Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Linear Algebra and its Applications 412 (2006) 380–395
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

A regularization algorithm for matrices of bilinear and sesquilinear forms

Roger A. Horn [a,*], Vladimir V. Sergeichuk [b,1]

[a] *Department of Mathematics, University of Utah, 155 South 1400 East, Salt Lake City, UT 84112-0090, USA*
[b] *Institute of Mathematics, Tereshchenkivska 3, Kiev, Ukraine*

Received 29 May 2005; accepted 6 July 2005
Available online 26 August 2005
Submitted by M. Goldberg

Abstract

Over a field or skew field $\mathbb{F}$ with an involution $a \mapsto \tilde{a}$ (possibly the identity involution), each singular square matrix A is *congruent to a direct sum

$$S^*AS = B \oplus J_{n_1} \oplus \cdots \oplus J_{n_p}, \quad 1 \leqslant n_1 \leqslant \cdots \leqslant n_p,$$

in which S is nonsingular and $S^* = \widetilde{S}^T$; B is nonsingular and is determined by A up to *congruence; and the $n_i \times n_i$ singular Jordan blocks J_{n_i} and their multiplicities are uniquely determined by A. We give a regularization algorithm that needs only elementary row operations to construct such a decomposition. If $\mathbb{F} = \mathbb{C}$ (respectively, $\mathbb{F} = \mathbb{R}$), we exhibit a regularization algorithm that uses only unitary (respectively, real orthogonal) transformations and a reduced form that can be achieved via a unitary *congruence or congruence (respectively, a real orthogonal congruence). The selfadjoint matrix pencil $A + \lambda A^*$ is decomposed by our regularization algorithm into the direct sum

$$S^*(A + \lambda A^*)S = (B + \lambda B^*) \oplus (J_{n_1} + \lambda J_{n_1}^*) \oplus \cdots \oplus (J_{n_p} + \lambda J_{n_p}^*)$$

with selfadjoint summands.

* Corresponding author.
E-mail addresses: rhorn@math.utah.edu (R.A. Horn), sergeich@imath.kiev.ua (V.V. Sergeichuk).
[1] The research was started while this author was visiting the University of Utah supported by NSF grant DMS-0070503.

0024-3795/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2005.07.004

AMS classification: 15A63; 15A21; 15A22

Keywords: Canonical matrices; Bilinear forms; Matrix pencils; Stable algorithms

1. Introduction

All of the matrices that we consider are over a field or skew field $\mathbb{F}$ with an involution $a \mapsto \tilde{a}$, that is, a bijection on $\mathbb{F}$ such that

$$\widetilde{a+b} = \tilde{a} + \tilde{b}, \quad \widetilde{ab} = \tilde{b}\tilde{a}, \quad \tilde{\tilde{a}} = a.$$

We refer to $\tilde{a}$ as the *conjugate* of a. If $\mathbb{F}$ is a field, the identity mapping $a \mapsto a$ on $\mathbb{F}$ is always an involution; over the complex field, complex conjugation $a \mapsto \bar{a}$ is an involution.

The entrywise conjugate of the transpose of a matrix $A = [a_{ij}]$ is denoted by

$$A^* = \widetilde{A}^T = [\tilde{a}_{ji}].$$

If there is a square nonsingular matrix S such that $S^*AS = B$, then A and B are said to be **congruent*; if the involution on $\mathbb{F}$ is the identity, i.e., $S^* = S^T$ and $S^*AS = S^TAS = B$, we say that A and B are *congruent*. Congruence of matrices (sometimes called T-*congruence*) is therefore a special type of *congruence in which the involution is the identity. Over the complex field with complex conjugation as the involution, *congruence is sometimes called *conjunctivity*. If A is nonsingular, we write A^{-*} for $(A^*)^{-1}$. Let

$$J_n = \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}$$

denote the $n \times n$ singular Jordan block.

For any $m \times n$ matrix A (that is, $A \in \mathbb{F}^{m\times n}$) we write $N(A) := \{x \in \mathbb{F}^n : Ax = 0\}$ (the *null space* of A) and denote its dimension by $\dim N(A) = \operatorname{nullity} A$. If A is square, we let

$$A^{[k]} := A \oplus \cdots \oplus A \ (k \text{ times}).$$

In Section 2 we describe a constructive *regularization algorithm* that determines a *regularizing decomposition*

$$B \oplus J_{n_1} \oplus \cdots \oplus J_{n_p}, \quad B \text{ nonsingular and } 1 \leqslant n_1 \leqslant \cdots \leqslant n_p \tag{1}$$

to which a given square singular matrix A is *congruent. The *congruence class of B (the *regular part* of A under *congruence) as well as the sizes and multiplicities of the direct summands $J_{n_1}, \ldots, J_{n_p}$ (the *singular part* of A under *congruence) are all uniquely determined by the *congruence class of A. If $\mathbb{F} = \mathbb{C}$ (respectively, $\mathbb{F} = \mathbb{R}$), the regularizing decomposition (1) can be determined using only unitary (respectively, real orthogonal) transformations. Our proof of the existence and uniqueness of the regularizing decomposition (1) uses two geometric *congruence invariants that we discuss in Section 3: $\dim N(A)$ and $\dim(N(A^*) \cap N(A))$.

In Section 4 we exhibit a canonical *sparse form* that is *congruent to A and determines the sizes and multiplicities of the nilpotent direct summands in the regularizing decomposition (1). The essential parameters of the sparse form are identical to those produced by our regularization algorithm, which verifies the validity of the algorithm. When $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, we describe a reduced form related to the canonical sparse form that can be achieved using only unitary *congruences or T-congruences.

The regularization algorithm reduces the problem of determining a *congruence canonical form to the nonsingular case. A complete set of *congruence canonical forms (up to classification of Hermitian forms) when $\mathbb{F}$ is a field with characteristic not equal to two is given in [7, Theorem 3]; see also [5, Theorem 2]. A nonalgorithmic reduction to the nonsingular case was given by Gabriel for bilinear forms [2]; his method was extended in [6] to sesquilinear forms, and in [7] to systems of sesquilinear forms and linear mappings. The form of the regularizing decomposition (1) is implicit in the statement of Proposition 3.1 in [1] when $\mathbb{F}$ is a field and the involution is the identity; the construction employed in its proof does not suggest a simple algorithm for identifying the parameters in (1).

If $A, B \in \mathbb{F}^{m\times n}$, then the polynomial matrix $A + \lambda B$ is called a *matrix pencil*. Two matrix pencils $A + \lambda B$ and $A' + \lambda B'$ are said to be *strictly equivalent* if there exist nonsingular matrices S and R such that $S(A + \lambda B)R = A' + \lambda B'$. In [9], Van Dooren described an algorithm that uses only unitary transformations and for each complex matrix pencil $A + \lambda B$ produces a strictly equivalent pencil

$$(C + \lambda D) \oplus (M_1 + \lambda N_1) \oplus \cdots \oplus (M_l + \lambda N_l) \tag{2}$$

in which C and D are nonsingular constituents of the *regular part* $C + \lambda D$ of the Kronecker canonical form of $A + \lambda B$; each $M_i + \lambda N_i$ is a singular direct summand of that canonical form; see [3, Section XII, Theorem 5]. Each $M_i + \lambda N_i$ has the form

$$I_n + \lambda J_n, \quad J_n + \lambda I_n, \quad F_n + \lambda G_n, \quad \text{or} \quad G_n^T + \lambda F_n^T$$

for some n, in which

$$F_n = \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix} \quad \text{and} \quad G_n = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix} \text{ are } (n-1)\times n.$$

The direct sum (2) is a *regularizing decomposition* of $A + \lambda B$; $C + \lambda D$ is the *regular part* of $A + \lambda B$. Van Dooren's algorithm was extended to cycles of linear mappings with arbitrary orientation of arrows in [8].

If Van Dooren's algorithm is used to construct a regularizing decomposition of a *selfadjoint matrix pencil $A + \lambda A^*$, the regular part produced need not be *selfadjoint. However, the regularizing decomposition of $A + \lambda A^*$ that we describe in Section 5 always produces a *selfadjoint regular part.

For any nonnegative integers m and n, we denote the $m \times n$ zero matrix by 0_{mn}, or by 0_m if $m = n$. The $n \times 0$ matrix 0_{n0} is understood to represent the linear mapping $0 \to \mathbb{F}^n$; the $0 \times n$ matrix 0_{0n} represents the linear mapping $\mathbb{F}^n \to 0$; the 0×0 matrix 0_0 represents the linear mapping $0 \to 0$. For every $p \times q$ matrix M_{pq} we have

$$M_{pq} \oplus 0_{m0} = \begin{bmatrix} M_{pq} & 0 \\ 0 & 0_{m0} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{p0} \\ 0_{mq} & 0_{m0} \end{bmatrix} = \begin{bmatrix} M_{pq} \\ 0_{mq} \end{bmatrix}$$

and

$$M_{pq} \oplus 0_{0n} = \begin{bmatrix} M_{pq} & 0 \\ 0 & 0_{0n} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{pn} \\ 0_{0q} & 0_{0n} \end{bmatrix} = [M_{pq} \quad 0_{pn}].$$

In particular,

$$0_{p0} \oplus 0_{0q} = 0_{pq}$$

and $J_n^{[0]} = 0_0$. Consistent with the definition of singularity, our convention is that *a* 0×0 matrix is nonsingular.

2. The regularization algorithm

The first stage in our *regularization algorithm* for a singular square matrix A is to reduce it by *congruence transformations in two steps that construct a smaller matrix $A_{(1)}$ and integers m_1 and m_2 as follows:

Step 1. Choose a nonsingular S such that the top rows of SA are linearly independent and the bottom m_1 rows are zero, then form $(SA)S^*$ and partition it so that the upper left block is square:

$$\begin{aligned} A \longmapsto\ & SA = \begin{bmatrix} A' \\ 0 \end{bmatrix} \quad \begin{array}{l} (S \text{ is nonsingular and the rows of} \\ \quad A' \text{ are linearly independent}), \end{array} \\ \longmapsto\ & SAS^* = \begin{bmatrix} A'S^* \\ 0 \end{bmatrix} = \left[\begin{array}{c|c} M & N \\ \hline 0 & 0_{m_1} \end{array}\right] \quad \begin{array}{l} (S \text{ is the same and} \\ \quad M \text{ is square}). \end{array} \end{aligned} \tag{3}$$

The integer m_1 is the nullity of A.

Step 2. Choose a nonsingular R such that the top rows of RN are zero and the bottom m_2 rows are linearly independent:

$$RN = \begin{bmatrix} 0 \\ E \end{bmatrix} \quad \begin{array}{l} (R \text{ is nonsingular and the rows of} \\ \quad E \text{ are linearly independent}). \end{array} \tag{4}$$

The integer m_2 is the rank of N. Now perform a *congruence of S^*AS with $R \oplus I$:

$$\left[\begin{array}{c|c} M & N \\ \hline 0 & 0 \end{array}\right] \longmapsto (R \oplus I) \left[\begin{array}{c|c} M & N \\ \hline 0 & 0 \end{array}\right] (R \oplus I)^*, \tag{5}$$

$$= \left[\begin{array}{c|c} RMR^* & RN \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_{(1)} & B & 0 \\ C & D & E \\ \hline \multicolumn{2}{c|}{0} & 0_{m_1} \end{array}\right] \begin{array}{l} \\ \}m_2. \\ \}m_1 \end{array} \tag{6}$$

The block RMR^* has been partitioned so that D is $m_2 \times m_2$. The size of the square matrix $A_{(1)}$ is strictly less than that of A.

If $A_{(1)}$ is nonsingular, the algorithm terminates. If $A_{(1)}$ is singular, the second stage of the regularization algorithm is to perform the two *congruences (3) and (5) on it and obtain integers m_3 (the nullity of $A_{(1)}$) and m_4, and a square matrix $A_{(2)}$ whose size is strictly less than that of $A_{(1)}$.

The regularization algorithm proceeds from stage k to stage $k + 1$ by performing the two *congruences (3) and (5) on the singular square matrix $A_{(k-1)}$ to obtain m_{2k-1}, m_{2k}, and $A_{(k)}$. When the algorithm terminates at stage τ with a square matrix $A_{(\tau)}$ that is nonsingular, we have in hand a non-increasing sequence of integers $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_{2\tau-1} \geqslant m_{2\tau} \geqslant 0$ and a nonsingular matrix $A_{(\tau)}$. Our main result is that these data determine the singular part of A under *congruence as well as the *congruence class of the regular part according to the following rule:

Theorem 1. *Let A be a given square singular matrix over $\mathbb{F}$ and apply the regularization algorithm to it. Then A is *congruent to $A_{(\tau)} \oplus M$, in which $A_{(\tau)}$ is nonsingular and*

$$M = J_1^{[m_1-m_2]} \oplus J_2^{[m_2-m_3]} \oplus J_3^{[m_3-m_4]} \oplus \cdots \oplus J_{2\tau-1}^{[m_{2\tau-1}-m_{2\tau}]} \oplus J_{2\tau}^{[m_{2\tau}]}. \tag{7}$$

*The integers $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_{2\tau-1} \geqslant m_{2\tau} \geqslant 0$, as well as the *congruence class of $A_{(\tau)}$, are uniquely determined by the *congruence class of A.*

In the next section we offer a geometric interpretation for the integers m_i in (7) and explain why they and the *congruence class of each of the square matrices $A_{(k)}$ produced by the regularization algorithm are *congruence invariants of A. Implicit in the regularization algorithm are certain reductions of A by *congruences that we refine in order to explain why the regularizing decomposition in (7) is valid.

The nonsingular matrices S and R in the two *congruence steps of the regularization algorithm can always be constructed with elementary row operations. For the complex (respectively, real) field, it can be useful for numerical implementation to know that S and R may be chosen to be unitary (respectively, real orthogonal).

Theorem 2. *Let A be a given square singular complex (respectively, real) matrix. The regularizing decomposition (7) of A can be determined using only unitary (respectively, real orthogonal) transformations.*

Proof. (a) Suppose $\mathbb{F} = \mathbb{C}$ with complex conjugation as the involution. Let $A = U^*\Sigma Z$ be a singular value decomposition in which $\Sigma = \Sigma_1 \oplus 0_{m_1}$, Σ_1 is positive diagonal, and U and Z are unitary. The choice $S = U$ achieves the required reduction in Step 1. In Step 2, let $N = \widehat{V}^*\widehat{\Sigma}W$ be a singular value decomposition in which $\widehat{V}$ and W are unitary, $\widehat{\Sigma} = \widehat{\Sigma}_1 \oplus 0$, and $\widehat{\Sigma}_1$ is positive diagonal and $m_2 \times m_2$. Let

$$P = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$$

be the *reversal matrix* whose size is the same as that of $\widehat{V}$. Then $N = (P\widehat{V})^*(P\widehat{\Sigma})W$, $(P\widehat{\Sigma})W$ has the block form (4), and $V := P\widehat{V}$ is unitary, so we may take $R = V$ in Step 2. Thus, A is unitarily *congruent (unitarily similar) to a block matrix of the form (6) in which D is square and each of E and $[A_{(1)}\ B]$ has linearly independent rows.

(b) Suppose $\mathbb{F} = \mathbb{C}$ with the identity involution. In Step 1, choose $S = \overline{U}$ from (a). In Step 2, choose $R = \overline{V}$ from (a). Thus, A is unitarily T-congruent to a block matrix of the form (6) in which D is square and each of E and $[A_{(1)}\ B]$ has linearly independent rows.

(c) Suppose $\mathbb{F} = \mathbb{R}$ with the identity involution. Proceed as in (a), choosing U, $\widehat{V}$, and W to be real orthogonal in the two singular value decompositions. Thus, A is real orthogonally congruent to a block matrix of the form (6) in which D is square and each of E and $[A_{(1)}\ B]$ has linearly independent rows. $\square$

The regularizing algorithm tells how to construct a sequence of pairs of transformations of the square matrices $A_{(k)}$ that are sufficient to determine the regularizing decomposition of A. Implicit in these transformations is a sequence of pairs of *congruences that reduce A in successive stages. After the first stage, the *congruences reduce A to the form (6). After the second stage, if we were to carry out the *congruences we would obtain a matrix of the form

$$\begin{bmatrix} A_{(2)} & \bigstar & 0 & \bigstar & 0 \\ \bigstar & \bigstar & \square & \bigstar & 0 \\ 0 & 0 & 0 & \square & 0 \\ \bigstar & \bigstar & \bigstar & \bigstar & \square \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \}m_4 \\ \}m_3, \\ \}m_2 \\ \}m_1 \end{matrix} \tag{8}$$

in which the diagonal blocks are square, the $\bigstar$ blocks are not necessarily zero, and each $\square$ block has linearly independent rows. Theorem 2 ensures that if A is complex, then there are unitary matrices U and V such that each of U^*AU and V^TAV has the form (8), with possibly different values for the parameters m_i. If A is real, there is a real orthogonal Q such that Q^TAQ has the form (8).

3. *Congruence invariants and a reduced form

Throughout this section, $A \in \mathbb{F}^{m\times m}$ and S is a nonsingular matrix. Of course, $\operatorname{nullity} A = \operatorname{nullity} S^*AS$, so nullity is a *congruence invariant. The relationships

$$N(S^*AS) = S^{-1}N(A) \quad \text{and} \quad N(S^*A^*S) = S^{-1}N(A^*) \tag{9}$$

between the null spaces of A and S^*AS, and those of A^* and S^*A^*S, imply that

$$N(S^*A^*S) \cap N(S^*AS) = S^{-1}(N(A^*) \cap N(A)). \tag{10}$$

We refer to $\zeta := \dim N(A^*) \cap N(A)$ as the **normal nullity* of A. We let $\nu := \operatorname{nullity} A$, refer to $\kappa := \nu - \zeta$ as the **non-normal nullity* of A, and let $\rho = m - \kappa - \nu$.

It follows from (9) and (10) that ν, ζ, κ, and ρ are *congruence invariants. Because

$$N(A^*) \cap N(A) = N\left(\begin{bmatrix} A \\ A^* \end{bmatrix}\right), \tag{11}$$

ν and ζ (and hence also κ and ρ) can be computed using elementary row operations.

The parameter m_1 produced by the regularization algorithm is the nullity of A, so it is a *congruence invariant: $m_1 = \nu$.

The parameter m_2 produced by the regularization algorithm is the rank of the block N in (3). Since N has m_1 columns and full row rank, its nullity is $m_1 - m_2$. Suppose $z \in \mathbb{F}^{m_1}$ and $Nz = 0$, let $y^* = [0\ z^*]$, and let $\mathscr{A} = SAS^*$ denote the block matrix in (3). Then $\mathscr{A}y = 0$ and $y^*\mathscr{A} = 0$, so $\zeta = \dim(N(\mathscr{A}^*) \cap N(\mathscr{A})) = \operatorname{nullity} N = m_1 - m_2$, and hence $m_1 - m_2 = \zeta$ is the *normal nullity of A. This means that $m_2 = m_1 - \zeta = \nu - \zeta = \kappa$ is the *non-normal nullity of A, so m_2 is also a *congruence invariant.

The following lemma ensures that the *congruence class of the square matrix $A_{(1)}$ in (6) is also a *congruence invariant.

Lemma 3. *Suppose that a singular square matrix A is *congruent to*

$$M = \begin{bmatrix} A_{(1)} & B & 0 \\ C & D & E \\ 0 & 0 & 0_\nu \end{bmatrix} \quad \text{and also to } \underline{M} = \begin{bmatrix} \underline{A}_{(1)} & \underline{B} & 0 \\ \underline{C} & \underline{D} & \underline{E} \\ 0 & 0 & 0_{\underline{\nu}} \end{bmatrix},$$

in which D is $\kappa \times \kappa$, $\underline{D}$ is $\underline{\kappa} \times \underline{\kappa}$, and each of E, $\underline{E}$, $[A_{(1)}\ B]$, and $[\underline{A}_{(1)}\ \underline{B}]$ has linearly independent rows. Then $\nu = \underline{\nu}$, $\kappa = \underline{\kappa}$, and $A_{(1)}$ is *congruent to $\underline{A}_{(1)}$; that is, ν, κ, ρ, and the *congruence class of the $\rho \times \rho$ matrix $A_{(1)}$ are *congruence invariants of A.

Proof. The form of M ensures that ν is its nullity and that κ is its *non-normal nullity; $\underline{\nu}$ is the nullity of $\underline{M}$ and $\underline{\kappa}$ is its *non-normal nullity. Since M and $\underline{M}$ are *congruent to A and hence to each other, their nullities and *non-normal nullities are the same, so $\nu = \underline{\nu}$ and $\kappa = \underline{\kappa}$.

Let

$$\widehat{M} = \begin{bmatrix} A_{(1)} & B & 0 \\ C & D & E \end{bmatrix} \quad \text{and} \quad \underline{\widehat{M}} = \begin{bmatrix} \underline{A}_{(1)} & \underline{B} & 0 \\ \underline{C} & \underline{D} & \underline{E} \end{bmatrix}.$$

If $S = [S_{ij}]_{i,j=1}^2$ is nonsingular, S_{22} is $\nu \times \nu$, and $SMS^* = \underline{M}$, then

$$SM = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} \widehat{M} \\ 0 \end{bmatrix} = \begin{bmatrix} \bigstar \\ S_{21}\widehat{M} \end{bmatrix} = \begin{bmatrix} \bigstar \\ 0 \end{bmatrix} = \underline{M}S^{-*},$$

so $S_{21}\widehat{M} = 0$. Full row rank of $\widehat{M}$ ensures that $S_{21} = 0$ and hence both S_{11} and S_{22} are nonsingular. If we write $S_{11} = [R_{ij}]_{i,j=1}^{2}$, in which R_{22} is $\kappa \times \kappa$, then equating the 1,2 blocks of SMS^* and $\underline{M}$ tells us that

$$S_{11}\begin{bmatrix}0\\E\end{bmatrix} = \begin{bmatrix}R_{12}E\\ \bigstar\end{bmatrix} = \begin{bmatrix}0\\ \bigstar\end{bmatrix} = \begin{bmatrix}0\\ \underline{E}\end{bmatrix}(S_{22})^{-*},$$

which ensures that $R_{12} = 0$, R_{11} and R_{22} are nonsingular, and $R_{11}A_{(1)}R_{11}^{*} = \underline{A}_{(1)}$. □

At each stage $k = 1, 2, \dots$ of the algorithm, Lemma 3 ensures that the *congruence class of A determines the *congruence class of the square matrix $A_{(k)}$ as well as the integers m_{2k-1} (the nullity of $A_{(k-1)}$) and m_{2k} (the *non-normal nullity of $A_{(k-1)}$). The number of stages τ in the algorithm until it terminates as well as the *congruence class of the final nonsingular square matrix $A_{(\tau)}$ are also determined by the *congruence class of A. All that remains to be shown is that these data determine the regularizing decomposition of A according to the rule in Theorem 1.

The block matrix (6) can be reduced to a sparser form by *congruence if $m_2 > 0$: the block E may be taken to be $[I_{m_2}\ 0]$ and the blocks C and D may be taken to be zero. To achieve these reductions, is is useful to realize that if $A \to AS$ adds linear combinations of a set of columns of A with index set α to certain columns, and if the rows of A with index set α are all zero, then $S^*A = A$, so $S^*AS = AS$.

Lemma 4. *If a singular square matrix A is *congruent to a block matrix $\mathscr{A}$ of the form* (6) *in which $m_2 > 0$, D is $m_2 \times m_2$, and E has linearly independent rows, then it is *congruent to*

$$\begin{bmatrix}A_{(1)} & B & 0\\ 0 & 0_{m_2} & [I_{m_2}\ 0]\\ 0 & 0 & 0_{m_1}\end{bmatrix}. \tag{12}$$

Proof. Since $\operatorname{rank} E = m_2$, there is a nonsingular V such that $EV = [I_{m_2}\ 0]$. For $S = I_{m-m_1} \oplus V$ we have

$$S^*\mathscr{A}S = \mathscr{A}S = \begin{bmatrix}A_{(1)} & B & 0\\ C & D & [I_{m_2}\ 0]\\ 0 & 0 & 0_{m_1}\end{bmatrix} := \mathscr{A}'.$$

Then, for

$$S = \begin{bmatrix}I_{m-m_1} & 0\\ X & I_{m_1}\end{bmatrix} \quad\text{and}\quad X = -\begin{bmatrix}C & D\\ 0 & 0\end{bmatrix},$$

$\mathscr{A}'S = S^*\mathscr{A}'S$ has the form (12). □

A block matrix of the form (12) is said to be a **congruence reduced form* of A if it is *congruent to A, $A_{(1)}$ is square, and $[A_{(1)}\ B]$ has linearly independent rows.

There are four possibilities for the $\rho \times \rho$ matrix $A_{(1)}$ in a *congruence reduced form of A:

- $\rho = 0$: Then A is *congruent to
$$\mathscr{A} = \begin{bmatrix} 0_{m_2} & [I_{m_2}\ 0] \\ 0 & 0_{m_1} \end{bmatrix}.$$
Since rank $\mathscr{A} = m_2$ and $\mathscr{A}^2 = 0$, its Jordan Canonical Form contains m_2 blocks J_2 and $m_1 - m_2$ blocks J_1. But $\mathscr{A}$ is similar to its Jordan Canonical Form via a permutation similarity, which is a *congruence, so $J_1^{[m_1-m_2]} \oplus J_2^{[m_2]}$ is the regularizing decomposition for A.
- $\rho > 0$ and $A_{(1)} = 0_\rho$, so $m_3 = \operatorname{nullity} A_{(1)} = \rho$: A is *congruent to
$$\mathscr{A} = \begin{bmatrix} 0_{m_3} & B & 0 \\ 0 & 0_{m_2} & [I_{m_2}\ 0] \\ 0 & 0 & 0_{m_1} \end{bmatrix},$$
in which B has full row rank. There is a nonsingular V such that $BV = [I_{m_3}\ 0]$, so if we let $S = I_{m_3} \oplus V \oplus I_{m_1}$, we have
$$S^*\mathscr{A}S = \begin{bmatrix} 0_{m_3} & [I_{m_3}\ 0] & 0 \\ 0 & 0_{m_2} & [V^*\ 0] \\ 0 & 0 & 0_{m_1} \end{bmatrix} := R.$$
Now let $S = I_{m_3+m_2} \oplus (V^{-*} \oplus I_{m_1-m_2})$ and compute
$$S^*RS = \begin{bmatrix} 0_{m_3} & [I_{m_3}\ 0] & 0 \\ 0 & 0_{m_2} & [I_{m_2}\ 0] \\ 0 & 0 & 0_{m_1} \end{bmatrix} := N.$$
Then rank $N = m_3 + m_2$, rank $N^2 = m_3$, and $N^3 = 0$, so the Jordan Canonical Form of N is $J_1^{[m_1-m_2]} \oplus J_2^{[m_2-m_3]} \oplus J_3^{[m_3]}$, which is the regularizing decomposition for A.
- $\rho > 0$ and $A_{(1)}$ is nonsingular: Let R denote the block matrix in (12), let
$$S = \begin{bmatrix} I_\rho & -(A_{(1)})^{-1}B \\ 0 & I_{m_2} \end{bmatrix} \oplus I_{m_1},$$
and compute
$$S^*RS = \begin{bmatrix} A_{(1)} & 0 & 0 \\ X & 0_{m_2} & [I_{m_2}\ 0] \\ 0 & 0 & 0_{m_1} \end{bmatrix},$$
in which $X = -B^*A_{(1)}^{-*}A_{(1)}$. Lemma 4 tells us that S^*RS is *congruent to (12) with $B = 0$, that is, to $A_{(1)} \oplus M$ with
$$M = \begin{bmatrix} 0_{m_2} & [I_{m_2}\ 0] \\ 0 & 0_{m_1} \end{bmatrix}.$$

Since rank $M = m_2$ and $M^2 = 0$, the regularizing decomposition of A is $A_{(1)} \oplus J_1^{[m_1-m_2]} \oplus J_2^{[m_2]}$.

- $\rho > 0$ and $A_{(1)}$ is singular but nonzero: We address this case in the next lemma.

Lemma 5. *Let $\rho > 0$ and let $A_{(1)}$ be the $\rho \times \rho$ upper left block in a *congruence reduced form* (12) *of A. Let m_3 and m_4 denote the nullity and *non-normal nullity, respectively, of $A_{(1)}$, and suppose that $m_3 > 0$. Then A is *congruent to*

$$\begin{bmatrix} A_{(2)} & B' & 0 & 0 & 0 \\ 0 & 0_{m_4} & [I_{m_4}\ 0] & 0 & 0 \\ 0 & 0 & 0_{m_3} & [I_{m_3}\ 0] & 0 \\ 0 & 0 & 0 & 0_{m_2} & [I_{m_2}\ 0] \\ 0 & 0 & 0 & 0 & 0_{m_1} \end{bmatrix}, \tag{13}$$

*in which $[A_{(2)}\ B']$ has linearly independent rows. The parameters m_1, m_2, m_3, and m_4, and the *congruence class of $A_{(2)}$ are *congruence invariants of A.*

Proof. Step 1: Lemma 4 ensures that there is a nonsingular S such that

$$S^*A_{(1)}S = \begin{bmatrix} A_{(2)} & B' & 0 \\ 0 & 0_{m_4} & [I_{m_4}\ 0] \\ 0 & 0 & 0_{m_3} \end{bmatrix}$$

is a *congruence reduced form of $A_{(1)}$. Let ρ' denote the size of $A_{(2)}$. Let $\widehat{S} = S \oplus I_{m_2+m_1}$ and observe that $\widehat{S}^*A\widehat{S}$ has the block form

$$\begin{bmatrix} S^*A_{(1)}S & S^*B & 0 \\ 0 & 0_{m_2} & [I_{m_2}\ 0] \\ 0 & 0 & 0_{m_1} \end{bmatrix} = \begin{bmatrix} A_{(2)} & B' & 0 & B_1 & 0 \\ 0 & 0_{m_4} & [I_{m_4}\ 0] & B_2 & 0 \\ 0 & 0 & 0_{m_3} & B_3 & 0 \\ 0 & 0 & 0 & 0_{m_2} & [I_{m_2}\ 0] \\ 0 & 0 & 0 & 0 & 0_{m_1} \end{bmatrix}, \tag{14}$$

in which

$$S^*B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}.$$

Step 2: Let M denote the upper left 2×3 block of the 5×5 block matrix in (14). The rows of M are linearly independent, so its columns span $\mathbb{F}^{\rho'+m_4}$. Add a linear combination of the *columns* of M to the fourth block column of (14) in order to put zeros in the blocks B_1 and B_2. Complete this column operation to a *congruence by adding the conjugate linear combination of *rows* of M to the fourth block row of (14); this spoils the zeros in the first four blocks of the fourth block row. Add linear combinations of the fifth block column to the first four block columns in order to re-establish the zero blocks there; the fifth block row is zero so completing this column operation to a *congruence with a conjugate row operation has no effect.

We have now achieved a *congruence of A that has the form

$$R = \begin{bmatrix} A_{(2)} & B' & 0 & 0 & 0 \\ 0 & 0_{m_4} & [I_{m_4}\ 0] & 0 & 0 \\ 0 & 0 & 0_{m_3} & B_3 & 0 \\ 0 & 0 & 0 & 0_{m_2} & [I_{m_2}\ 0] \\ 0 & 0 & 0 & 0 & 0_{m_1} \end{bmatrix}, \tag{15}$$

in which B_3 has linearly independent rows.

Step 3: Whenever one has a block matrix like that in (15), in which some of the superdiagonal blocks below the first block row (all with linearly independent rows) do not have the standard form $[I\ 0]$, there is a finite sequence of *congruences that restores it to a standard form like that in (13). For example, B_3 in (15) has linearly independent rows, so there is a nonsingular V such that $B_3V = [I_{m_3}\ 0]$. Right-multiply the fourth block column of R by V and left-multiply the 4th block row of the result by V^*. This restores the standard form of the block in position 3,4 but spoils the $[I\ 0]$ block in position 4,5, which still has linearly independent rows. Now right-multiply the fifth block column by a factor that restores it to standard form (in this case, the right multiplier is $V^{-*} \oplus I_{m_1-m_2}$) and then left-multiply the fifth block row by the conjugate transpose of that factor. If there are more than five block rows, continue this process down the block superdiagonal to the block in the last block column; all of the superdiagonal blocks below the first block row will then be restored to standard form since the last block row is zero. Of course, this finite sequence of transformations is a *congruence of R. $\square$

The preceding lemma clarifies the nature of the block B in a *congruence reduced form (12) of A: except for the requirement that $[A_{(1)}\ B]$ have full row rank, B is otherwise arbitrary.

If there are different involutions on $\mathbb{F}$, the same matrix may have a different regularizing decomposition for each involution. For example, take $\mathbb{F} = \mathbb{C}$ and consider

$$A = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}.$$

If the involution is complex conjugation, then $N(A) = N(A^*)$ since A is Hermitian, $\zeta = m_1 = 1$, $\kappa = m_2 = 0$, and $\rho = 1$; the regularizing decomposition of A is $[1] \oplus J_1$. However, if the involution is the identity, then $N(A) \cap N(A^T) = \{0\}$, $\zeta = m_1 = 0$, $\kappa = m_2 = 1$, and $\rho = 0$; the regularizing decomposition of A is J_2.

4. The regularizing decomposition

Reduction of A to a sparse form that reveals all of its singular structure under *congruence can be achieved by repeating the three steps in Lemma 5 to obtain successively smaller blocks $A_{(3)}, A_{(4)}, \ldots, A_{(\tau)}$ (with successively smaller nullities) in which $A_{(\tau)}$ is the first block that is nonsingular. The final reduced form achieved after these reduction steps is described in the following theorem.

Theorem 6 (*Regularizing Decomposition*). *Let A be a given square singular matrix over $\mathbb{F}$. Perform the regularization algorithm on A and obtain the integers $\tau, m_1, m_2, \ldots, m_{2\tau}$ and a nonsingular matrix $A_{(\tau)}$. Then $\tau, m_1, m_2, \ldots, m_{2\tau}$ and the *congruence class of $A_{(\tau)}$ are *congruence invariants of A. Moreover,*

(a) (*Canonical sparse form*) *A is *congruent to $A_{(\tau)} \oplus N$, in which*

$$N = \begin{bmatrix} 0_{m_{2\tau}} & [I_{m_{2\tau}}\ 0] & & & & & \\ & 0_{m_{2\tau-1}} & [I_{m_{2\tau-1}}\ 0] & & & & \\ & & \ddots & \ddots & & & \\ & & & 0_{m_3} & [I_{m_3}\ 0] & & \\ & & & & 0_{m_2} & [I_{m_2}\ 0] \\ & & & & & 0_{m_1} \end{bmatrix} \tag{16}$$

has all of its nonzero entries in the first block superdiagonal, and each block $[I_{m_k}\ 0]$ is $m_k \times m_{k-1}$, $k = 2, 3, \ldots, 2\tau$.

(b) (*Existence*) *A is *congruent to $A_{(\tau)} \oplus M$, in which*

$$M = J_1^{[m_1-m_2]} \oplus J_2^{[m_2-m_3]} \oplus J_3^{[m_3-m_4]} \oplus \cdots \oplus J_{2\tau-1}^{[m_{2\tau-1}-m_{2\tau}]} \oplus J_{2\tau}^{[m_{2\tau}]}. \tag{17}$$

(c) (*Uniqueness*) *Suppose A is *congruent to $B \oplus C$, in which B is nonsingular and C is a direct sum of nilpotent Jordan blocks. Then B is *congruent to $A_{(\tau)}$ and some permutation of the direct summands of C gives M.*

(d) (*Unitarily reduced form*) (*i*) *If $\mathbb{F} = \mathbb{C}$ and the involution is complex conjugation, there is a complex unitary U such that U^*AU has the form*

$$\begin{array}{c}\left[\begin{array}{c|cc|cc|cc|cc}
B_{2\tau+1} & 0 & 0 & & & & & & \\
\bigstar & \bigstar & B_{2\tau} & & & & & & \\
0 & 0 & 0 & \bigstar & 0 & & & & \\
\hline
 & \ddots & & & & \bigstar & 0 & & \\
 & & \ddots & B_7 & 0 & & & \bigstar & 0 \\
\hline
 & \bigstar & & \bigstar & B_6 & & & & \\
 & 0 & & 0 & 0 & B_5 & 0 & & \\
\hline
 & & \bigstar & & & \bigstar & B_4 & & \\
 & & 0 & & & 0 & 0 & B_3 & 0 \\
\hline
 & & & \bigstar & & & & \bigstar & B_2 \\
 & & & 0 & & & & 0 & 0
\end{array}\right]\end{array}
\begin{array}{l} \\ \}m_{2\tau} \\ \}m_{2\tau-1} \\ \\ \vdots \\ \vdots \\ \\ \}m_4 \\ \}m_3 \\ \}m_2 \\ \}m_1 \end{array} \tag{18}$$

in which the $\bigstar$ blocks are not necessarily zero; all $2\tau + 1$ diagonal blocks $B_{2\tau+1}, \bigstar, 0, \ldots, \bigstar, 0, \bigstar, 0$ are square; $B_{2\tau+1}$ is nonsingular; and each of $B_2, \ldots, B_{2\tau}$ has linearly independent rows. The integers $\tau, m_1, \ldots, m_{2\tau}$ are the same as those in (16) and (17). The *congruence class of $B_{2\tau+1}$ is the same *as that of* $A_{(\tau)}$. *In the principal submatrix of* (18) *obtained by deleting the block row and column containing* $B_{2\tau+1}$, *replacing all* ★ *blocks with zero blocks and replacing each* B_i *with* $[I\ 0]$ *produces the matrix* N *in* (16).

(ii) *If* $\mathbb{F} = \mathbb{C}$ *and the involution is the identity, the complex* T*-congruence class of* $B_{2\tau+1}$ *is the same as that of* $A_{(\tau)}$, *and all the other statements in* (i) *are correct.*

(iii) *If* $\mathbb{F} = \mathbb{R}$ *and the involution is the identity, there is a real orthogonal* Q *such that* $Q^T AQ$ *has the form* (18), *the real* T*-congruence class of* $B_{2\tau+1}$ *is the same as that of* $A_{(\tau)}$, *and all the other statements in* (i) *are correct.*

Proof. The *congruence invariance of the parameters m_i and τ, as well as the *congruence class of $A_{(\tau)}$ have already been established. The form of N is the outcome of repeating the reduction described in Lemma 5 until it terminates with a block $A_{(\tau)}$ that is nonsingular. The only issue is the explicit description of the Jordan block structure in (17).

Notice that

$$\underbrace{[I_{m_k}\ 0]}_{m_{k-1}}\underbrace{[I_{m_{k-2}}\ 0]}_{m_{k-2}} = \underbrace{[I_{m_k}\ 0]}_{m_{k-2}}$$

and hence

$$N^2 = \begin{bmatrix} 0_{m_{2\tau}} & 0 & [I_{m_{2\tau}}\ 0] & & & \\ & 0_{m_{2\tau-1}} & 0 & [I_{m_{2\tau-1}}\ 0] & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0_{m_3} & 0 & [I_{m_3}\ 0] \\ & & & & 0_{m_2} & 0 \\ & & & & & 0_{m_1} \end{bmatrix}$$

has all of its nonzero entries in the *second* block superdiagonal. In general, N^k is a 0-1 matrix that has its nonzero entries in the blocks $[I_{m_{2\tau}}\ 0], \dots, [I_{m_{k+1}}\ 0]$ in the kth block superdiagonal. The structure of the powers N^k ensures that the rank of each is equal to the number of its nonzero entries, so

$$\operatorname{rank} N^k = m_{k+1} + \cdots + m_{2\tau}, \quad \kappa = 1, \dots, 2\tau - 1 \tag{19}$$

and $N^{2\tau} = 0$. The list of multiplicities of the nilpotent Jordan blocks in the Jordan Canonical Form of N (arranged in order of increasing size) is given by the sequence of second differences of the sequence $\{\operatorname{rank} N^k\}_{k=1}^{2\tau}$ [4, Exercise, p. 127], which is $m_1 - m_2, m_2 - m_3, m_3 - m_4$, etc. The direct sum of nilpotent Jordan blocks in (17) is therefore the Jordan Canonical Form of N.

The final step in proving (17) is to show that the Jordan Canonical Form of N can be achieved via a permutation similarity, which is a *congruence. A conceptual way to do this is to show that the directed graphs of the two matrices M and N are isomorphic.

The directed graph of J_k is a linear chain with k nodes $P_1, \ldots, P_k$ in which there is an arc from P_i to P_{i+1} for each $i = 1, \ldots, k-1$, so the directed graph of M is a disjoint union of such linear chains. There are $m_k - m_{k+1}$ chains with k nodes for each $k = 1, \ldots, 2\tau$.

To understand the directed graph of N one can begin with any node corresponding to any row in the first block row. Each of these $m_{2\tau}$ nodes is the first in a linear chain with 2τ nodes. In the second block row, the nodes corresponding to the first $m_{2\tau}$ rows are members of the linear chains associated with the first block row, but the nodes corresponding to the last $m_{2\tau-1} - m_{2\tau}$ rows begin their own linear chains, each with $2\tau - 1$ nodes. Proceeding in this way downward through the block rows of N we identify a set of disjoint linear chains that is identical to the set of disjoint linear chains associated with M. A permutation of labels of nodes that identifies the directed graphs of M and N gives a permutation matrix that achieves the desired permutation similarity between M and N.

The uniqueness assertion follows from our identification of all the relevant parameters as *congruence invariants of A and from the uniqueness of the Jordan Canonical Form.

Finally, the assertions about the unitarily reduced form (18) follow from the regularizing algorithm in Section 2 and the proof of Theorem 2. When the regularizing algorithm is carried out with unitary transformations, the result is a matrix of the form (18), of which (8) is a special case. □

5. Regularization of a *selfadjoint pencil

Theorem 6 implies that every *selfadjoint matrix pencil $A + \lambda A^*$ has a regularizing decomposition (2) with a *selfadjoint regular part. The algorithm in Section 2 can be used to construct the regularizing decomposition, and if $\mathbb{F} = \mathbb{C}$ with either the identity or complex conjugation as the involution (respectively, $\mathbb{F} = \mathbb{R}$ with the identity involution), the construction can be carried out using only unitary (respectively, real orthogonal) transformations. We emphasize that the involution on $\mathbb{F}$ may be the identity, so the assertions in the following theorem are valid for matrix pencils of the form $A + \lambda A^T$.

Theorem 7. *Let $A + \lambda A^*$ be a *selfadjoint matrix pencil over $\mathbb{F}$ and let A be *congruent to $A_{(\tau)} \oplus M$, in which $A_{(\tau)}$ is nonsingular and M is the direct sum of nilpotent Jordan blocks in* (17). *Then there is a nonsingular S such that $S(A + \lambda A^*)S^* = (A_{(\tau)} + \lambda A_{(\tau)}^*) \oplus K$ and*

$$K = (J_1 + \lambda J_1^T)^{[m_1 - m_2]} \oplus (J_2 + \lambda J_2^T)^{[m_2 - m_3]} \oplus \cdots \oplus (J_{2\tau} + \lambda J_{2\tau}^T)^{[m_{2\tau}]}.$$

Moreover, each singular block $J_k + \lambda J_k^T$ *may be replaced by*

$$\begin{cases} (F_\ell + \lambda G_\ell) \oplus (G_\ell^T + \lambda F_\ell^T) & \text{if } k = 2\ell - 1 \text{ is odd,} \\ (J_\ell + \lambda I_\ell) \oplus (I_\ell + \lambda J_\ell) & \text{if } k = 2\ell \text{ is even.} \end{cases} \tag{20}$$

Use of the blocks (20) instead of the corresponding Jordan blocks is justified by the following lemma.

Lemma 8. $J_k + \lambda J_k^T$ *is strictly equivalent to* (20).

Proof. If there is a permutation matrix S such that

$$SJ_kS^T = \begin{cases} M_\ell := \begin{bmatrix} 0 & G_\ell^T \\ F_\ell & 0 \end{bmatrix} & \text{if } k = 2\ell - 1 \text{ is odd,} \\ \\ N_\ell := \begin{bmatrix} 0 & I_\ell \\ J_m & 0 \end{bmatrix} & \text{if } k = 2\ell \text{ is even,} \end{cases}$$

then $S(J_k + \lambda J_k^T)S^T$ is strictly equivalent to (20). To prove the existence of such an S, we need to prove that M_ℓ and N_ℓ can be obtained from J_k by simultaneous permutations of rows and columns; that is, there exists a permutation f on $\{1, 2, \dots, k\}$ that transforms the positions

$$(1, 2), (2, 3), \dots, (k - 1, k)$$

of the unit entries in J_k to the positions

$$(f(1), f(2)), (f(2), f(3)), \dots, (f(k - 1), f(k)) \tag{21}$$

of the unit entries in M_ℓ if $k = 2\ell - 1$ or in N_ℓ if $k = 2\ell$. To obtain the sequence (21), we arrange the indices of the units in

$$M_\ell = \left[\begin{array}{cccc|cccc} & & & & 0 & & & 0 \\ & & 0 & & 1 & \ddots & & \\ & & & & & & \ddots & 0 \\ & & & & 0 & & & 1 \\ \hline 1 & 0 & & 0 & & & & \\ & \ddots & \ddots & & & 0 & & \\ 0 & & 1 & 0 & & & & \end{array}\right] \quad ((2\ell - 1) \times (2\ell - 1))$$

as follows:

$$(\ell, 2\ell - 1), (2\ell - 1, \ell - 1), (\ell - 1, 2\ell - 2), (2\ell - 2, \ell - 2),$$
$$\dots, (2, \ell + 1), (\ell + 1, 1),$$

and the indices of the units in N_k as follows:

$$(1, \ell + 1), (\ell + 1, 2), (2, \ell + 2), (\ell + 2, 3), \dots, (2\ell - 1, \ell), (\ell, 2\ell). \quad \square$$

References

[1] D.Ž. Đoković, F. Szechtman, K. Zhao, An algorithm that carries a square matrix into its transpose by an involutory congruence transformation, Electron. J. Linear Algebra 10 (2003) 320–340.

[2] P. Gabriel, Appendix: degenerate bilinear forms, J. Algebra 31 (1974) 67–72.

[3] F.R. Gantmacher, The Theory of Matrices, Chelsea, New York, 2000.

[4] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, New York, 1987.

[5] R.A. Horn, V.V. Sergeichuk, Congruence of a square matrix and its transpose, Linear Algebra Appl. 389 (2004) 347–353.

[6] C. Riehm, M. Shrader-Frechette, The equivalence of sesquilinear forms, J. Algebra 42 (1976) 495–530.

[7] V.V. Sergeichuk, Classification problems for system of forms and linear mappings, Math. USSR, Izvestiya 31 (3) (1988) 481–501.

[8] V.V. Sergeichuk, Computation of canonical matrices for chains and cycles of linear mappings, Linear Algebra Appl. 376 (2004) 235–263.

[9] P. Van Dooren, The computation of Kronecker's canonical form of a singular pencil, Linear Algebra Appl. 27 (1979) 103–140.