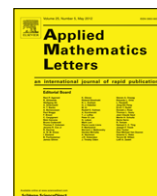




Contents lists available at SciVerse ScienceDirect

## Applied Mathematics Letters

journal homepage: [www.elsevier.com/locate/aml](http://www.elsevier.com/locate/aml)

# Mixing properties for nonautonomous linear dynamics and invariant sets

M. Murillo-Arcila<sup>a</sup>, A. Peris<sup>b,\*</sup><sup>a</sup> IUMPA, Universitat Politècnica de València, Edifici 8G, 46022 València, Spain<sup>b</sup> IUMPA, Universitat Politècnica de València, Departament de Matemàtica Aplicada, Edifici 7A, 46022 València, Spain

## ARTICLE INFO

## Article history:

Received 13 May 2012

Received in revised form 22 August 2012

Accepted 22 August 2012

## Keywords:

Nonautonomous discrete systems

Linear dynamics

Mixing properties

Hypercyclic operators

## ABSTRACT

We study mixing properties (topological mixing and weak mixing of arbitrary order) for nonautonomous linear dynamical systems that are induced by the corresponding dynamics on certain invariant sets. The kinds of nonautonomous systems considered here can be defined using a sequence  $(T_i)_{i \in \mathbb{N}}$  of linear operators  $T_i : X \rightarrow X$  on a topological vector space  $X$  such that there is an invariant set  $Y$  for which the dynamics restricted to  $Y$  satisfies a certain mixing property. We then obtain the corresponding mixing property on the closed linear span of  $Y$ . We also prove that the class of nonautonomous linear dynamical systems that are weakly mixing of order  $n$  contains strictly the corresponding class with the weak mixing property of order  $n + 1$ .

© 2012 Elsevier Ltd. All rights reserved.

## 1. Introduction

Given a sequence of linear and continuous maps (for short, operators)  $T_i : X \rightarrow X$ ,  $i \in \mathbb{N}$ , defined on a topological vector space  $X$  (usually a Banach space or a complete and metrizable space), we consider the corresponding *nonautonomous discrete system* (NDS)  $(X, T_\infty) = (X, (T_n \circ T_{n-1} \cdots \circ T_1)_{n \in \mathbb{N}})$  and we study the behavior of the orbits  $\text{Orb}(x, T_\infty) = \{T^{(k)}x; k \geq 0\}$ ,  $x \in X$ , where  $T^{(k)} := T_k \circ \cdots \circ T_1$ ,  $k \in \mathbb{N}$ ,  $T^{(0)} = \text{Id}_X$ . More precisely, we are interested in topological mixing and weak mixing properties. We say that  $Y \subset X$  is an *invariant set* for the NDS  $(X, T_\infty)$  if  $T_n(Y) \subset Y$  for all  $n \in \mathbb{N}$ . We selected this formulation to be consistent with the concept of the NDS, since then  $(Y, T_\infty|_Y)$ , induced by the sequence  $T_i|_Y$ ,  $i \in \mathbb{N}$ , is also an NDS. The dynamical properties that we are interested in will be obtained on the closure of the linear span of an invariant subset  $Y \subset X$  under general assumptions about the dynamics of  $T_\infty|_Y$ . Also, the class of weakly mixing linear NDS of order  $n$  will be shown to contain strictly the corresponding one with the weak mixing property of order  $n + 1$ .

Chaotic behavior for nonautonomous discrete systems has been studied by several authors [1–5]. Very recently, Balibrea and Oprocha [6] obtained several results concerning weak mixing and chaos in nonautonomous discrete systems on compact sets. Some of their results will be used to consider the corresponding dynamical behavior on linear nonautonomous systems. The theory of linear dynamics is well established in the case of iterations of a single operator (an autonomous dynamical system). We refer the reader to the recent books on the subject [7,8]. The case of nonautonomous linear dynamics is not yet developed, although a more general concept of universality of a sequence of operators  $(T_n)_{n \in \mathbb{N}}$  where the orbits are defined as  $\{T_n x; n \in \mathbb{N}\}$ ,  $x \in X$ , has been treated by several authors (see, e.g., [9–13]).

Discretizations of  $C_0$ -semigroups of operators on Banach spaces provide a natural source of nonautonomous linear systems, and they deserve special attention since linear PDEs and infinite systems of linear ODEs involving the time variable usually yield a  $C_0$ -semigroup of operators as a solution semigroup. Therefore, the asymptotic behavior of the solutions

\* Corresponding author.

E-mail addresses: [mamuar1@posgrado.upv.es](mailto:mamuar1@posgrado.upv.es) (M. Murillo-Arcila), [aperis@mat.upv.es](mailto:aperis@mat.upv.es) (A. Peris).

depends on the behavior of the  $C_0$ -semigroup. We refer the reader to [14–18] for several results concerning chaotic behavior of discretizations of  $C_0$ -semigroups. Most of these results are collected in Section 7.3 of [8]. One type of linear universality which has attracted attention in recent years is the dynamics of tuples of operators introduced by Feldman [19]. More precisely, given a commuting tuple  $(T_1, \dots, T_n)$  of operators defined on a certain topological vector space  $X$ , he studied the existence of (somewhere) dense orbits  $\{(T_n^{k_n} \circ \dots \circ T_1^{k_1})x; k_i \geq 0\}$ . The subsystems that correspond to increasing sequences in  $\mathbb{N}^n$ , with its natural order, can be written as nonautonomous discrete systems.

We will essentially follow the notation of [6]. An NDS  $(X, f_\infty)$  is *weakly mixing* of order  $n$  if, for any nonempty open sets  $U_1, \dots, U_n, V_1, \dots, V_n$  and for any  $N > 0$ , there is  $k > N$  such that  $f^{(k)}(U_i) \cap V_i \neq \emptyset$  for  $i = 1, \dots, n$ . If  $(X, f_\infty)$  is weakly mixing of order  $n$  for every  $n \geq 2$  then we say that it is weakly mixing of all orders.  $(X, f_\infty)$  is said to be *mixing* if for any nonempty open sets  $U, V \subset X$  there exists  $N > 0$  such that  $f^{(k)}(U) \cap V \neq \emptyset$  for all  $k \geq N$ . These notions can be extended naturally to a system  $(X, (f_k)_k)$  of sequences of maps  $f_k : X \rightarrow X, k \in \mathbb{N}$ , by substituting  $f^{(k)}$  by  $f_k$ .

## 2. Mixing properties on linear NDS induced by invariant sets

The purpose of this section is, given a linear NDS  $(X, T_\infty)$ , where  $X$  is a topological vector space, with an invariant set  $Y \subset X$ , to obtain mixing properties on the closure of  $\text{span}(Y)$ , the linear span of  $Y$ , induced by the corresponding ones in  $(Y, T_\infty|_Y)$ . Actually, the main result in this section will be given for sequences of operators, so we will obtain as a consequence the results for linear NDS and for tuples of operators.

**Theorem 1.** *Let  $X$  be a topological vector space and let the system  $(X, (T_n)_n)$ , where  $\{T_n : X \rightarrow X; n \in \mathbb{N}\}$ , be a sequence of operators such that  $T_n(Y) \subset Y$  for every  $n \in \mathbb{N}$  and for a certain  $Y \subset X$  with  $0 \in Y$ . We consider  $Z := \overline{\text{span}(Y)}$ .*

- (1) *If  $(Y, (T_n|_Y)_n)$  is weakly mixing of all orders then  $(Z, (T_n|_Z)_n)$  is also weakly mixing of all orders.*
- (2) *If  $(Y, (T_n|_Y)_n)$  is mixing then  $(Z, (T_n|_Z)_n)$  is also mixing.*

**Proof.** We will show (1). (2) follows from an easier argument.

Suppose then that  $(Y, (T_n|_Y)_n)$  is weakly mixing of all orders. Given any  $m \in \mathbb{N}$ , we have to show that  $(Z, (T_n|_Z)_n)$  is weakly mixing of order  $m$ . Let  $U_j, V_j \subset Z$  be nonempty open sets,  $j = 1, \dots, m$ . We find  $n \in \mathbb{N}, \alpha_{i,j}, \beta_{i,j} \in \mathbb{K}$  and  $u_{i,j}, v_{i,j} \in Y, i = 1, \dots, n, j = 1, \dots, m$ , such that  $u_j := \sum_{i=1}^n \alpha_{i,j} u_{i,j} \in U_j$  and  $v_j := \sum_{i=1}^n \beta_{i,j} v_{i,j} \in V_j, j = 1, \dots, m$ . There are nonempty open sets  $U_{i,j}, V_{i,j}, W \subset Y$  with  $0 \in W, u_{i,j} \in U_{i,j}, v_{i,j} \in V_{i,j}, i = 1, \dots, n, j = 1, \dots, m$ , such that

$$\sum_{i=1}^n \alpha_{i,j} U_{i,j} + \sum_{i=1}^n \gamma_i W \subset U_j, \quad \sum_{i=1}^n \beta_{i,j} V_{i,j} + \sum_{i=1}^n \gamma_i W \subset V_j$$

for any  $\gamma_i \in \{\alpha_{i,j}, i = 1, \dots, n, j = 1, \dots, m\} \cup \{\beta_{i,j}, i = 1, \dots, n, j = 1, \dots, m\}, i = 1, \dots, n, j = 1, \dots, m$ . Since  $(Y, (T_n|_Y)_n)$  is weakly mixing of all orders there are  $y_{i,j} \in U_{i,j}, w_{i,j} \in W$ , and  $k \in \mathbb{N}$  such that  $T_k(y_{i,j}) \in W$  and  $T_k(w_{i,j}) \in V_{i,j}, i = 1, \dots, n, j = 1, \dots, m$ . The above conditions yield that  $y_j := \sum_{i=1}^n (\alpha_{i,j} y_{i,j} + \beta_{i,j} w_{i,j}) \in U_j$  and  $T_k y_j \in V_j, j = 1, \dots, m$ .  $\square$

The result for linear NDS follows now from [Theorem 1](#).

**Corollary 2.** *Let  $X$  be a topological vector space and let  $(X, T_\infty)$  be a linear NDS with an invariant set  $Y \subset X$  such that  $0 \in Y$ . We consider  $Z := \overline{\text{span}(Y)}$ .*

1. *If  $(Y, T_\infty|_Y)$  is weakly mixing of all orders then  $(Z, T_\infty|_Z)$  is also weakly mixing of all orders.*
2. *If  $(Y, T_\infty|_Y)$  is mixing then  $(Z, T_\infty|_Z)$  is also mixing.*

In [19] examples were given of somewhere dense orbits for tuples of operators that are not dense, and sufficient conditions under which a somewhere dense orbit under a tuple of operators must be everywhere dense were obtained. The kinds of sufficient conditions given by Feldman were “operator-theoretic”. We can also derive conditions implying that, when there is a somewhere dense orbit, it must be everywhere dense. We recall that, for linear autonomous systems, no extra assumptions are needed to show that somewhere dense orbits are everywhere dense, as shown by Bourdon and Feldman [20] answering a question in [21] (see also [22] for the corresponding version for  $C_0$ -semigroups).

**Corollary 3.** *Let  $T = (T_1, \dots, T_n)$  be a commuting tuple of operators defined on a topological vector space  $X$ . Let  $x \in X$  be such that  $\text{Orb}(x, T) := \{(T_n^{k_n} \circ \dots \circ T_1^{k_1})x; k_i \geq 0 \text{ for all } i\}$  is somewhere dense in  $X$ . Let  $(R_n)_{n \in \mathbb{N}}$  be an enumeration of  $\{T_n^{k_n} \circ \dots \circ T_1^{k_1}; k_i \geq 0 \text{ for all } i\}$  and let  $Y := \overline{\text{Orb}(x, T)}$ . If  $(Y, (R_n|_Y)_n)$  is weakly mixing of all orders then  $\text{Orb}(x, T)$  is everywhere dense.*

**Proof.** By [Theorem 1](#),  $(X, (R_n)_n)$  is weakly mixing of all orders since  $\text{span}(Y) = X$  because  $Y$  contains a nonempty open set. In particular, given an arbitrary nonempty open set  $V \subset X$  and a nonempty open set  $U \subset \text{Orb}(x, T)$ , there exists  $k \in \mathbb{N}$  such that  $R_k(U) \cap V \neq \emptyset$ . By continuity, we find a nonempty open set  $\tilde{U} \subset U$  such that  $R_k(\tilde{U}) \subset V$ . Let  $j_1, \dots, j_n \geq 0$  be such that  $(T_n^{j_n} \circ \dots \circ T_1^{j_1})x \in \tilde{U}$ , and  $j'_1, \dots, j'_n \geq 0$  with  $R_k = T_n^{j_n} \circ \dots \circ T_1^{j'_1}$ . For  $k_i := j_i + j'_i, i = 1, \dots, n$ , we get  $(T_n^{k_n} \circ \dots \circ T_1^{k_1})x \in V$ , so  $\text{Orb}(x, T)$  is everywhere dense.  $\square$

The following example links the nonlinear dynamics in dimension 1 with the linear infinite-dimensional dynamics. The corresponding autonomous version was obtained in [23]. The idea follows what is called Carleman linearization, and it is inspired in [24].

**Example 4.** Let  $\{p_n : I \rightarrow I; n \in \mathbb{N}\}$  be a sequence of polynomials on an interval  $I$  that contains 0 such that  $p_n(0) = 0, n \in \mathbb{N}$ , and the corresponding generated NDS  $(I, p_\infty)$  is weakly mixing of order 3. By [6, Theorem 11] we know that  $(I, p_\infty)$  is weakly mixing of all orders. We will embed  $(I, p_\infty)$  in a linear NDS  $(X, T_\infty)$  via a map  $\phi$  such that  $T_n \circ \phi = \phi \circ p_n$  for every  $n \in \mathbb{N}$  and  $\text{span}(\phi(I)) = X$ . To do so we set

$$X = \left\{ (x_i)_i \in \mathbb{C}^{\mathbb{N}}; \exists r > 0 \text{ such that } \sup_i |x_i| r^i < \infty \right\}.$$

Actually, we can identify via Taylor expansion at  $0 X = \mathcal{H}(0)$ , the space of holomorphic germs at 0. That is,  $X$  consists of the functions that are (defined and) holomorphic on a neighbourhood of 0.  $X$  is endowed with the natural topology as the inductive limit. We refer the reader to, e.g., [25] for the details.

We define the embedding  $\phi : I \rightarrow X$  as  $\phi(x) = (x, x^2, x^3, \dots)$ . Given  $n \in \mathbb{N}$ , we set the operator  $T_n : X \rightarrow X$  such that the  $k$ th coordinate of  $T_n x$  is

$$T_n(x_1, x_2, \dots)_k = \sum_{j=k}^{km_n} \alpha_{k,j} x_j, \quad k \in \mathbb{N}, x = (x_1, x_2, \dots) \in X,$$

where  $m_n = \text{deg}(p_n)$  and  $p_n(x)^k = \sum_{j=k}^{km_n} \alpha_{k,j} x^j$ . The selection of the sequence space  $X$  easily gives that  $T_n$  is a well-defined operator on  $X$ . Also, a simple computation shows that  $T_n \circ \phi = \phi \circ p_n$ . Let  $Y := \phi(I)$ . We observe that  $\text{span}(Y)$  is dense in  $X$  by the Hahn–Banach theorem. Indeed, since the dual of  $X$  is

$$X' = \left\{ (y_i)_i \in \mathbb{C}^{\mathbb{N}}; \sum_{i=1}^{\infty} |y_i| R^i < \infty \text{ for all } R > 0 \right\},$$

which can be identified with the space of entire functions, we have that  $\langle \phi(x), (y_i)_i \rangle = \sum_i y_i x^i = 0$  for some  $(y_i)_i \in X'$  and for all  $x \in I$  implies that  $y_i = 0$  for every  $i \in \mathbb{N}$  because an entire function that is annihilated on a set with accumulation points should be identically 0. The hypotheses of Corollary 2 are satisfied, and  $(X, T_\infty)$  is weakly mixing of all orders.

### 3. Weak mixing properties of different orders

We will prove that it is possible to obtain examples of linear NDS which show the strict inclusion of the different orders for the weak mixing property. This contrasts with the case for nonautonomous interval maps, where it was shown that there are examples which are weakly mixing of order 2 which are not weakly mixing of order 3 [6, Theorem 9], but once an interval NDS is weakly mixing of order 3, then it follows immediately that it is of arbitrary order  $n \geq 2$  [6, Theorem 11].

**Theorem 5.** *Given any  $n \geq 2$  there is a linear NDS  $(\ell^2, T_\infty)$  defined on the Hilbert space  $\ell^2$  which is weakly mixing of order  $n$ , but it is not weakly mixing of order  $n + 1$ .*

**Proof.** We consider an arbitrary mixing operator on  $\ell^2$  like, for instance, the weighted backward shift  $T := 2B, T(x_1, x_2, \dots) = (2x_2, 2x_3, \dots)$ . Since every mixing map is weakly mixing of all orders, given  $n \in \mathbb{N}$ , let  $(w_1, \dots, w_n) \in \ell^2 \times \dots \times \ell^2$  be a vector whose orbit is dense on  $\ell^2 \times \dots \times \ell^2$  for the operator  $T \times \dots \times T$ . For any  $k \geq 0$ , let  $X_k := \text{span}\{T^k w_1, \dots, T^k w_n\}$  and let  $P_k : \ell^2 \rightarrow X_k$  be the corresponding orthogonal projection. We observe that  $\dim(X_k) = n$  for every  $k \geq 0$  since, otherwise, there would be  $k_0 \geq 0$  such that  $\dim(X_k) \leq \dim(X_{k_0}) < n$  for all  $k \geq k_0$ , which contradicts the fact that  $\{(T^k w_1, \dots, T^k w_n); k \geq k_0\}$  is dense in the  $n$ -product of  $\ell^2$ . We set  $T_1 = P_0$  and  $T_{k+1} = P_k \circ T, k \in \mathbb{N}$ .

The linear NDS  $(\ell^2, T_\infty)$  is clearly not weakly mixing of order  $n + 1$ . Indeed, let  $V_1, \dots, V_{n+1}$  be nonempty open sets of  $\ell^2$  such that any  $n + 1$ -tuple  $(v_1, \dots, v_{n+1}) \in V_1 \times \dots \times V_{n+1}$  is linearly independent. Then, since  $(T_k \circ \dots \circ T_1)(\ell^2)$  is  $n$ -dimensional, it cannot intersect all the  $V_i$ 's,  $i = 1, \dots, n + 1$ , and therefore  $(\ell^2, T_\infty)$  is not weakly mixing of order  $n + 1$ .

On the other hand, given any collection  $U_i, V_i \subset \ell^2$  of nonempty open sets,  $i = 1, \dots, n$ , since  $P_0$  is an orthogonal projection, and thus an open mapping, we find vectors  $u_i \in U_i, i = 1, \dots, n$ , such that

$$\{T_1 u_1, \dots, T_1 u_n\} = \{P_0 u_1, \dots, P_0 u_n\}$$

is linearly independent. Let  $P_0 u_i = \sum_{j=1}^n \alpha_{i,j} w_j, i = 1, \dots, n$ . By definition of the  $T_k$ 's we obtain

$$(T_{k+1} \circ \dots \circ T_1) u_i = \sum_{j=1}^n \alpha_{i,j} T^k w_j, \quad i = 1, \dots, n,$$

for all  $k \in \mathbb{N}$ . We consider the matrix  $A := (\alpha_{i,j})_{i,j}$ , which is invertible since the  $P_0 u_i$ 's are linearly independent. Let  $B = A^{-1} = (\beta_{i,j})_{i,j}$ . We fix  $v_i \in V_i$ ,  $i = 1, \dots, n$ , and a 0-neighbourhood  $W$  such that

$$v_i + \sum_{j=1}^n \alpha_{i,j} W \subset V_i, \quad i = 1, \dots, n.$$

By the selection of the  $w_j$ 's, there is  $k \in \mathbb{N}$  such that  $T^k w_i \in \sum_{j=1}^n \beta_{i,j} v_j + W$ ,  $i = 1, \dots, n$ . Therefore, since  $B = A^{-1}$ , we have

$$T_1^{(k+1)} u_i = \sum_{j=1}^n \alpha_{i,j} T^k w_j \in v_i + \sum_{j=1}^n \alpha_{i,j} W \subset V_i, \quad i = 1, \dots, n,$$

and we conclude that  $(\ell^2, T_\infty)$  is weakly mixing of order  $n$ .  $\square$

**Remark 6.** It is known that a system  $(T_n)_n$  of commuting operators that is weakly mixing of order 2 is necessarily weakly mixing of all orders [10,11]. In particular, sequences of operators generated by commuting tuples of operators [19] and discretizations of  $C_0$ -semigroups of operators, which are weakly mixing of order 2, immediately turn out to be weakly mixing of all orders. This means that there is no hope of finding examples like the one in Theorem 5 within this framework. We do not know whether it is possible to obtain counterexamples of this type for linear NDS that are not artificially constructed.

### Acknowledgments

This work was supported in part by MEC and FEDER, Project MTM2010-14909, and by GV, Project PROMETEO/2008/101. The first author was also supported by a grant from the FPU Program of MEC. We thank the referees whose reports led to an improvement in the presentation of this work.

### References

- [1] J.S. Cánovas, On  $\omega$ -limit sets of non-autonomous discrete systems, *J. Difference Equ. Appl.* 12 (2006) 95–100.
- [2] J.S. Cánovas, Li–Yorke chaos in a class of nonautonomous discrete systems, *J. Difference Equ. Appl.* 17 (2011) 479–486.
- [3] S. Kolyada, L. Snoha, On topological dynamics of sequences of continuous maps, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 5 (1995) 1437–1438.
- [4] S. Kolyada, L. Snoha, Topological entropy of nonautonomous dynamical systems, *Random Comput. Dynam.* 4 (1996) 205–233.
- [5] S. Kolyada, L. Snoha, M. Misiurewicz, Topological entropy of nonautonomous piecewise monotone dynamical systems on the interval, *Fund. Math.* 160 (1999) 161–181.
- [6] F. Balibrea, P. Oprocha, Weak mixing and chaos in nonautonomous discrete systems, *Appl. Math. Lett.* 25 (2012) 1135–1141.
- [7] F. Bayart, É. Matheron, *Dynamics of Linear Operators*, Cambridge University Press, 2009.
- [8] K.-G. Grosse-Erdmann, A. Peris Manguillot, *Linear Chaos*, in: Universitext, Springer, 2011.
- [9] T. Bermúdez, A. Bonilla, A. Peris, On hypercyclicity and supercyclicity criteria, *Bull. Aust. Math. Soc.* 70 (2004) 45–54.
- [10] L. Bernal-González, K.-G. Grosse-Erdmann, The hypercyclicity criterion for sequences of operators, *Studia Math.* 157 (2003) 17–32.
- [11] J. Bès, A. Peris, Hereditarily hypercyclic operators, *J. Funct. Anal.* 167 (1999) 94–112.
- [12] S. Grivaux, Construction of operators with prescribed behaviour, *Arch. Math. (Basel)* 81 (2003) 291–299.
- [13] F. León-Saavedra, V. Müller, Hypercyclic sequences of operators, *Studia Math.* 175 (2006) 1–18.
- [14] F. Bayart, T. Bermúdez, Semigroups of chaotic operators, *Bull. Lond. Math. Soc.* 41 (2009) 823–830.
- [15] T. Bermúdez, A. Bonilla, J.A. Conejero, A. Peris, Hypercyclic, topologically mixing and chaotic semigroups on Banach spaces, *Studia Math.* 170 (2005) 57–75.
- [16] L. Bernal-González, K.-G. Grosse-Erdmann, Existence and nonexistence of hypercyclic semigroups, *Proc. Amer. Math. Soc.* 135 (2007) 755–766.
- [17] J.A. Conejero, A. Peris, Linear transitivity criteria, *Topology Appl.* 153 (2005) 767–773.
- [18] J.A. Conejero, A. Peris, Hypercyclic translation  $C_0$ -semigroups on complex sectors, *Discrete Contin. Dyn. Syst.* 25 (2009) 1195–1208.
- [19] N. Feldman, Hypercyclic tuples of operators and somewhere dense orbits, *J. Math. Anal. Appl.* 346 (2008) 82–98.
- [20] P.S. Bourdon, N.S. Feldman, Somewhere dense orbits are everywhere dense, *Indiana Univ. Math. J.* 52 (2003) 811–819.
- [21] A. Peris, Multi-hypercyclic operators are hypercyclic, *Math. Z.* 236 (2001) 779–786.
- [22] G. Costakis, A. Peris, Hypercyclic semigroups and somewhere dense orbits, *C. R. Math. Acad. Sci. Paris* 335 (2002) 895–898.
- [23] M. Murillo-Arcila, A. Peris, Chaotic behaviour of linear operators on invariant sets, Preprint.
- [24] V. Protopopescu, Linear vs. nonlinear and infinite vs. finite: an interpretation of chaos, Oak Ridge National Laboratory Report, TM-11667, Oak Ridge, Tennessee, USA, 1990.
- [25] S. Dineen, *Complex Analysis on Infinite-Dimensional Spaces*, Springer, London, 1999.