n} \geq \lim_{n \to \infty} \|T'(x^*)\|^n^{1/n} = \rho(T'(x^*))
\]

3. MAIN RESULTS

In this section, we use \(\text{Lip}(T, D, x^*)\) to quantitatively describe the global exponential asymptotic stability of equilibrium point \(x^*\) on \(D\). First, we give a Lemma.

**Lemma 1.** If iterative sequence \(\{T^n\}x\) converges uniformly to \(x^*\) for any initial value \(x \in D\), then \(\text{Lip}(T, D, x^*) = \rho(T'(x^*))\).

**Proof.** For arbitrary \(\varepsilon > 0\), there must exist an equivalent norm \(\|\|\) such that \(\|T'(x^*)\| \leq \rho(T'(x^*)) + \varepsilon/2\). In addition, \(T\) is continuously
differentiable at \( x^* \); therefore, there exists a neighborhood \( U \) of \( x^* \) such that \( \|T'(x)\| \leq \|T'(x^*)\| + \varepsilon/2 \leq \rho(T'(x^*)) + \varepsilon \) holds for any \( x \in U \). By the midvalue theorem, we have

\[
\|Tx - Ty\| \leq (\rho(T'(x^*)) + \varepsilon)\|x - y\|, \quad \forall x, y \in U.
\]

Let \( \alpha = \rho(T'(x^*)) + \varepsilon \). Since \( \{T^n x\} \) converges uniformly to \( x^* \), there exists a positive integer \( N \) such that \( T^n z \in U \) for any \( z \in D \) and \( n \geq N \). Thus,

\[
\|T^n z - T^n x^*\| \leq \alpha \|T^{n-1} z - T^{n-1} x^*\| \leq \cdots \leq \alpha^{n-N} \|T^N z - T^N x^*\| \\
\leq \alpha^{n-N} \|T^N|_{(D, x^*)} \||x - x^*\|.
\]

This implies that \( |T^n|_{(D, x^*)} \leq \alpha^{n-N} |T^N|_{(D, x^*)} \) for any \( n > N \). Therefore,

\[
Lip(T, D, x^*) = \lim_{n \to \infty} \left( \frac{|T^n|_{(D, x^*)}}{n} \right)^{1/n} \leq \alpha = \rho(T'(x^*)) + \varepsilon.
\]

Since \( \varepsilon \) is an arbitrary positive number, \( Lip(T, D, x^*) \leq \rho(T'(x^*)) \). We proved \( Lip(T, D, x^*) \geq \rho(T'(x^*)) \) in Section 2; therefore, \( Lip(T, D, x^*) = \rho(T'(x^*)) \).

**Theorem 1.** Equilibrium point \( x^* \) is globally exponentially asymptotically stable on \( D \) if and only if \( Lip(T, D, x^*) < 1 \). Moreover, if \( x^* \) is globally exponentially asymptotically stable, then \( Lip(T, D, x^*) = \rho(T'(x^*)) \), and the value on either side of the equation is the infimum of the exponential bounds.

**Proof** (Necessary condition). If \( x^* \) is globally exponentially asymptotically stable with exponential bound \( \alpha \in (0, 1) \), then \( \{T^n x\} \) converges uniformly to \( x^* \) for any \( x \in D \). This implies that \( Lip(T, D, x^*) = \rho(T'(x^*)) \). Since for any positive integer \( n \), \( T^n \) is continuously differentiable at \( x^* \), we have

\[
\lim_{\|x - x^*\| \to 0} \frac{\|T^n x - T^n x^* - (T^n)'(x^*)(x - x^*)\|}{\|x - x^*\|} = 0.
\]

Therefore, for any \( \varepsilon > 0 \), there exists a neighborhood \( U(\varepsilon) \) of \( x^* \) such that for any \( x \in U(\varepsilon) \),

\[
\|T^n x - T^n x^* - (T^n)'(x^*)(x - x^*)\| \\
= \|T^n x - T^n x^* - (T^n)'(x^*)(x - x^*)\| \leq \varepsilon \|x - x^*\|.
\]

Let

\[
T'(x^*) \cdot (x - x^*) \\
= (T^n x - T^n x^*) - (T^n x - T^n x^* - T'(x^*) \cdot (x - x^*)).
\]
Then for any \( x \in U(\varepsilon) \),
\[
\|T'(x^*)^n \cdot (x - x^*)\|
\leq \|T^n x - T^n x^*\| + \|T^n x - T^n x^* - T'(x^*)^n \cdot (x - x^*)\|
\leq M \cdot \alpha^n \|x - x^*\| + \varepsilon \|x - x^*\|.
\]

This means that
\[
\|T'(x^*)^n\| = \sup_{x \in U(\varepsilon), x \neq x^*} \|T'(x^*)^n \cdot (x - x^*)\|/\|x - x^*\| \leq M \cdot \alpha^n + \varepsilon.
\]

Since \( \varepsilon \) is an arbitrary positive number, \( \|T'(x^*)^n\| \leq M \cdot \alpha^n \). Therefore, we have
\[
Lip(T, D, x^*) = \rho(T'(x^*)) = \lim_{n \to \infty} \|T'(x^*)^n\|^{1/n} \leq \alpha < 1.
\]

(Sufficient condition). Since \( D \) is a bounded and closed subset and \( T \) is a continuous differentiable operator, we have \( \sup_{z \in D} \|T'(z)\| < \infty \). By the midvalue theorem,
\[
\|Tx - Ty\| \leq \sup_{z \in D} \|T'(z)\| \cdot |x - y|, \quad \forall x, y \in D.
\]

Thus \( T \) satisfies the Lipschitzian condition. Let
\[
L(T) = \sup_{x, y \in D, x \neq y} \|Tx - Ty\|/\|x - y\|.
\]

Since \( L(T^{n+m}) \leq L(T^n) \cdot L(T^m) \) holds for any positive integer \( n \) and \( m \), the limitation \( \lim_{n \to \infty} L(T^n)^{1/n} \) exists [5].

Since \( Lip(T, D, x^*) < 1 \), \( T^n x \) converges uniformly to \( x^* \) for any \( x \in D \) and \( \rho(T'(x^*)) < 1 \). For any \( \varepsilon \in (0, 1 - \rho(T'(x^*))) \), there exists an equivalent norm (without loss of generality, we assume \( \| \| \) and \( \| \| \) for any \( x, y \in U \) such that
\[
\|Tx - Ty\| \leq (\rho(T'(x^*)) + \varepsilon) \|x - y\|, \quad \forall x, y \in U
\]
(see the proof of Lemma 1). Denote \( \rho(T'(x^*)) + \varepsilon \) as \( \alpha \). Since \( \{T^n x\} \) converges uniformly to \( x^* \) for any \( x \in D \), there exists a positive integer \( N \) such that \( T^N z \in U \) for any \( z \in D \) and \( n > N \). Thus, for any \( x, y \in D \),
\[
\|T^n x - T^n y\| \leq \alpha \|T^{n-1} x - T^{n-1} y\| \leq \cdots \leq \alpha^{n-N} \|T^N x - T^N y\|
\leq \alpha^{n-N} L(T^N) \|x - y\|.
\]

This implies that \( L(T^n) \leq \alpha^{n-N} L(T^N) \leq \alpha^{n-N} L(T)^N \) for any \( n > N \), and \( \lim_{n \to \infty} L(T^n)^{1/n} \leq \alpha \). Since \( \lim_{n \to \infty} L(T^n)^{1/n} \) is a constant independent of
the choice of the equivalent norms and \( \varepsilon > 0 \) is an arbitrary positive number, we have

\[
\lim_{n \to \infty} L(T^n)^{1/n} \leq \rho(T'(x^*)) = \text{Lip}(T, D, x^*).
\]

Additionally, for any positive integer \( n \), \( L(T^n) \geq |T^n|_{(D,x^*)} \), therefore, \( \lim_{n \to \infty} L(T^n)^{1/n} \geq \text{Lip}(T, D, x^*) \). This concludes that \( \lim_{n \to \infty} L(T^n)^{1/n} = \text{Lip}(T, D, x^*) \). For arbitrary \( \varepsilon \in (0, 1 - \text{Lip}(T, D, x^*)) \), there exists an integer \( N > 0 \) such that \( L(T^N) \leq (\text{Lip}(T, D, x^*) + \varepsilon)^N \). Let

\[
d(x, y) = \sum_{k=1}^{N} (\text{Lip}(T, D, x^*) + \varepsilon)^{k-1} \cdot ||T^{N-k}x - T^{N-k}y||,
\]

\[\forall x, y \in D.
\]

Then \( d(x, y) \) is a metric function that satisfies the following inequalities:

\[
(\text{Lip}(T, D, x^*) + \varepsilon)^{N-1} \cdot ||x - y|| \leq d(x, y) \leq \sum_{k=1}^{N} \left( (\text{Lip}(T, D, x^*) + \varepsilon)^{k-1} \cdot (L(T)^{N-k}) \right) \cdot ||x - y||,
\]

\[\forall x, y \in D.
\]

Let \( C_1 = (\text{Lip}(T, D, x^*) + \varepsilon)^{N-1} \) and \( C_2 = \sum_{k=1}^{N} (\text{Lip}(T, D, x^*) + \varepsilon)^{k-1} \cdot (L(T)^{N-k}) \). We have the following derivations:

\[
d(Tx, Ty) = \sum_{k=1}^{N} (\text{Lip}(T, D, x^*) + \varepsilon)^{k-1} \cdot ||T^{N-(k-1)}x - T^{N-(k-1)}y|| \leq (\text{Lip}(T, D, x^*) + \varepsilon) \cdot \sum_{k=1}^{N} (\text{Lip}(T, D, x^*) + \varepsilon)^{k-1}
\]

\[
\cdot ||T^{N-k}x - T^{N-k}y|| + ||T^{N}x - T^{N}y|| - (\text{Lip}(T, D, x^*) + \varepsilon)^{N} \cdot ||x - y|| \leq (\text{Lip}(T, D, x^*) + \varepsilon) \cdot d(x, y).
\]

Thus for any positive integer \( n \),

\[
d(T^n x, T^n x^*) \leq (\text{Lip}(T, D, x^*) + \varepsilon) \cdot d(T^{n-1} x, T^{n-1} x^*) \leq \cdots \leq (\text{Lip}(T, D, x^*) + \varepsilon)^n \cdot d(x, x^*)
\]
and

$$\|T^n x - x^*\| \leq \frac{d(T^n x, T^n x^*)}{C_1} \leq \frac{(\text{Lip}(T, D, x^*) + \varepsilon)^n \cdot d(x, x^*)}{C_1}$$

$$\leq \frac{C_2}{C_1} \cdot (\text{Lip}(T, D, x^*) + \varepsilon)^n \cdot \|x - x^*\|,$$

i.e., $x^*$ is globally exponentially asymptotically stable, and the exponential bound is $\text{Lip}(T, D, x^*) + \varepsilon$.

When $x^*$ is globally exponentially asymptotically stable, $\text{Lip}(T, D, x^*) \leq \beta$ holds for any exponential bound $\beta$. Meanwhile, $\text{Lip}(T, D, x^*) + \varepsilon$ is an exponential bound for any positive number $\varepsilon$. Thus $\text{Lip}(T, D, x^*)$ is the infimum of the exponential bounds.

**Corollary 1.** $x^*$ is globally exponentially asymptotically stable on $D$ if and only if one of the following conditions holds:

1. There exists a positive integer $N_0$ such that $T^{N_0}$ is a contractive operator.
2. $T$ is a contractive operator with respect to a metric function $d(x, y)$ satisfying the inequality $C_1 \cdot \|x - y\| \leq d(x, y) \leq C_2 \cdot \|x - y\|$, $\forall x, y \in D$, $0 < C_1 \leq C_2$ are two constants.

**Remark 1.** It is known that a matrix $A$ is exponentially stable if and only if spectral radius $\rho(A) < 1$ [3]. Obviously, Theorem 1 generalizes this well-known result to a nonlinear operator. Corollary 1 actually presents a converse to the Banach contraction map theorem.

From Theorem 1, we can obtain easily global exponential asymptotic stability condition for a class of special nonlinear discrete dynamical systems.

**Corollary 2.** If the original $x^* = \theta$ is the unique equilibrium point of $T$ on $D$, and $T(\lambda x) = |\lambda| \cdot T(x)$, $\lambda \in R$, $x \in D$, then $\theta$ is globally exponentially asymptotically stable if and only if $\rho(T'(\theta)) < 1$.

**Proof.** By Theorem 1, we have $\rho(T'(\theta)) < 1$ if $\theta$ is globally exponentially asymptotically stable. Conversely, suppose $\rho(T'(\theta)) < 1$. In Theorem 1, we proved that $T$ satisfies Lipschitzian conditions on $D$. Therefore, there exists a sufficient large positive integer $B > 1$ such that $G = T/B$ is a contraction map on $D$. Since $D$ is a closed, convex subset and $T(\theta) = \theta$, $G(D) \subset D$ results. By the Banach contraction theorem [6], the iterative sequence $(G^n x)$ converges uniformly to $\theta$ for any initial value $x \in D$. 
Therefore, we have
\[
\frac{\text{Lip}(T)}{B} = \text{Lip}(G) = \rho(G'(\theta)) = \rho\left(\frac{T'(\theta)}{B}\right) = \rho(T'(\theta)).
\]

Obviously, \(\text{Lip}(T) = \rho(T'(\theta)) < 1\) and \(\theta\) is globally exponentially asymptotically stable.

In finite-dimensional space, the stability and exponential asymptotic stability of a linear system are equivalent [3, Theorem 4.2.2], but this generally is not the case for a nonlinear system. About the relation between stability and global exponential asymptotic stability of a nonlinear system, we have the following result.

**Theorem 2.** Equilibrium point \(x^*\) is globally exponentially asymptotically stable on \(D\) if and only if the two following conditions are satisfied:

1. Iterative sequence \(\{T^n x\}\) converges to \(x^*\) for any \(x \in D\).
2. \(\rho(T^n(x^*)) < 1\).

**Proof.** We need only consider the sufficient condition. First, we prove that \(\{T^n x\}\) converges uniformly to \(x^*\) for any \(x \in D\). In Lemma 1 we proved that for any \(\varepsilon \in (0, 1 - \rho(T'(x^*)))\), there exists an equivalent norm \(\|\|\|\) and a neighborhood \(U\) of \(x^*\) such that \(\|Tx - Ty\| \leq (\rho(T'(x^*)) + \varepsilon)|x - y|\), \(\forall x, y \in U\). Therefore, \(T\) is a contractive operator on \(U\) and \(T^n(U) \rightarrow \{x^*\}\) as \(n \rightarrow \infty\). Since \(\{T^n x\}\) converges to \(x^*\) for any \(x \in D\) and \(T^n(U) \rightarrow \{x^*\}\), there exists a topologically equivalent metric \(d(x, y)\) on \(D\) such that \(T\) is a contractive operator with respect to this metric. Since \(D\) is a compact set (bounded closed set) with respect to norm \(\|\|\|\), and the metric \(d(x, y)\) is topologically equivalent to the norm \(\|\|\), \(D\) is also a compact set with respect to metric \(d(x, y)\). This leads to the conclusion that \(D\) is a bounded closed set with respect to metric \(d(x, y)\) and \(diam(D) < \infty\). Let \(A_n = T^n D\), \(M = \bigcap_{n=0}^\infty A_n\). Then \(A_n \subset A_{n-1}\) and \(x^* \in M\). Suppose there is another point \(x_n \in M\); then \(\forall n, \exists y_n \in D\) such that \(T^n y_n = x_n\). For any \(n\),
\[
d(x^*, x_n) = d(T^n x^*, T^n y_n) \leq \alpha^n \cdot d(x^*, y_n) \leq \alpha^n \cdot diam(D) \rightarrow 0.
\]
Therefore, \(d(x^*, x_n) = 0\), \(x^* = x_n\), \(\bigcap_{n=0}^\infty A_n = \{x^*\}\).

Let \(diam(A_n) = \text{sup}\{d(x, y) : x, y \in A_n\}\) and \(diam(A_n) = \text{sup}\{\|x - y\| : x, y \in A_n\}\). Then \(\lim_{n \rightarrow \infty} diam(A_n) = \lim_{n \rightarrow \infty} diam(T^n D) = \lim_{n \rightarrow \infty} \alpha^n \cdot diam(D) = 0\). Since the metric \(d(x, y)\) and the norm \(\|\|\) are topologically equivalent, for any given \(\varepsilon > 0\), \(B(x^*, \varepsilon) = \{x \in D : \|x - x^*\| < \varepsilon\}\) is the
neighborhood of \( x^* \) with respect to both the norm \( \| \cdot \| \) and the metric \( d(x, y) \). Therefore, there exists an integer \( N > 0 \) such that \( A_n \subset B(x^*, \varepsilon) \) as \( n > N \). Obviously, this leads to the conclusion that \( \lim_{n \to \infty} \text{diam}(A_n) = 0 \) and \( (T^n) \text{x} \) converges uniformly to \( x^* \) for any \( x \in D \). By Lemma 1 and Theorem 1, \( \text{Lip}(T, D, x^*) = \rho(T'(x^*)) < 1 \) and \( x^* \) is globally exponentially asymptotically stable on \( D \).

**Corollary 3.** Suppose iterative sequence \( (T^n) \text{x} \) converges to \( x^* \) for any \( x \in D \). Then:

1. \( x^* \) is globally exponentially asymptotically stable on \( D \) if and only if \( x^* \) is locally exponentially stable. Here \( x^* \) is locally exponentially stable if there is a local neighborhood \( U \) of \( x^* \) and constants \( M > 0 \) and \( 0 < \alpha < 1 \) such that for any initial value \( x_0 \in U \),

\[
\|x_k - x^*\| \leq M \cdot \alpha^k \cdot \|x_0 - x^*\|, \quad k = 0, 1, 2, \ldots
\]

2. \( x^* \) is globally uniformly asymptotically stable on \( D \) if and only if \( x^* \) is locally uniformly stable. Here \( x^* \) is globally uniformly asymptotically stable on \( D \) if for any neighborhood \( U \) of \( x^* \) there exist an integer \( N > 0 \) such that \( T^nz \in U \) for any \( z \in D \) and \( n > N \). We say that \( x^* \) is locally uniformly stable if there is a local neighborhood \( \tilde{U} \) of \( x^* \) such that for any neighborhood \( U \) of \( x^* \), there exist an integer \( N > 0 \) such that \( T^nz \in \tilde{U} \) for any \( z \in \tilde{U} \) and \( n > N \).

In the following, we offer an example to illustrate the difference between the globally exponentially asymptotically stable property of nonlinear discrete dynamic system \( x_n = T_{n-1}x + b \) and \( x_{k+1} = Tx_k \). Here \( b \in \mathbb{R}^n \) is a given constant vector. For a matrix \( A \), it is known the linear difference equations \( x_n = Ax_{n-1} \) and \( x_n = Ax_{n-1} + b \) share the same stable property. However, for a nonlinear operator, this again does not hold. Suppose that the equation \( x_{k+1} = Tx_k \) is globally exponentially asymptotically stable but \( x_n = T_{n-1}x + b \) might or might not be globally exponentially asymptotically stable. Furthermore, even if \( x_n = T_{n-1}x + b \) is globally exponentially asymptotically stable, the infimum of its exponential bounds of convergent trajectories is different from that of \( x_{k+1} = Tx_k \). For example, suppose \( f(x) = .5 \cdot \sin x \) and \( \tilde{f}(x) = .5 \cdot \sin x + 1, \quad x \in [-2, +2] \). It is easy to observe that both \( f \) and \( \tilde{f} \) are contraction operators. The fixed point of \( f \) is 0, and the fixed point of \( \tilde{f} \) is \( x^* \in (5, 1) \). Therefore, both 0 and \( x^* \) are globally exponentially asymptotically stable. But, by Theorem 1, \( \text{Lip}(f) = .5 \cdot \cos 0 = .5 \), \( \text{Lip}(\tilde{f}) = .5 \cdot \cos(x^*) < .5 \); i.e., the infimums of the exponential bounds of convergent trajectories of both equations are different from each other.
4. CONCLUSION

In this paper we have introduced a characteristic function that can be used to quantitatively describe the global exponential asymptotic stability of nonlinear discrete dynamical system on a bounded, closed, and convex subset. Several sufficient and necessary conditions have been presented. The result generalizes a well-known fact in matrix stability theory. Based on Theorem 2 and Corollary 3, we give a method for distinguishing global exponential asymptotic stability from global uniform asymptotic stability. We note that many sufficient conditions obtained in literature for global asymptotic stability are actually the sufficient conditions for global asymptotic exponential stability.

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