Journal of Algebra 344 (2011) 114–136
Contents lists available at ScienceDirect
Journal of Algebra
www.elsevier.com/locate/jalgebra

Cohomology of toroidal orbifold quotients

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ARTICLE INFO

Article history:
Received 16 March 2010
Available online 23 August 2011
Communicated by Michel Broué

Keywords:
Toroidal orbifolds
Crystallographic groups
Serre spectral sequence
Group cohomology

ABSTRACT

Let $\varphi : \mathbb{Z}/p \to GL_n(\mathbb{Z})$ denote an integral representation of the cyclic group of prime order p. This induces a $\mathbb{Z}/p$-action on the torus $X = \mathbb{R}^n/\mathbb{Z}^n$. The goal of this paper is to explicitly compute the cohomology groups $H^*(X/\mathbb{Z}/p;\mathbb{Z})$ for any such representation. As a consequence we obtain an explicit calculation of the integral cohomology of the classifying space associated to the family of finite subgroups for any crystallographic group $\Gamma = \mathbb{Z}^n \rtimes \mathbb{Z}/p$ with prime holonomy.

1. Introduction

Let G be a finite group and $\varphi : G \to GL_n(\mathbb{Z})$ an integral representation of G. In this way G acts linearly on $\mathbb{R}^n$ preserving the integral lattice $\mathbb{Z}^n$, hence inducing a G-action on the torus $X_\varphi = X := \mathbb{R}^n/\mathbb{Z}^n$. The quotient $X \to X/G$ has the natural structure of a global quotient orbifold and is an example of what is often called a toroidal orbifold. The goal of this paper is to explicitly compute the cohomology groups $H^*(X/G;\mathbb{Z})$ for the particular case where $G = \mathbb{Z}/p$ is the cyclic group of prime order.

The indecomposable integral representations of $\mathbb{Z}/p$ have been completely classified (see [6]). In general, if L is a $\mathbb{Z}G$-lattice then there are unique integers r, s and t and an isomorphism

$$L \cong \left(\bigoplus_r A_i\right) \oplus \left(\bigoplus_s P_j\right) \oplus \left(\bigoplus_t \mathbb{Z}\right),$$

where $\mathbb{Z}$ is the trivial $\mathbb{Z}/p$-module, each A_i is an indecomposable module that corresponds to an element of the ideal class group of $\mathbb{Z}[\zeta_p]$, where ζ_p is a primitive p-th root of unity, and each P_j

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[1] Partially supported by NSERC.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.08.004

is a projective indecomposable module of rank p as an abelian group. In this case L is said to be a $\mathbb{Z}/p$-module of type (r,s,t). The situation simplifies after localizing at the prime p. Let $\mathbb{Z}_{(p)}$ be the ring of integers localized at the prime p. Then (see [6]), there are only three distinct isomorphism classes of indecomposable $\mathbb{Z}_{(p)}G$-lattices, namely the trivial module $\mathbb{Z}_{(p)}$, the augmentation ideal IG and the group ring $\mathbb{Z}_{(p)}G$. Moreover, if L is any finitely generated $\mathbb{Z}G$-lattice, then there is a $\mathbb{Z}G$-lattice $L' \cong IG^r \oplus \mathbb{Z}G^s \oplus \mathbb{Z}^t$ and a $\mathbb{Z}G$-homomorphism $f : L' \to L$ such that f is an isomorphism after tensoring with $\mathbb{Z}_{(p)}$.

In this paper it is shown that given a $\mathbb{Z}G$-lattice L induced by an integral representation φ, the cohomology groups $H^*(X_\varphi/G;\mathbb{Z})$ only depends on the type of L. Moreover, if L is of type (r,s,t) then explicit descriptions for these cohomology groups are obtained in terms of r, s and t. More precisely the goal of this paper is to prove the following theorem.

Theorem 1. *Let $G=\mathbb{Z}/p$ where p is a prime number. Suppose that X is the G-space induced by a $\mathbb{Z}G$-lattice L of type (r,s,t) and rank n. Then*

$$H^k(X/G;\mathbb{Z}) \cong \mathbb{Z}^{\alpha_k} \oplus (\mathbb{Z}/p)^{\beta_k},$$

and the coefficients α_k and β_k are given as follows: consider the formal power series in x

$$F_L(x) = \left(\frac{1-(\alpha x)^p}{1-\alpha x}\right)^r \left(1+\epsilon_p x^p\right)^s (1+x)^t$$

subject to the relations $\alpha^2=1$, $\epsilon_2=\alpha$ and $\epsilon_p=1$ for $p>2$. Using these relations, $F_L(x)$ can be written in the form

$$F_L(x) = \sum_{i\geqslant 0} f_i x^i + \sum_{i\geqslant 0} g_i \alpha x^i,$$

then

$$\alpha_k = \frac{1}{p}\left[\binom{n}{k} + (p-1)(f_k-g_k)\right].$$

Similarly, β_k is obtained by writing the formal series in x

$$T_L(x) = \frac{x(1+x)^t}{1-x^2}\left[p^r x^2(1+x)^s - x^2 + 1 - (1+\alpha x)\left(1+\epsilon_p x^p\right)^s \left(\frac{1-(\alpha x)^p}{1-\alpha x}\right)^r\right]$$

in the form $T_L(x)=\sum_{i\geqslant 0}\beta_i x^i + \sum_{i\geqslant 0}\gamma_i \alpha x^i$, where α and ϵ_p subject to the same relations as above.

Toroidal orbifolds naturally appear in different geometric contexts. In dimension six they are a source of key examples connected to mathematical aspects of orbifold string theory (see [3]). They also arise in the context of spaces of representations; it can be shown that the moduli space of isomorphism classes of flat connections on principal stable symplectic bundles over the torus $(\mathbb{S}^1)^n$, can be described as the infinite symmetric product of a toroidal orbifold,

$$Rep\left(\mathbb{Z}^n, Sp\right) := \operatorname*{colim}_{m\to\infty} Rep\left(\mathbb{Z}^n, Sp(m)\right) \cong SP^\infty\left(\left(\mathbb{S}^1\right)^n/\mathbb{Z}/2\right)$$

where $\mathbb{Z}/2$ acts diagonally by complex conjugation, an action which arises from the direct sum of copies of the sign representation. In fact this space turns out to be a product of Eilenberg–MacLane spaces determined precisely by the homology of the quotient orbifold $(\mathbb{S}^1)^n/\mathbb{Z}/2$ (see [2] for details).

Similarly, recall that given a topological space Y, the m-th cyclic product of Y is defined to be the quotient $CP^m(Y) := Y^m/\mathbb{Z}/m$, where $\mathbb{Z}/m$ acts on Y^m by a cyclic permutation. The calculations here provide a complete computation for the homology of the p-th cyclic powers of any torus $(\mathbb{S}^1)^n$, as the permutation action corresponds to a direct sum of copies of the regular representation of $\mathbb{Z}/p$. Note that a method for such calculations was formulated long ago by Swan [10]; the approach outlined here is of course much more explicit.

However the most important motivation for these calculations arises from the study of topological invariants of crystallographic groups with prime holonomy. Given a rank n integral representation of $\mathbb{Z}/p$, it can be easily seen that this gives rise to an action of the semi-direct product $\Gamma = \mathbb{Z}^n \rtimes \mathbb{Z}/p$ on $Y = \mathbb{R}^n$ with the following crucial properties: for a subgroup $H \subset \Gamma$, Y^H is non-empty if and only if H is a finite subgroup of Γ, and furthermore in that case Y^H is contractible. Thus Y is a universal space for the family of finite subgroups in Γ, denoted by $\underline{E}\Gamma$ (see [9] for definitions), and the associated classifying space is $\underline{B}\Gamma = \underline{E}\Gamma/\Gamma$, which in this case is precisely the orbifold quotient $(\mathbb{S}^1)^n/\mathbb{Z}/p$. Thus our main result together with the results in [5] can be reformulated as follows:

Theorem 2. *Let Γ be a crystallographic group with holonomy of prime order p, expressed as an extension*

$$1 \to L \to \Gamma \to \mathbb{Z}/p \to 1. \tag{1}$$

Then the cohomology of the classifying space for the family of all finite subgroups in Γ can be explicitly computed and depends only on the representation type of L over the ring of integers localized at p as follows:

- *If Γ is torsion-free then Γ is a Bieberbach group, $\underline{B}\Gamma = B\Gamma$ and $H^*(\underline{B}\Gamma;\mathbb{Z})$ can be computed using* [5, *Theorem* 2].
- *If Γ is not torsion-free, then the sequence* (1) *splits, $\Gamma = L \rtimes \mathbb{Z}/p$ and $H^*(\underline{B}\Gamma;\mathbb{Z})$ can be computed using Theorem* 1.

Using methods from equivariant K-theory and an analysis analogous to that done in this paper, these computations will serve as important input for the calculation of the complex K-theory of $\underline{B}\Gamma$ for Γ a crystallographic group with prime holonomy.

2. Preliminaries

Let G be a finite group and $\varphi : G \to GL_n(\mathbb{Z})$ an integral representation of G. Consider $X = X_\varphi$ the G-space induced by the representation φ. Then the fibration sequence

$$X \to X \times_G EG \to BG \tag{2}$$

induces a long exact sequence of homotopy groups

$$\cdots \to \pi_i(X) \to \pi_i(X \times_G EG) \to \pi_i(BG) \to \pi_{i-1}(X) \to \cdots.$$

This sequence is trivial for $i > 1$ and thus $X \times_G EG$ is an Eilenberg–MacLane space of type $K(\Gamma, 1)$, where $\Gamma := \pi_1(X \times_G EG)$ fits into a short exact sequence

$$1 \to \pi_1(X) \to \Gamma \to G \to 1. \tag{3}$$

The action of G on X makes

$$L := \pi_1(X) \cong H_1(X;\mathbb{Z}) \cong \mathbb{Z}^n$$

into a $\mathbb{Z}G$-module that corresponds to the representation φ. Moreover, $[0] \in \mathbb{R}^n/\mathbb{Z}^n = X$ is a fixed point for this action and thus (2) has a section. This implies that the extension (3) splits and thus

$\Gamma \cong L \rtimes G$. For example, when the representation φ is faithful the group Γ is a crystallographic group.

The cohomology groups of groups of the form $\Gamma \cong L \rtimes G$, for $G = \mathbb{Z}/p$ with p a prime number, were computed in [3, Theorem 1.1]. In there it was proved that the Lyndon–Hochschild–Serre spectral sequence associated to the short exact sequence

$$1 \to L \to \Gamma \to G \to 1$$

collapses on the E_2-term without extension problems. This can be seen as follows. Suppose first that L is a $\mathbb{Z}G$-lattice of the form $L = IG^r \oplus \mathbb{Z}G^s \oplus \mathbb{Z}^t$. For such lattices it follows by [3, Theorem 3.2] and [3, Proposition 3.3] that there is a special free resolution $\epsilon : F \to \mathbb{Z}$ of $\mathbb{Z}$ as a $\mathbb{Z}[L]$-module admitting an action of G compatible with φ. Thus by [3, Theorem 2.4] the corresponding Lyndon–Hochschild–Serre spectral sequence collapses in this particular case. Suppose now that L is any $\mathbb{Z}G$-lattice. Then we can find a $\mathbb{Z}G$-lattice $L' \cong IG^r \oplus \mathbb{Z}G^s \oplus \mathbb{Z}^t$ and a $\mathbb{Z}G$-homomorphism $f : L' \to L$ such that f is an isomorphism after tensoring with $\mathbb{Z}_{(p)}$. By comparing the spectral sequences corresponding to L and L' as done in [3, Theorem 4.1] it can be seen that the Lyndon–Hochschild–Serre corresponding to L also collapses on the E_2-term without extension problems. Therefore, for any $k \geqslant 0$

$$H^k(\Gamma; \mathbb{Z}) \cong \bigoplus_{i+j=k} H^i\Big(G; \bigwedge^j (L^*)\Big).$$

Here L^*, as usual, denotes the dual G-module $\mathrm{Hom}(L, \mathbb{Z})$. As an application of this, by [3, Theorem 1.2], if $G = \mathbb{Z}/p$ acts on X via a representation $\varphi : G \to GL_n(\mathbb{Z})$, then for each $k \geqslant 0$

$$H^k_G(X; \mathbb{Z}) \cong \bigoplus_{i+j=k} H^i\big(G; H^j(X; \mathbb{Z})\big). \tag{4}$$

This completely describes the additive structure of the equivariant cohomology groups $H^k_G(X; \mathbb{Z})$. Moreover, these groups can explicitly be computed as observed in [5] in the following way. Suppose that L is a $\mathbb{Z}G$-lattice of type (r, s, t) and rank n. Consider the formal power series in x

$$F_L(x) = \left(\frac{1 - (\alpha x)^p}{1 - \alpha x}\right)^r \big(1 + \epsilon_p x^p\big)^s (1 + x)^t$$

subject to the relations $\alpha^2 = 1$, $\epsilon_2 = \alpha$ and $\epsilon_p = 1$ for $p > 2$. Using these relations, the formal series $F_L(x)$ can be written in the form

$$F_L(x) = \sum_{i \geqslant 0} f_i x^i + \sum_{i \geqslant 0} g_i \alpha x^i$$

for integer numbers f_i and g_i for $i \geqslant 0$. The coefficients f_i and g_i determine the type of the $\mathbb{Z}G$-module $\bigwedge^i L$. Indeed, by [5, Corollary 5.8] if

$$h_i = \frac{1}{p}\left[\binom{n}{i} + (p-1)(f_i - g_i)\right]$$

then $\bigwedge^i L$ is of type $(g_i, h_i - f_i, f_i)$ and thus

$$H^0\Big(G, \bigwedge^i L\Big) = \mathbb{Z}^{h_i}, \qquad H^1\Big(G, \bigwedge^i L\Big) = (\mathbb{Z}/p)^{g_i}, \qquad H^2\Big(G, \bigwedge^i L\Big) = (\mathbb{Z}/p)^{f_i}.$$

As a corollary the following is obtained.

Corollary 3. *Suppose $G = \mathbb{Z}/p$ acts on X via a representation $\varphi : G \to GL_n(\mathbb{Z})$ inducing a $\mathbb{Z}G$-lattice L of type (r, s, t). Then for each $k \geqslant 0$*

$$H^k_G(X;\mathbb{Z}) \cong \mathbb{Z}^{a_k} \oplus (\mathbb{Z}/p)^{b_k}$$

and the coefficients a_k and b_k are given as follows: write

$$F_L(x) = \sum_{i \geqslant 0} f_i x^i + \sum_{i \geqslant 0} g_i \alpha x^i$$

then

$$a_k = \frac{1}{p}\left[\binom{n}{k} + (p-1)(f_k - g_k)\right].$$

Similarly the coefficients b_k can be obtained in the following way. Fix $k \geqslant 0$ and write the formal power series

$$G_{L,k}(x) = \alpha x\left(\frac{1-(\alpha x)^k}{1-\alpha x}\right)\left(\frac{1-(\alpha x)^p}{1-\alpha x}\right)^r \left(1+\epsilon_p x^p\right)^s (1+x)^t$$

in the form $G_{L,k}(x) = \sum_{i \geqslant 0} c_{i,k} x^i + \sum_{i \geqslant 0} d_{i,k} \alpha x^i$ where α and ϵ_p subject to the same relations as above. Then $b_k = c_{k,k}$ is the coefficient of x^k in $G_{L,k}(x)$.

This corollary can be used as a first step towards the computation of the cohomology groups of the form $H^k(X/G;\mathbb{Z})$ as is shown next.

Corollary 4. *Let X be induced by an integral representation $\varphi : G \to GL_n(\mathbb{Z})$ inducing a $\mathbb{Z}G$-lattice of type (r, s, t). Then*

$$H^k(X/G;\mathbb{Z}) = F_k \oplus T_k,$$

where F_k is a free abelian group of rank

$$\alpha_k = \frac{1}{p}\left[\binom{n}{k} + (p-1)(f_k - g_k)\right]$$

and T_k is a p-torsion abelian group.

Proof. Consider the map $\phi : X \times_G EG \to X/G$ obtained by mapping EG to a point. Given $x \in X$, $\phi^{-1}([x]) \cong BG_x$, where G_x is the isotropy group of G at x. Since $G = \mathbb{Z}/p$, then G_x is either the trivial subgroup or G. In either case BG_x has trivial cohomology with coefficients in $\mathbb{Q}$ and also with coefficients in $\mathbb{F}_q$, the field with q-elements for a prime q different from p. The Vietoris–Begle theorem shows that ϕ induces isomorphisms

$$\phi^* : H^*(X/G;\mathbb{Q}) \xrightarrow{\cong} H^*_G(X;\mathbb{Q}),$$
$$\phi^* : H^*(X/G;\mathbb{F}_q) \xrightarrow{\cong} H^*_G(X;\mathbb{F}_q).$$

Using the previous corollary and the universal coefficient theorem we see that

$$H^k_G(X;\mathbb{Q}) \cong \mathbb{Q}^{\alpha_k} \quad \text{and} \quad H^k_G(X;\mathbb{F}_q) \cong \mathbb{F}_q^{\alpha_k}$$

and the corollary follows. □

The previous corollary reduces our problem to computing T_k, the p-torsion subgroup of $H^k(X/G;\mathbb{Z})$. The strategy that we will follow to compute these groups is as follows. Consider $\phi : X \times_G EG \to X/G$ the map defined above and let F be the fixed point set of X under the action of G. Then [4, Proposition VII 1.1] shows that ϕ induces an isomorphism

$$\phi^* : H^*(X/G, F;\mathbb{Z}) \to H^*_G(X, F;\mathbb{Z}).$$

Via this isomorphism, the groups T_k can be computed using the following steps. First we compute the p-torsion subgroups of $H^*_G(X, F;\mathbb{Z})$. Then we use this information together with the long exact sequence in cohomology associated to the pair $(X/G, F)$ to deduce the structure of T_k.

We establish now some notation. Let $R^* = H^*(G,\mathbb{Z})$. Then R^* can be seen as a graded commutative ring whose structure is given by $R^* = \mathbb{Z}[t]/(pt)$, where $deg(t) = 2$. Graded R^*-modules of the form $M = \bigoplus_{n\geqslant 0} M^n$, where M^n is a finite dimensional $\mathbb{F}_p$-vector space for $n > 0$, appear naturally in our computations. For such modules we have the following definition.

Definition 5. Given a graded R^*-module $M = \bigoplus_{n\geqslant 0} M^n$ as above, define the formal power series in $\mathbb{Z}[[x]]$

$$q_M(x) := a_1x + a_2x^2 + a_3x^3 + \cdots$$

where $a_n = \dim_{\mathbb{F}_p} M^n$ for $n > 0$.

If M is a graded R^*-module M as above, the series $q_M(x)$ together with M^0 completely determine the structure of M as an abelian group. For example, if L is a $\mathbb{Z}G$-lattice, then $H^*(G, L)$ is a graded R^*-module of this kind and the series $q_{H^*(G,L)}(x)$ can be explicitly computed in the following way. Suppose first that A is an indecomposable module of rank $p-1$ that corresponds to an element of the ideal class group of $\mathbb{Z}[\zeta_p]$. Then by [1, Corollary 1.7], it follows that $S^* := H^*(G, A)$ is a graded R^*-module such that $S^n = 0$ for n even and $S^n = \mathbb{Z}/p$ for $n > 0$ odd. Also, given any projective indecomposable module P of rank p then by [1, Proposition 1.5], $H^*(G, P) = \mathbb{Z}$ is the trivial graded R^*-module concentrated on degree 0. Therefore, given a $\mathbb{Z}G$-lattice L of type (r, s, t) there is an isomorphism of graded R^*-modules

$$H^*(G, L) \cong \mathbb{Z}^s \oplus \left(R^*\right)^t \oplus \left(S^*\right)^r.$$

In particular

$$q_{H^*(G,L)}(x) = rx + tx^2 + rx^3 + tx^4 \cdots \tag{5}$$

$$= \frac{rx + tx^2}{1 - x^2}. \tag{6}$$

To compute the p-torsion subgroup of $H^k_G(X, F;\mathbb{Z})$ we use the Serre spectral sequence

$$E_2^{i,j} = H^i\big(G, H^j(X, F;\mathbb{Z})\big) \Longrightarrow H^{i+j}_G(X, F;\mathbb{Z}) \tag{7}$$

associated to the pair (X, F). In this spectral sequence when j and r are fixed, $E_r^{*,j}$ is a graded R^*-module of the kind considered in Definition 5 and each differential

$$d_r : E_r^{*,j} \to E_r^{*,j-r+1}$$

is homomorphism of graded R^*-modules of degree r. In here we will show that the different formal power series $q_{E_\infty^{*,j}}(x)$, for $j \geqslant 0$, determine the p-torsion subgroups of $H_G^k(X, F; \mathbb{Z})$. This will be done by determining the nontrivial differentials in the spectral sequence (7). To do this we first consider the particular cases of $\mathbb{Z}G$-lattices of type $(r, 0, 0)$ in Section 4 and type $(0, s, 0)$ in Section 5. Then we use this information to handle the general case in Section 6.

The following lemma plays a key role in our computations.

Lemma 6. *Suppose that p is a prime number. Let $G = \mathbb{Z}/p$ act on a finistic space X with fixed point set F. If there is an integer N such that $H^k(X, F; \mathbb{Z}) = 0$ for $k > N$, then $H_G^k(X, F; \mathbb{Z}) = 0$ for $k > N$.*

Proof. This follows by applying [4, Exercise III.9] and [4, Proposition VII 1.1]. □

3. Structure of the fixed points

In this section we investigate the nature of the fixed point set of the action of G on a torus X induced by a general $\mathbb{Z}G$-lattice L.

To start, note that if L and M are two $\mathbb{Z}G$-lattices then as G-spaces

$$X_{L \oplus M} = X_L \times X_M. \tag{8}$$

In particular this shows that $F_{L\oplus M} = F_L \times F_M$. Here F_L denotes the fixed point set of the action of G on X_L for a given $\mathbb{Z}G$-lattice L.

We consider next the particular cases $L = A$, $L = P$ and $L = \mathbb{Z}$, where A and P are indecomposable modules as of rank $p - 1$ and rank p respectively, as described before.

Lemma 7. *Let A be an indecomposable module of rank $p - 1$ corresponding to an element of the ideal class group of $\mathbb{Z}[\zeta_p]$. Then the fixed point set F of the G-action on the induced torus X is a discrete set with p points.*

Proof. Consider the short exact sequence of G-modules defining the G-space X

$$0 \to A \to A \otimes \mathbb{R} \to (A \otimes \mathbb{R})/A = X \to 0.$$

This short exact sequence induces a long exact sequence on the level of group cohomology

$$0 \to H^0(G, A) \to H^0(G, A \otimes \mathbb{R}) \to H^0(G, X) \to H^1(G, A) \to H^1(G, A \otimes \mathbb{R}) \to \cdots.$$

Note that $H^1(G, A \otimes \mathbb{R}) = 0$ and $H^0(G, A) = H^0(G, A \otimes \mathbb{R}) = 0$, thus there is an isomorphism

$$F = H^0(G, X) \cong H^1(G, A) \cong \mathbb{Z}/p. \qquad \square$$

Lemma 8. *Let P be a projective indecomposable module of rank p. Then $F_P \cong \mathbb{S}^1$. Moreover, there is a commutative diagram*

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r, "\Delta"] \arrow[d, "h_1"'] & (\mathbb{S}^1)^p \arrow[d, "h_2"] \\ F_P \arrow[r, "i"] & X_P, \end{tikzcd} \tag{9}$$

where Δ denotes the diagonal inclusion of $\mathbb{S}^1$ into $(\mathbb{S}^1)^p$ and h_1, h_2 are covering maps of degree relatively prime with p.

Proof. Consider the exact sequence defining X_P

$$0 \to P \to P \otimes \mathbb{R} \to X_P \to 0.$$

This yields a short exact sequence of the form

$$0 \to P^G \to (P \otimes \mathbb{R})^G \to (X_P)^G \to 0$$

as $H^1(G, P) = 0$. Note that $P^G \cong \mathbb{Z}$ and $(P \otimes \mathbb{R})^G \cong \mathbb{R}$, therefore $F = X_P^G \cong \mathbb{S}^1$. To prove the second assertion, suppose that $L = \mathbb{Z}G$. Then in this particular case it is easy to see that the inclusion $i : F_{\mathbb{Z}G} \to X_{\mathbb{Z}G}$ corresponds to the diagonal inclusion $\Delta : \mathbb{S}^1 \to (\mathbb{S}^1)^p$. Let $f : \mathbb{Z}G \to P$ be homomorphism of $\mathbb{Z}G$-modules such that f is an isomorphism after tensoring with $\mathbb{Z}_{(p)}$. In particular, f is an isomorphism after tensoring with $\mathbb{R}$ and we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "f^G"'] & \mathbb{R} \arrow[r] \arrow[d, "(f\otimes 1)^G"'] & (X_{\mathbb{Z}G})^G \arrow[r] \arrow[d, "\tilde{f}"'] & 0 \\
0 \arrow[r] & P^G \arrow[r] & (P \otimes \mathbb{R})^G \arrow[r] & (X_P)^G \arrow[r] & 0.
\end{tikzcd}$$

The map $f^G : \mathbb{Z} = (\mathbb{Z}G)^G \to P^G \cong \mathbb{Z}$ must be multiplication by a number q, where q is relatively prime to p. This proves that the map

$$h_1 : \mathbb{S}^1 \to F_P$$

induced by f is a degree q covering map. Similarly, $h_2 : (\mathbb{S}^1)^p \to X_P$ is a covering map of degree relatively prime to p. $\square$

Notice that when $L = \mathbb{Z}$ is the trivial G-module then $X_{\mathbb{Z}} = F_{\mathbb{Z}} = \mathbb{S}^1$. As a corollary we obtain.

Corollary 9. *Let X be the G-space induced by a $\mathbb{Z}G$-lattice of type (r, s, t). Then*

$$F := X^G \cong \coprod_{p^r} (\mathbb{S}^1)^{s+t}.$$

Lemma 10. *Let L and M be two $\mathbb{Z}G$-lattices. Assume that $f : L \to M$ is a $\mathbb{Z}G$-homomorphism that is an isomorphism after tensoring with $\mathbb{Z}_{(p)}$. Then f induces an isomorphism*

$$f^* : H_G^*(X_M, F_M; \mathbb{Z}_{(p)}) \to H_G^*(X_L, F_L; \mathbb{Z}_{(p)}).$$

Proof. Let $K = \mathbb{Z}_{(p)}$. Consider the Serre spectral sequence with K-coefficients

$$E_2^{i,j} = H^i\big(G, H^j(X_M, F_M; K)\big) \Longrightarrow H_G^{i+j}(X_M, F_M; K).$$

Similarly we obtain a spectral sequence $\tilde{E}_r^{i,j}$ associated to L. The map $f : L \to M$ induces a map of spectral sequences

$$f_r^{i,j} : E_r^{i,j} \to \tilde{E}_r^{i,j}.$$

We will prove the lemma by showing that f induces an isomorphism on the corresponding E_2-terms. Let $L_K^* = L^* \otimes K$ and $M_K^* = M^* \otimes K$. To start note that $H^j(X_L; K) \cong \bigwedge^j L_K^*$ and similarly $H^j(X_M; K) \cong \bigwedge^j M_K^*$. By hypothesis $\bigwedge^j L_M^*$ and $\bigwedge^j L_M^*$ are isomorphic KG-modules, with isomorphism induced by f. Thus

$$f^*: H^j(X_M; K) \to H^j(X_L; K) \tag{10}$$

is an isomorphism of KG-modules. On the other hand, since $f: L \to M$ is an isomorphism after tensoring with K, then L can be seen as a sub-lattice of M of finite index q, with q relatively prime with p. Consider the map $f^G: F_L \to F_M$ induced by f on the level of fixed points. An argument similar to that in Lemma 8 can be used to show that f^G is a covering map of degree relatively prime to p. In particular

$$\left(f^G\right)^*: H^j(F_M; K) \to H^j(F_L; K) \tag{11}$$

is an isomorphism of KG-modules. Finally, note that f induces a morphism between the long exact sequences in cohomology with K-coefficients associated to the pairs (X_L, F_L) and (X_M, F_M). By (10), (11) and the 5-lemma we conclude that

$$f^*: H^*(X_M, F_M; K) \to H^*(X_L, F_L; K)$$

is an isomorphism of KG-modules. This proves the lemma. $\square$

4. Modules of type $(r, 0, 0)$

In this section we consider the particular case of a $\mathbb{Z}G$-lattice L of type $(r, 0, 0)$ determined by an integral representation φ.

Suppose that L is such a lattice. Then as an abelian group L has rank $n = r(p-1)$ and thus the associated torus X has rank $r(p-1)$. Consider the Serre spectral associated to the pair (X, F)

$$E_2^{i,j} = H^i\big(G, H^j(X, F; \mathbb{Z})\big) \Longrightarrow H_G^{i+j}(X, F; \mathbb{Z}). \tag{12}$$

In this case F is a finite set with p^r points by Corollary 9. In particular $H^k(X, F) = 0$ if $k > r(p-1)$, thus Lemma 6 implies that $H_G^k(X, F) = 0$ for $k > r(p-1)$. We will use this fact to show that in the spectral sequence (12) all the differentials are trivial except for the differentials of the form

$$d_j: E_j^{*,j} \to E_j^{*,1},$$

whenever $1 \leqslant j \leqslant r(p-1)$. Moreover, we will see that for such j the differential d_j is injective on positive degrees. To show this, note that as F is a discrete set then $H^0(X, F) = 0$ and for $j \geqslant 2$

$$H^j(X, F) \cong H^j(X) \cong \bigwedge^j L^*.$$

Since L^* is of type $(r, 0, 0)$, if

$$F_L(x) = \left(\frac{1-(\alpha x)^p}{1-\alpha x}\right)^r = \sum_{i \geqslant 0} f_i x^i + \sum_{i \geqslant 0} g_i \alpha x^i$$

and $j \geqslant 2$, then $H^j(X,F)$ is a $\mathbb{Z}G$-lattice of type $(g_j, h_j - f_j, f_j)$ with

$$h_j = \frac{1}{p}\left[\binom{n}{j} + (p-1)(f_j - g_j)\right].$$

In particular there is an isomorphism of graded R^*-modules

$$H^*\big(G, H^j(X,F)\big) \cong \mathbb{Z}^{h_j - f_j} \oplus \big(R^*\big)^{f_j} \oplus \big(S^*\big)^{g_j}. \tag{13}$$

Define $p_r(j)$ to be the number of all possible sequences of integers $l_1, \dots, l_r$ such that $0 \leqslant l_i \leqslant p-1$ and $l_1 + \cdots + l_r = j$. Then in this case it is easy to see that for j even $f_j = p_r(j)$ and $g_j = 0$ and for j odd $f_j = 0$ and $g_j = p_r(j)$. Let's compute now $H^*(G, H^1(X,F))$. The long exact sequence in cohomology associated to the pair (X,F) gives a short exact sequence of $\mathbb{Z}G$-modules

$$0 \to \mathbb{Z}^{p^r - 1} \to H^1(X,F) \to H^1(X) \to 0.$$

In particular, as a group, $H^1(X,F)$ is a free abelian group and there is a long exact sequence

$$\cdots \to H^i\big(G, \mathbb{Z}^{p^r-1}\big) \to H^i\big(G, H^1(X,F)\big) \to H^i\big(G, H^1(X)\big) \xrightarrow{\delta} H^{i+1}\big(G, \mathbb{Z}^{p^r-1}\big) \to \cdots. \tag{14}$$

We claim that when i is odd $H^i(G, H^1(X,F)) = 0$. To see this it is enough to show that if i is odd and sufficiently large $H^i(G, H^1(X,F)) = 0$. Pick i odd with $i > r(p-1)$ so that $H^{i+1}_G(X,F) = 0$ for such i. In particular there are no nontrivial permanent cocycles in total degree $i+1$ in the spectral sequence (12). Trivially all the differentials with source $E_2^{i,1} = H^i(G, H^1(X,F))$ are zero and therefore any element in $E_2^{i,1}$ must be in the image of some differential. However, (13) implies that any differential with target $E_k^{i,1}$ has a trivial source. This shows that $H^i(G, H^1(X,F)) = 0$ for i odd. In particular, the long exact sequence (14) reduces to the short exact sequence

$$0 \to H^i\big(G, H^1(X)\big) \to H^{i+1}\big(G, \mathbb{Z}^{p^r-1}\big) \to H^{i+1}\big(G, H^1(X,F)\big) \to 0, \tag{15}$$

for $i > 0$ odd. Since $H^1(X) \cong L^*$ is a $\mathbb{Z}G$-module of type $(r,0,0)$ it follows that $H^1(X,F)$ is a $\mathbb{Z}G$-module of type $(0, r, p^r - r - 1)$ and

$$H^*\big(G, H^1(X,F)\big) \cong \mathbb{Z}^r \oplus \Bigg(\bigoplus_{p^r - r - 1} R^*\Bigg).$$

This describes the E_2-term of the spectral sequence (12). Consider now the Serre spectral sequence

$$\tilde{E}_2^{i,j} = H^i\big(G, H^j(X)\big) \Longrightarrow H_G^{i+j}(X)$$

associated to the fibration sequence $X \to X \times_G EG \to BG$. As it was pointed out before this sequence collapses on the E_2-term. The inclusion $f: X \to (X,F)$ defines a map of spectral sequences $f_k^{i,j}: E_k^{i,j} \to \tilde{E}_k^{i,j}$. Notice that $f_2^{i,j}$ is an isomorphism when $j \geqslant 2$. This shows that the only possibly nontrivial differentials in (12) are of the form

$$d_j : E_j^{*,j} \to E_j^{*,1}$$

for $2 \leqslant j \leqslant r(p-1)$. We determine next the nature of these differentials. Note that the factor $\mathbb{Z}^{h_j - f_j} \subset (H^j(X,F))^G$ lies in the image of the norm map. Consider the transfer map associated to the trivial subgroup $\{1\} \hookrightarrow G$. This map preserves the filtrations that induce the Serre spectral sequence and thus it induces a map of the corresponding spectral sequences

$$\tau_{\{1\}}^G : H^i\big(\{1\}, H^j(X,F)\big) \to H^i\big(G, H^j(X,F)\big).$$

Since the image of the transfer map $\tau_{\{1\}}^G : H^0(\{1\}, H^j(X,F)) \to H^0(G, H^j(X,F))$ consists of elements in the image of the norm map, it follows that all the differentials in the Serre spectral sequence (12) are trivial on the summand $\mathbb{Z}^{h_j - f_j}$. Let's show now that the differential d_j is injective on positive degrees. To see this it suffices to show that $d_j : E_j^{i,j} \to E_j^{i+j,1}$ is injective for i big enough. If $i > r(p-1)$ then $H_G^i(X,F) = 0$, therefore in total degree i with $i > r(p-1)$ there are no nontrivial permanent cocycles. Since all the differentials landing in $E_j^{*,j}$ are trivial for $j \geqslant 2$ this forces $d_j : E_j^{i,j} \to E_j^{i+j,1}$ to be injective when $i > r(p-1)$.

The above can be summarized in the following way. If $j \neq 1$ then

$$q_{E_\infty^{*,j}}(x) = 0. \tag{16}$$

On the other hand, since $H^1(X,F)$ is of type $(0, r, p^r - r - 1)$ then by (5)

$$q_{E_2^{*,1}}(x) = \frac{(p^r - r - 1)x^2}{1 - x^2}.$$

Also, $d_j : E_j^{*,j} \to E_j^{*,1}$ is a homomorphism of graded R^*-modules of degree j that is injective on positive degrees and $E_j^{*,j} = E_2^{*,j}$ is a $\mathbb{Z}G$-lattice of type $(g_j, h_j - f_j, f_j)$, it follows that

$$q_{E_{j+1}^{*,1}}(x) = q_{E_j^{*,1}}(x) - \left(\frac{f_j + x g_j}{1 - x^2}\right) x^j.$$

Therefore

$$\begin{aligned} q_{E_\infty^{*,1}}(x) &= \frac{(p^r - r - 1)x^2}{1 - x^2} - \sum_{j \geqslant 2} \left(\frac{f_j + x g_j}{1 - x^2}\right) x^j && (17) \\ &= \frac{1}{1 - x^2}\left[\big(p^r - 1\big)x^2 + 1 - \sum_{j \geqslant 0} (f_j + x g_j) x^j\right]. && (18) \end{aligned}$$

This completely characterizes the E_∞-term in the spectral sequence (12) and determines $H_G^*(X,F;\mathbb{Z})$ up to extension problems. Let's assume for a moment that there are no extension problems in this case. If this is true then

$$H_G^k(X,F;\mathbb{Z}) \cong \mathbb{Z}^{\kappa_k} \oplus (\mathbb{Z}/p)^{\lambda_k},$$

for some integers κ_k and λ_k. Moreover, the integers λ_k are determined by the formal series

$$\bar{Q}_L(x) := \sum_{j \geqslant 0} \lambda_j x^j = \sum_{j \geqslant 0} x^j q_{E_\infty^{*,j}}(x).$$

From (16) and (18) it follows that

$$\bar{Q}_L(x) = \sum_{j \geqslant 1} x^j q_{E_\infty^{*,j}}(x) = \frac{x}{1 - x^2}\left[p^r x^2 - x^2 + 1 - \sum_{j \geqslant 0} (f_j + x g_j) x^j\right].$$

Let's show now that indeed there are no extension problems in the spectral sequence (12). To see this we need to consider the Serre spectral sequence computing $H_G^*(X, F; \mathbb{F})$, where $\mathbb{F} = \mathbb{Q}$, $\mathbb{F} = \mathbb{F}_p$ and $\mathbb{F} = \mathbb{F}_q$, for a prime number q different from p. Arguments similar to those formulated above can be applied in these cases to obtain explicit descriptions of the E_∞-term in these spectral sequences. This way we obtain

$$\begin{aligned} H_G^k(X, F; \mathbb{Q}) &\cong \mathbb{Q}^{\kappa_k}, \\ H_G^k(X, F; \mathbb{F}_q) &\cong \mathbb{F}_q^{\kappa_k}, \\ H_G^k(X, F; \mathbb{F}_p) &\cong \mathbb{F}_p^{\kappa_k+\lambda_k+\lambda_{k+1}}. \end{aligned}$$

The only way that this is possible is that indeed there are no extension problems in the spectral sequence (12). To finish note that in Lemma 16 it is proved that the p-torsion of $H^k(X/G; \mathbb{Z})$ is an $\mathbb{F}_p$-vector space of dimension β_k that in this case equals λ_k. Thus these groups are determined by the formal power series

$$Q_L(x) := \sum_{j \geqslant 0} \beta_j x^j = \bar{Q}_L(x).$$

To find the explicit description for the series $Q_L(x)$ notice that

$$(1+\alpha x)F_L(x) = (1+\alpha x)\left(\frac{1-(\alpha x)^p}{1-\alpha x}\right)^r = \sum_{i \geqslant 0}(f_i + x g_i)x^i + \sum_{i \geqslant 0} \alpha(x f_i + g_i)x^i.$$

Thus if

$$T_L(x) = \frac{x}{1-x^2}\left[p^r x^2 - x^2 + 1 - (1+\alpha x)\left(\frac{1-(\alpha x)^p}{1-\alpha x}\right)^r\right]$$

then $T_L(x) = Q_L(x) + \alpha R_L(x)$, for some $R_L(x) \in \mathbb{Z}[[x]]$. This together with Corollary 4 prove the following theorem.

Theorem 11. [2] *Suppose that X is induced by a $\mathbb{Z}G$-lattice L of type $(r, 0, 0)$. Then*

$$H^k(X/G; \mathbb{Z}) \cong \mathbb{Z}^{\alpha_k} \oplus (\mathbb{Z}/p)^{\beta_k},$$

where the coefficients α_k and β_k are given as follows: using the same relations as before, write the formal power series in x in the form

$$F_L(x) = \left(\frac{1-(\alpha x)^p}{1-\alpha x}\right)^r = \sum_{i \geqslant 0} f_i x^i + \sum_{i \geqslant 0} g_i \alpha x^i$$

then

$$\alpha_k = \frac{1}{p}\left[\binom{r(p-1)}{k} + (p-1)(f_k - g_k)\right].$$

[2] This result forms part of the Doctoral dissertation of the second author (see [8]).

Similarly, β_k is obtained by writing the formal series in x

$$T_L(x) = \frac{x}{1-x^2}\left[p^r x^2 - x^2 + 1 - (1+\alpha x)\left(\frac{1-(\alpha x)^p}{1-\alpha x}\right)^r\right]$$

in the form $T_L(x) = \sum_{i\geqslant 0} \beta_i x^i + \sum_{i\geqslant 0} \gamma_i \alpha x^i$, where α and ϵ_p subject to the same relations as before.

Example. It is easy to see that in Theorem 11 the coefficients β_k are given by

$$\beta_k = \begin{cases} \sum_{j=k}^{r(p-1)} p_r(j) & \text{if } k \text{ is odd and } k > 1, \\ 0 & \text{else.} \end{cases}$$

Here, as defined above, $p_r(j)$ is the number of all possible sequences of integers $l_1, \dots, l_r$ such that $0 \leqslant l_i \leqslant p-1$ and $l_1 + \dots + l_r = j$. For example when $p = 2$, if X is induced by a $\mathbb{Z}G$-lattice of type $(r, 0, 0)$ then

$$H^k(X/G; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{\binom{r}{k}} & \text{if } k \text{ is even},\ 0 \leqslant k \leqslant r, \\ (\mathbb{Z}/2)^{\binom{r}{k}+\binom{r}{k+1}+\dots+\binom{r}{r}} & \text{if } k \text{ is odd},\ 1 < k \leqslant r, \\ 0 & \text{else.} \end{cases}$$

Remark. Independently and by different methods, Davis and Lück in [7] obtained the same answers for the cohomology groups of the form $H^*(X/G; \mathbb{Z})$, where X is the G-space induced by a $\mathbb{Z}G$-lattice of type $(r, 0, 0)$.

5. Modules of type $(0, s, 0)$

In this section we compute the cohomology groups of the form $H^*(X/G; \mathbb{Z})$, where now X is induced by a module L of type $(0, s, 0)$. We begin by considering the particular case $L = (\mathbb{Z}G)^s$ and later extend the computations for a general module.

Lemma 12. *Suppose that $L = (\mathbb{Z}G)^s$. Consider the formal power series in x*

$$F(x) = \left(1 + \epsilon_p x^p\right)^s = \sum_{i\geqslant 0} f_i x^i + \sum_{i\geqslant 0} g_i \alpha x^i,$$

$$O(x) = (1+x)^s = \sum_{i\geqslant 0} v_i x^i$$

subject to the same relations as before. Then for $j \geqslant 1$, $H^j(X, F)$ is a $\mathbb{Z}G$-lattice of type $(g_j + v_j, w_j - f_j, f_j)$, where

$$w_j = \frac{1}{p}\left[\binom{sp}{j} - \binom{s}{j} + (p-1)(f_j - g_j - v_j)\right].$$

Proof. Write $L = P_1 \oplus \dots \oplus P_s$, where $P_j = \mathbb{Z}G$ for $1 \leqslant j \leqslant s$ and let $i : F \to X$ be the inclusion map. Then, up to homeomorphism, i can be identified with the diagonal inclusion of $(\mathbb{S}^1)^s$ into $((\mathbb{S}^1)^p)^s$. This shows that

$$i^* : H^j(X) \to H^j(F)$$

is surjective $j \geqslant 0$. Because of this, the long exact sequence in cohomology associated to the pair (X, F) reduces to the different short exact sequences

$$0 \to H^j(X, F) \to H^j(X) \to H^j(F) \to 0.$$

In particular, $H^j(X, F)$ is a free abelian group (ignoring the G-action) and there is a long exact sequence

$$\cdots \to H^k\big(G, H^j(X, F)\big) \to H^k\big(G, H^j(X)\big) \xrightarrow{i^*} H^k\big(G, H^j(F)\big) \xrightarrow{\delta} \cdots. \tag{19}$$

We claim that the map $i^*: H^k(G, H^j(X)) \to H^k(G, H^j(F))$ is trivial when $k > 0$. To see this note that

$$H^j(X) \cong \bigwedge^j H^1(X) \cong \bigwedge^j L^* \cong \bigwedge^j \big(P_1^* \oplus \cdots \oplus P_s^*\big)$$

and by the Kunneth theorem there is a commutative diagram

$$\begin{tikzcd} H^k(G, H^j(X)) \arrow[r, "i^*"] \arrow[d, "\cong"'] & H^k(G, H^j(F)) \arrow[d, "\cong"] \\ \bigoplus_{n_1+\cdots n_s=j} H^k(G, \bigwedge^{n_1} P_1^* \otimes \cdots \otimes \bigwedge^{n_s} P_s^*) \arrow[r, "\epsilon_{n_1,\ldots,n_s}"] & \bigoplus_{n_1+\cdots n_s=j} H^k(G, \bigwedge^{n_1} \mathbb{Z} \otimes \cdots \otimes \bigwedge^{n_s} \mathbb{Z}), \end{tikzcd}$$

where $\epsilon_{n_1,\ldots,n_s}$ is induced by the inclusion map $i: F \to X$. Notice that the G-module $\bigwedge^{n_1} \mathbb{Z} \otimes \cdots \otimes \bigwedge^{n_s} \mathbb{Z}$ is trivial unless $0 \leqslant n_q \leqslant 1$ for all q. Suppose then that $0 \leqslant n_q \leqslant 1$ for all q. Since $n_1 + \cdots + n_s = j \geqslant 1$ it follows that $n_q = 1$ for some q and therefore $\bigwedge^{n_1} P_1^* \otimes \cdots \otimes \bigwedge^{n_s} P_s^*$ has trivial cohomology. This shows that in any case the map $i^*: H^k(G, H^j(X)) \to H^k(G, H^j(F))$ is trivial for $k > 0$ and (19) reduces to the short exact sequence

$$0 \to H^{k-1}\big(G, H^j(F)\big) \xrightarrow{\delta} H^k\big(G, H^j(X, F)\big) \xrightarrow{f^*} H^k\big(G, H^j(X)\big) \to 0 \tag{20}$$

for $k > 1$. This sequence splits for $k > 1$ as it is a short exact sequence of vector spaces over $\mathbb{F}_p$. The lemma follows using the fact that $H^j(X)$ is a $\mathbb{Z}G$-lattice of type $(g_j, h_j - f_j, f_j)$ where

$$F(x) = \big(1 + \epsilon_p x^p\big)^s = \sum_{i \geqslant 0} f_i x^i + \sum_{i \geqslant 0} g_i \alpha x^i$$

and the fact that $H^j(F)$ is of type $(0, 0, v_j)$ with $(1+x)^s = \sum_{j \geqslant 0} v_j x^j$. $\square$

The previous lemma can be used to determine the p-torsion of $H_G^k(X, F; \mathbb{Z})$ in this case. The following theorem is then obtained.

Theorem 13. *Suppose that X is induced by the $\mathbb{Z}G$-lattice $L = (\mathbb{Z}G)^s$. Then the p-torsion subgroup of $H_G^k(X, F; \mathbb{Z})$ is an $\mathbb{F}_p$-vector space of dimension λ_k, where λ_k is obtained by writing the formal series in x*

$$\bar{T}_L(x) = \frac{x}{1 - x^2}\big[(1+x)^s - (1 + \alpha x)\big(1 + \epsilon_p x^p\big)^s\big]$$

in the form $\bar{T}_L(x) = \sum_{i \geqslant 0} \lambda_i x^i + \sum_{i \geqslant 0} \kappa_i \alpha x^i$, where α and ϵ_p subject to the same relations as above.

Proof. As before write $L = P_1 \oplus \cdots \oplus P_s$, where $P_j = \mathbb{Z}G$ for $1 \leqslant j \leqslant s$. The proof of the theorem follows by a careful study of the differentials in the Serre spectral sequence

$$E_2^{i,j} = H^i\big(G, H^j(X, F)\big) \Longrightarrow H_G^{i+j}(X, F). \tag{21}$$

Consider the natural map of pairs $f : X \to (X, F)$. If $j > s$ then

$$f^* : H^j(X, F) \to H^j(X)$$

is an isomorphism and the Serre spectral sequence

$$\tilde{E}_2^{i,j} = H^i\big(G, H^j(X)\big) \Longrightarrow H_G^{i+j}(X)$$

collapses on the E_2-term. Therefore any nontrivial differential in (21) lands in $E_k^{i,j}$ for $1 \leqslant j \leqslant s$. By the previous lemma for $j \geqslant 1$

$$H^*\big(G, H^j(X, F)\big) \cong \mathbb{Z}^{w_j - f_j} \oplus \big(R^*\big)^{f_j} \oplus \big(S^*\big)^{g_j + v_j}$$

where f_j, g_j, w_j and v_j are as described there. Note that in particular $v_j = \binom{s}{j}$ for all $j \geqslant 1$ and $f_j = g_j = 0$ if $p \nmid j$. As in the previous section all the differentials are trivial in the summand $\mathbb{Z}^{w_j - f_j} \subset H^0(G, H^j(X, F))$ as it consists of elements in the image of the norm map. We are going to show that in this case all the differentials are trivial except for the ones of the form

$$d_{l(p-1)+1} : E_{l(p-1)+1}^{*,lp} \to E_{l(p-1)+1}^{*,l}, \tag{22}$$

whenever $1 \leqslant l \leqslant s$. Moreover, we are going to show that

$$E_{l(p-1)+1}^{*,lp} \cong \mathbb{Z}^{w_{lp} - f_{lp}} \oplus \big(R^*\big)^{f_{lp}} \oplus \big(S^*\big)^{g_{lp} + v_{lp}}, \tag{23}$$

$$E_{l(p-1)+1}^{*,l} \cong \mathbb{Z}^{w_l} \oplus \big(S^*\big)^{v_l}, \tag{24}$$

and the homomorphism $d_{l(p-1)+1}$ maps the factor $(R^*)^{f_{lp}} \oplus (S^*)^{g_{lp}}$ injectively on positive degrees. Suppose for a moment that this is true. Then for all $l \geqslant 1$

$$q_{E_{l(p-1)+1}^{*,l}}(x) = \frac{v_l x}{1 - x^2}$$

and since $d_{l(p-1)+1}$ is a homomorphism of graded R^*-modules of degree $l(p-1)+1$ then

$$q_{E_\infty^{*,l}}(x) = q_{E_{l(p-1)+2}^{*,l}}(x) = \frac{v_l}{1 - x^2} x - \left(\frac{f_{lp} + g_{lp} x}{1 - x^2}\right) x^{l(p-1)+1}.$$

A similar argument to the one provided below to study the differentials in the spectral sequence (21) can be used to handle the case where the coefficients in the sequence (21) are $\mathbb{Q}$, $\mathbb{F}_p$ and $\mathbb{F}_q$, where q is a prime different to p. This can be used to show that there are no extension problems in the spectral sequence (21). Therefore, the p-torsion of $H_G^k(X, F)$ is an $\mathbb{F}_p$-vector space of dimension λ_k and

$$\begin{aligned}
\bar{Q}_L(x) &:= \sum_{l\geqslant 0}\lambda_l x^l = \sum_{l\geqslant 0} x^l q_{E_\infty^{*,l}}(x) \\
&= \sum_{l\geqslant 1}\left(\frac{v_l}{1-x^2}\right)x^{l+1} - \sum_{l\geqslant 1}\left(\frac{f_{lp}+g_{lp}x}{1-x^2}\right)x^{lp+1} \\
&= \frac{x}{1-x^2}\left[(1+x)^s - \sum_{l\geqslant 0}(f_{lp}+g_{lp}x)x^{lp}\right].
\end{aligned}$$

Notice that

$$F_L(x) = \left(1+\epsilon_p x^p\right)^s = \sum_{i\geqslant 0}(f_i+\alpha g_i)x^i = \sum_{l\geqslant 0}(f_{lp}+\alpha g_{lp})x^{lp},$$

and thus

$$(1+\alpha x)F_L(x) = \sum_{i\geqslant 0}(f_i + xg_i)x^i + \sum_{i\geqslant 0}\alpha(xf_i+g_i)x^i.$$

Therefore

$$\bar{T}_L(x) := \frac{x}{1-x^2}\left[(1+x)^s - (1+\alpha x)\left(1+\epsilon_p x^p\right)^s\right],$$

can be written in the form $\bar{T}_L(x) = \bar{Q}_L(x) + \alpha\bar{R}_L(x)$, for some formal power series with integer coefficients $\bar{R}_L(x)$. This together with Corollary 4 proves the theorem.

Induction on s will be used to prove statements (22), (23) and (24). When $s=1$ then $v_1=1$, $f_p=1$ and $g_p=0$ for $p>2$. When $p=2$, $f_2=0$ and $g_2=1$. In any case, by Lemma 12 the only possible nontrivial differential is

$$d_p : E_p^{*,p} \cong \mathbb{Z}^{w_p-f_p}\oplus\left(R^*\right)^{f_p}\oplus\left(S^*\right)^{g_p} \to E_p^{*,1}\cong\mathbb{Z}^{w_1}\oplus S^*.$$

In this case, Lemma 8 implies that $H^i(X,F)=0$ if $i>sp$. Using Lemma 6 for such i we have $H^i_G(X,F)=0$. Since there are no nontrivial differentials landing in $E_p^{*,p}$ it follows that d_p is injective for sufficiently high degrees and the statements follows as d_p is a homomorphism of graded R^*-modules of degree p. Suppose that the statements are true for $s\geqslant 1$ and suppose

$$L = P_1\oplus\cdots\oplus P_{s+1},$$

where $P_i=\mathbb{Z}G$ for all $1\leqslant j\leqslant s+1$. For every $1\leqslant k\leqslant s+1$ let

$$L(k) := P_1\oplus\cdots\oplus\hat{P}_k\oplus\cdots\oplus P_{s+1},$$

where as usual $\hat{P}_k$ means that the factor P_k is not included. For such k we have natural projection and inclusion maps

$$i_k : L(k)\to L \quad\text{and}\quad \pi_k : L\to L(k).$$

These are homomorphisms of $\mathbb{Z}G$-modules that satisfy $\pi_k\circ i_k = 1$. Moreover, these maps induce G-equivariant maps

$$i_k : X_{L(k)}\to X_L \quad\text{and}\quad \pi_k : X_L\to X_{L(k)}$$

such that $\pi_k \circ i_k = 1$. By comparing the Serre spectral sequences associated to the pairs $(X_{L(k)}, F_{L(k)})$ and (X_L, F_L) using the maps π_k and i_k for $1 \leqslant k \leqslant s+1$, it follows by induction that

$$d_{l(p-l)+1} : E^{*,lp}_{l(p-1)+1} \to E^{*,l}_{l(p-1)+1}$$

is as claimed in (22) for $1 \leqslant l \leqslant s$. Also, we can conclude that if $1 \leqslant j \leqslant sp$ and $p \nmid j$, then

$$d_r : E^{*,j}_r \to E^{*,j-r+1}_r$$

is trivial for all r. Consider now the graded R^*-modules of the form $E^{*,s+1}_r$, for $r \geqslant 2$. There are two possibilities depending whether p divides $s+1$ or not. Suppose first that $p \nmid (s+1)$. In this case $f_{s+1} = g_{s+1} = 0$ and $v_{s+1} = 1$. Thus by Lemma 12

$$E^{*,s+1}_2 \cong \mathbb{Z}^{w_{s+1} - f_{s+1}} \oplus \left(R^*\right)^{f_{s+1}} \oplus \left(S^*\right)^{g_{s+1}+v_{s+1}} \cong \mathbb{Z}^{w_{s+1}} \oplus S^*.$$

Note that $1 \leqslant s+1 \leqslant sp$ and $p \nmid (s+1)$, then by the previous comment it follows that all the differentials starting at $E^{*,s+1}_r$ are trivial for all $r \geqslant 2$. By Lemma 6 there are no nontrivial permanent cocycles in total degree bigger than $(s+1)p$. Therefore if i is odd with $i > (s+1)p$ then all the elements in $E^{i,s+1}_2 \neq 0$ must be in the image of some differential $d_r : E^{i-r,s+r} \to E^{i,s+1}_r$ with $r \geqslant 2$. Note that if $sp < j < (s+1)p$ then $E^{*,j}_2 \cong \mathbb{Z}^{w_j}$, in particular $E^{*,j}_2$ is a graded R^*-module concentrated on degree 0 and consists of elements in the image of the norm map. Thus all differentials starting at $E^{*,j}_r$ are trivial for all $r \geqslant 2$. Because of this and the induction hypothesis, the only possible nonzero differential that has $E^{*,s+1}_r$ as target for some r is

$$d_{(s+1)(p-l)+1} : E^{*,(s+1)p}_{(s+1)(p-1)+1} \to E^{*,s+1}_{(s+1)(p-1)+1}.$$

This shows that $d_{(s+1)(p-l)+1}$ must be nontrivial,

$$E^{*,(s+1)p}_{(s+1)(p-l)+1} \cong E^{*,(s+1)p}_2 \cong \mathbb{Z}^{w_{(s+1)p} - f_{(s+1)p}} \oplus \left(R^*\right)^{f_{(s+1)p}} \oplus \left(S^*\right)^{g_{(s+1)p}}$$

and the restriction of $d_{l(p-1)+1}$ to the factor $(R^*)^{f_{(s+1)p}} \oplus (S^*)^{g_{(s+1)p}}$ is injective on positive degrees. This proves the induction hypothesis in this case. The case $p \mid (s+1)$ is handled in a similar way. $\square$

Suppose now that L is a $\mathbb{Z}G$-lattice of type $(0, s, 0)$. Then we can find a $\mathbb{Z}G$-homomorphism

$$f : (\mathbb{Z}G)^s \to L$$

that is an isomorphism after tensoring with $\mathbb{Z}_{(p)}$. By Lemma 10 f induces an isomorphism

$$f^* : H^*_G(X_L, F_L; \mathbb{Z}_{(p)}) \to H^*_G(X_{(\mathbb{Z}G)^s}, F_{(\mathbb{Z}G)^s}; \mathbb{Z}_{(p)}).$$

This shows that the p-torsion of $H^*_G(X_L, F_L; \mathbb{Z})$ is an $\mathbb{F}_p$-vector space of dimension λ_k, where λ_k is determined by the formal power series $\bar{Q}_L(x) = \sum_{j \geqslant 0} \lambda_j x^j$. Finally, in Lemma 16 the long exact sequence in cohomology associated to the pair $(X/G, F)$ is studied to conclude that p-torsion of $H^k(X/G; \mathbb{Z})$ is also $\mathbb{F}_p$-vector space of dimension β_k and the coefficients β_k are determined by

$$Q_L(x) := \sum_{j \geqslant 0} \beta_j x^j = \bar{Q}_L(x) - x\big[(1+x)^s - 1\big].$$

This proves the following theorem.

Theorem 14. *Suppose that X is induced by a $\mathbb{Z}G$-lattice L of type $(0,s,0)$. Then*

$$H^k(X/G;\mathbb{Z}) \cong \mathbb{Z}^{\alpha_k} \oplus (\mathbb{Z}/p)^{\beta_k},$$

where the coefficients α_k and β_k are given as follows: write the formal power series in x

$$F_L(x) = \left(1+\epsilon_p x^p\right)^s = \sum_{i\geqslant 0} f_i x^i + \sum_{i\geqslant 0} g_i \alpha x^i$$

where $\epsilon_2 = \alpha$ and $\epsilon_p = 1$ for $p > 2$. Then

$$\alpha_k = \frac{1}{p}\left[\binom{sp}{k} + (p-1)(f_k - g_k)\right].$$

Similarly, β_k is obtained by writing the formal series in x

$$T_L(x) = \frac{x}{1-x^2}\left[x^2(1+x)^s - x^2 + 1 - (1+\alpha x)\left(1+\epsilon_p x^p\right)^s\right]$$

in the form $T_L(x) = \sum_{i\geqslant 0} \beta_i x^i + \sum_{i\geqslant 0} \gamma_i \alpha x^i$, where α and ϵ_p subject to the same relations as above.

6. The general case

The information collected in the previous two sections is now assembled to compute the cohomology groups $H^*(X/G;\mathbb{Z})$, where the G-space X is induced by a general $\mathbb{Z}G$-lattice L.

We start by computing the following special case.

Theorem 15. *Suppose that X is the G-space associated to a $\mathbb{Z}G$-lattice L of type $(r,s,0)$. Then the p-torsion subgroup of $H^k_G(X,F;\mathbb{Z})$ is an $\mathbb{F}_p$-vector space of dimension λ_k. The coefficients λ_k are obtained by writing the formal series in x*

$$\bar{T}_L(x) = \frac{x}{1-x^2}\left[(1+x)^s\left(p^r x^2 - x^2 + 1\right) - (1+\alpha x)\left(1+\epsilon_p x^p\right)^s \left(\frac{1-(\alpha x)^p}{1-\alpha x}\right)^r\right]$$

in the form $\bar{T}_L(x) = \sum_{i\geqslant 0} \lambda_i x^i + \sum_{i\geqslant 0} \kappa_i \alpha x^i$, where α and ϵ_p subject to the same relations as before.

Proof. For simplicity we are going to consider the case $p > 2$. The case $p = 2$ is handled in a similar way. It is enough to prove the theorem for a $\mathbb{Z}G$-lattice of the form $L = L_1 \oplus L_2$ where $L_1 = \bigoplus_r A_i$ and $L_2 = \bigoplus_s P_j$ with A_i an indecomposable module of rank $p-1$ as before and $P_j = \mathbb{Z}G$ for all $1 \leqslant j \leqslant s$. This reduction is possible by Lemma 10 and the fact that given any $\mathbb{Z}G$-lattice M of type $(r,s,0)$ then we can find a homomorphism of G-modules $f : L \to M$ that is an isomorphism after tensoring with $\mathbb{Z}_{(p)}$. In this case, by (8) $X = X_{L_1} \times X_{L_2}$ and $F = F_{L_1} \times F_{L_2}$. Also F_{L_1} is a finite set with p^r points, $F_{L_2} \cong (\mathbb{S}^1)^s$ and the inclusion $i_2 : F_{L_2} \to X_{L_2}$ can be identified, up to homeomorphism, with the diagonal inclusion $\Delta : (\mathbb{S}^1)^s \to ((\mathbb{S}^1)^p)^s$. Consider the Serre spectral sequence

$$E_2^{i,j} := H^i\left(G, H^j(X,F;\mathbb{Z})\right) \Longrightarrow H^{i+j}_G(X,F;\mathbb{Z}). \tag{25}$$

We are going to study this spectral sequence by determining explicitly it's E_2-term and the nontrivial differentials. We start investigating the $\mathbb{Z}G$-module $H^j(X,F)$ for $j \geqslant 1$. When $j = 1$ it is easy to see that

$$H^1(X,F) \cong H^1(X_{L_1},F_{L_1}) \oplus H^1(X_{L_2},F_{L_2}),$$

in particular, by the work done in Sections 4 and 5 it follows that $H^1(X,F)$ is a $\mathbb{Z}G$-lattice of type (s,r,p^r-r-1). Suppose now that $j \geqslant 2$ and consider the long exact sequence

$$\cdots \to H^{j-1}(F) \xrightarrow{\delta} H^j(X,F) \to H^j(X) \xrightarrow{i^*} H^j(F) \to \cdots$$

induced by the pair (X,F). Note that $H^j(F) \cong \mathbb{Z}^{p^r\binom{s}{j}}$ and $\mathrm{Im}(i^*) \cong \mathbb{Z}^{\binom{s}{j}}$, therefore the previous sequence reduces to the different short exact sequences

$$0 \to \mathbb{Z}^{(p^r-1)\binom{s}{j-1}} \to H^j(X,F) \to M_j \to 0, \tag{26}$$

where M_j fits into the short exact sequence of $\mathbb{Z}G$-modules

$$0 \to M_j \to H^j(X) \xrightarrow{i^*} \mathbb{Z}^{\binom{s}{j}} \to 0. \tag{27}$$

Since M_j is a subgroup of a free abelian group, then M_j is a $\mathbb{Z}G$-lattice. To determine it's type recall that $H^j(X)$ is of type $(g_j, h_j - f_j, f_j)$, where

$$F_L(x) = \left(1+\epsilon_p x^p\right)^s \left(\frac{1-(\alpha x)^p}{1-\alpha x}\right)^r = \sum_{i\geqslant 0} f_i x^i + \sum_{i \geqslant 0} g_i \alpha x^i$$

and h_j is determined by f_j, g_j and the rank of $H^j(X)$. Explicitly,

$$h_j = \frac{1}{p}\left[\binom{n}{j} + (p-1)(f_j - g_j)\right].$$

Using the same method that was used in Lemma 12, it can be proved that

$$i^* : H^k\big(G, H^j(X)\big) \to H^k\big(G, \mathbb{Z}^{\binom{s}{j}}\big)$$

is the trivial map for $k>0$. Therefore, the long exact sequence associated in group cohomology to the short exact sequence (27) reduces to the short exact sequences

$$0 \to H^{k-1}\big(G, \mathbb{Z}^{\binom{s}{j}}\big) \to H^k(G, M_j) \to H^k\big(G, H^j(X)\big) \to 0,$$

for $k>1$. This shows that M_j is of type $(g_j + \binom{s}{j}, m_j - f_j, f_j)$ and m_j is determined by f_j, g_j and the rank of M_j as above. On the other hand, the short exact sequence (26) yields a long exact sequence

$$\cdots \to H^k\big(G, \mathbb{Z}^{(p^r-1)\binom{s}{j-1}}\big) \to H^k\big(G, H^j(X,F)\big) \to H^k(G, M_j) \xrightarrow{\partial} \cdots. \tag{28}$$

When $j \geqslant 2$ the connecting homomorphism

$$\partial : H^k(G, M_j) \to H^{k+1}\big(G, \mathbb{Z}^{(p^r-1)\binom{s}{j-1}}\big)$$

is trivial for $k>0$. This can be seen by comparing the long exact sequence (28) with that associated to the pair $(X_{L_1} \times F_{L_2}, F_{L_1} \times F_{L_2})$. Therefore (28) reduces to the short exact sequence

$$0 \to H^k\big(G, \mathbb{Z}^{(p^r-1)\binom{s}{j-1}}\big) \to H^k\big(G, H^j(X,F)\big) \to H^k(G, M_j) \to 0$$

for $k > 0$. This sequence splits as it is a short exact sequence of $\mathbb{F}_p$-vector spaces. This shows that $H^j(X, F)$ is a $\mathbb{Z}G$-lattice of type

$$(g_j + v_j, w_j - f_j - u_j, f_j + u_j),$$

where $u_j = (p^r - 1)\binom{s}{j-1}$ and $v_j = \binom{s}{j}$ and the coefficient w_j is determined in the same way as h_j. In particular, there is an isomorphism of graded R^*-modules

$$\begin{aligned} E_2^{*,1} &\cong \mathbb{Z}^r \oplus (R^*)^{p^r-r-1} \oplus (S^*)^s \\ &\cong \mathbb{Z}^r \oplus (R^*)^{u_1-r} \oplus (S^*)^{v_1} \end{aligned}$$

and for $j \geqslant 2$

$$E_2^{*,j} \cong \mathbb{Z}^{w_j-f_j-u_j} \oplus (R^*)^{f_j+u_j} \oplus (S^*)^{g_j+v_j}.$$

This completely describes the E_2-term of the spectral sequence (25). The differentials in this spectral sequence can be determined explicitly in a similar fashion as it was done in Theorems 11 and 13. As a general rule, on positive degrees the terms of the form $(R^*)^{f_j} \oplus (S^*)^{g_j}$ are the source of a non-trivial differential, hence these terms do not survive to the E_∞-term. Also the terms of the form $(R^*)^{u_j} \oplus (S^*)^{v_j}$ are the target of nontrivial differentials. More precisely, write

$$\left(\frac{1-(\alpha x)^p}{1-\alpha x}\right)^r = \sum_{i\geqslant 0} y_i x^i + \sum_{i\geqslant 0} z_i \alpha x^i.$$

Then an inductive argument on s while keeping r fixed similar to the one provided in Theorem 13 shows that differentials in this spectral sequence are given as follows:

- Suppose that $1 \leqslant l \leqslant s$, then just as in Theorem 13

  $$d_{l(p-1)+1} : E_{l(p-1)+1}^{*,lp} \to E_{l(p-1)+1}^{*,l}$$

  is such that

  $$q_{\operatorname{Im} d_{l(p-1)+1}}(x) = \frac{\binom{s}{l}}{1-x^2} x^{l(p-1)+1}.$$

- Suppose that $1 \leqslant l \leqslant s+1$ and $1 \leqslant k \leqslant rp$ are such that $(l-1)(p-1)+k \geqslant 2$, then

  $$d_{(l-1)(p-1)+k} : E_{(l-1)(p-1)+k}^{*,(l-1)p+k} \to E_{(l-1)(p-1)+k}^{*,l}$$

  is such that

  $$q_{\operatorname{Im} d_{(l-1)(p-1)+k}}(x) = \binom{s}{l-1}\left(\frac{y_k + z_k x}{1-x^2}\right) x^{(l-1)(p-1)+k}.$$

- All the other differentials are trivial.

Let $h(x)=\sum_{k\geqslant 1}(y_k+z_kx)x^k$. Then

$$\begin{aligned}
q_{E_\infty^{*,1}}(x)&=\frac{1}{1-x^2}\Bigl[\bigl(sx-sx^p\bigr)+\bigl(p^r-r-1\bigr)x^2-\sum_{k\geqslant 2}(y_k+z_kx)x^k\Bigr]\\
&=\frac{1}{1-x^2}\Bigl[s\bigl(x-x^p\bigr)+\bigl(p^r-1\bigr)x^2-\sum_{k\geqslant 1}(y_k+z_kx)x^k\Bigr]\\
&=\frac{1}{1-x^2}\bigl[s\bigl(x-x^p\bigr)+\bigl(p^r-1\bigr)x^2-h(x)\bigr].
\end{aligned}$$

Also, for $j\geqslant 2$

$$\begin{aligned}
q_{E_\infty^{*,j}}(x)&=\frac{1}{1-x^2}\Bigl[v_jx+u_jx^2-\binom{s}{l}x^{l(p-1)+1}\Bigr]-\frac{1}{1-x^2}\binom{s}{j-1}\Bigl[\sum_{k\geqslant 1}(y_k+z_kx)x^{(j-1)(p-1)+k}\Bigr]\\
&=\frac{1}{1-x^2}\Bigl[\binom{s}{j}\bigl(x-x^{j(p-1)+1}\bigr)+\binom{s}{j-1}\bigl(\bigl(p^r-1\bigr)x^2-h(x)x^{(j-1)(p-1)}\bigr)\Bigr].
\end{aligned}$$

In the spectral sequence (25) there are no extensions problems. This can be seen by studying in the same way the sequence (25) with coefficients in $\mathbb{Q}$, $\mathbb{F}_p$ and $\mathbb{F}_q$, for a prime q different from p. Therefore the p-torsion of $H^k_G(X,F;\mathbb{Z})$ is an $\mathbb{F}_p$-vector space of dimension λ_p and these coefficients are determined by the formal power series

$$\begin{aligned}
\bar{Q}_L(x)&:=\sum_{j\geqslant 0}\lambda_jx^j=\sum_{j\geqslant 1}x^jq_{E_\infty^{*,j}}(x)\\
&=\frac{1}{1-x^2}\Bigl[\sum_{j\geqslant 1}\binom{s}{j}\bigl(x^{j+1}-x^{jp+1}\bigr)\Bigr]\\
&\quad+\frac{1}{1-x^2}\Bigl[\bigl(p^r-1\bigr)x^2\Bigl(\sum_{j\geqslant 1}\binom{s}{j-1}x^j\Bigr)-h(x)\Bigl(\sum_{j\geqslant 1}\binom{s}{j-1}x^{(j-1)p+1}\Bigr)\Bigr]\\
&=\frac{x}{1-x^2}\bigl[\bigl(\bigl(p^r-1\bigr)x^2+1\bigr)(1+x)^s-\bigl(1+x^p\bigr)^s\bigl(h(x)+1\bigr)\bigr].
\end{aligned}$$

Note that $h(x)+1=\sum_{k\geqslant 0}(y_k+z_kx)x^k$ and

$$\left(\frac{1-(\alpha x)^p}{1-\alpha x}\right)^r=\sum_{i\geqslant 0}(y_i+\alpha z_i)x^i.$$

Therefore

$$(1+\alpha x)\left(\frac{1-(\alpha x)^p}{1-\alpha x}\right)^r=\sum_{k\geqslant 0}(y_k+z_kx)x^k+\sum_{k\geqslant 0}\alpha(y_kx+z_k)x^k.$$

This shows that

$$\bar{T}_L(x):=\frac{x}{1-x^2}\Bigl[(1+x)^s\bigl(\bigl(p^r-1\bigr)x^2+1\bigr)-(1+\alpha x)\bigl(1+\epsilon_px^p\bigr)^s\left(\frac{1-(\alpha x)^p}{1-\alpha x}\right)^r\Bigr]$$

can be written in the form $\bar{T}_L(x) = \bar{Q}_L(x) + \alpha \bar{R}_L(x)$, for some formal power series with integer coefficients $\bar{R}_L(x)$. □

Lemma 16. *Let X be the G-space induced by a $\mathbb{Z}G$-lattice L of type $(r, s, 0)$. Then the p-torsion subgroup of $H^k(X/G;\mathbb{Z})$ is a finite dimensional vector space over $\mathbb{F}_p$ of dimension β_k. The coefficients β_k, are determined by the formal power series*

$$Q_L(x) := \sum_{k\geqslant 0} \beta_k x^k = \bar{Q}_L(x) - x\big[(1+x)^s - 1\big].$$

Here $\bar{Q}_L(x) := \sum_{k\geqslant 0} \lambda_k x^k$ and λ_k is the dimension of the p-torsion subgroup of $H^k_G(X, F;\mathbb{Z})$ as an $\mathbb{F}_p$-vector space.

Proof. Let us consider first the particular case where the $\mathbb{Z}G$-lattice L is of the form $L = (IG)^r \oplus (\mathbb{Z}G)^s$. The previous lemma shows that the p-torsion subgroup of $H^k_G(X, F;\mathbb{Z})$ is an $\mathbb{F}_p$-vector space of dimension λ_k. The natural map

$$\phi : X \times_G EG \to X/G$$

induces an isomorphism

$$\phi^* : H^*(X/G, F;\mathbb{Z}) \to H^*_G(X, F;\mathbb{Z}) \tag{29}$$

by [4, Proposition VII 1.1]. Therefore the same is true for $H^k(X/G, F;\mathbb{Z})$. To handle $H^*(X/G;\mathbb{Z})$ consider the long exact sequence in cohomology associated to the pair $(X/G, F)$

$$\cdots \to H^{j-1}(F;\mathbb{Z}) \xrightarrow{\delta} H^j(X/G, F;\mathbb{Z}) \to H^j(X/G;\mathbb{Z}) \to H^j(F;\mathbb{Z}) \to \cdots. \tag{30}$$

By Corollary 9

$$F \cong \coprod_{p^r} (\mathbb{S}^1)^s,$$

in particular $H^j(F;\mathbb{Z}) \cong \mathbb{Z}^{p^r\binom{s}{j}}$. By comparing the long exact sequence (30) with the exact sequences in cohomology and equivariant cohomology associated to the pair (X, F), it can be proved that (30) reduces to the following exact sequences

$$0 \to \mathbb{Z}^{p^r-1} \to H^1(X/G, F) \to H^1(X/G) \to \mathbb{Z}^s \to 0,$$

for $j = 1$ and

$$0 \to (\mathbb{Z}/p)^{\binom{s}{j-1}} \oplus \mathbb{Z}^{(p^r-1)\binom{s}{j-1}} \xrightarrow{\delta} H^j(X/G, F) \to H^j(X/G) \to \mathbb{Z}^{\binom{s}{j}} \to 0,$$

for $j \geqslant 2$. Also it follows that $\mathbb{Z}^{(p^r-1)\binom{s}{j-1}}$ splits off $H^j(X/G, F;\mathbb{Z})$. Therefore we conclude that the p-torsion subgroup of $H^k(X/G;\mathbb{Z})$ is a finite dimensional $\mathbb{F}_p$-vector space of dimension β_k, where $\beta_0 = \lambda_0 = 0$, $\beta_1 = \lambda_1$ and

$$\beta_k = \lambda_k - \binom{s}{k-1}$$

for $k \geqslant 2$. This shows that

$$Q_L(x) = \bar{Q}_L(x) - x\big[(1+x)^s - 1\big]$$

and the lemma is true in this case. Suppose now that L is a general $\mathbb{Z}G$-lattice of type $(r, s, 0)$. Then we can find a $\mathbb{Z}G$-homomorphism

$$f : L' \to L$$

that is an isomorphism after tensoring with $\mathbb{Z}_{(p)}$ and with L' of the kind discussed above so the lemma is true for L'. Using Lemma 10 we see that f induces an isomorphism

$$f^* : H_G^*(X_L, F_L; \mathbb{Z}_{(p)}) \to H_G^*(X_{L'}, F_{L'}; \mathbb{Z}_{(p)}).$$

This in turn shows that f induces an isomorphism

$$f^* : H^*(X_L/G, F_L; \mathbb{Z}_{(p)}) \to H^*(X_{L'}/G, F_{L'}; \mathbb{Z}_{(p)})$$

by (29). Finally, by comparing the long exact sequences in cohomology with $\mathbb{Z}_{(p)}$-coefficients associated to the pairs $(X_L/G, F_L)$ and $(X_{L'}/G, F_{L'})$ it follows that f induces an isomorphism

$$f^* : H^*(X_L/G; \mathbb{Z}_{(p)}) \to H^*(X_{L'}/G; \mathbb{Z}_{(p)})$$

and thus using the universal coefficient theorem we see that the lemma is also true for L. $\square$

We are now ready to prove the main theorem.

Proof of Theorem 1. Suppose that L is a $\mathbb{Z}G$-lattice of type (r, s, t). Then we can write $L \cong L' \oplus \mathbb{Z}^t$ where L' is of type $(r, s, 0)$. By (8) we have

$$X_L \cong X_{L'} \times \big(\mathbb{S}^1\big)^t,$$

where G acts trivially on the factor $(\mathbb{S}^1)^t$. Then the Kunneth theorem gives an isomorphism

$$H^*(X_L/G) \cong H_G^*(X_{L'}/G) \otimes H^*\big(\big(\mathbb{S}^1\big)^t; \mathbb{Z}\big).$$

Using this isomorphism and Corollary 4, Theorem 15 and Lemma 16 the theorem is proved. $\square$

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