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Regularization by Surgery in the Restricted Three-Body Problem

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1. INTRODUCTION

The restricted problem of three bodies has been already regularized on surfaces of constant Jacobi function by Levi-Civita and Birkhoff. In this paper we prove that collisions in the restricted three-body problem are actually regularizable by surgery, following Easton's methods and ideas. Easton [3, 4], has proved the corresponding assertions for the two-body problem, and for binary collisions in the planar three-body problem. Since regularization by surgery is achieved for all Jacobi levels at once, it gives a more global outlook of the problem.

The idea of regularizing vector fields by surgery consists of excising from the manifold on which the vector field is defined, some neighborhood "isolating" all the integral curves which tend to or come from the singularity. We then identify the endpoints of orbits crossing the neighborhood, and show that this identification has a continuous unique extension, which pairs the endpoints of orbits entering the neighborhood and ending in the singularity, with the endpoints of orbits leaving the neighborhood coming from the singularity.

We will first recall the general ideas about regularization by surgery, define an isolating block for the restricted three-body problem, showing the collisions are regularizable. In the final section we check how the topology of the regularized levels by surgery agrees with the one obtained by direct methods. Birkhoff [1] first described the topology of the regularized levels in this problem.

At the end of this work we also give some properties of the set of initial conditions leading to collision. For example, it is an embedded cylinder on each level surface.

2. ISOLATING BLOCKS AND REGULARIZATION

The kind of neighborhoods which can be conveniently excised to achieve regularization of a vector field by surgery are called isolating blocks, and were first studied by Conley and Easton [2]. They are usually constructed by means

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of some suitable differentiable function generalizing the classical Lyapunov functions, in that its first and second derivates along integral curves must have certain properties (see [7]).

Let M be a C^∞ manifold of dimension n, let S be a closed subset of M, and let X be a C^∞ vector field defined on $M - S$.

If $p \in M - S$ and γ is an integral curve of X satisfying $\gamma(0) = p$, we denote $\gamma(t)$ by $p \cdot t$.

Let $f: M - S \to \mathbb{R}$ be a smooth function and define $\dot{f} = df(X): M - S \to \mathbb{R}$, the function obtained by taking directional derivatives along integral curves of X. Also define $\ddot{f} = \dot{g}$, where $g = \dot{f}$.

DEFINITION 1. We say that f is a *(hyperbolic) Lyapunov function* for X, if whenever $f(p) = 0$ and $\dot{f}(p) = 0$, it is the case that $\ddot{f}(p) > 0$.

This means, roughly speaking, that the only critical points of f along integral curves are minima.

Let B be an n-dimensional smooth submanifold with boundary of M, that is, B is the union of an open subset of M and its boundary $b = \partial B$, where b is a smooth $(n - 1)$ submanifold of M. Assume $S \cap b = \varnothing$. Define the sets:

$$\begin{aligned}
b^+ &= \{p \in b \ : \text{integral curve } p \cdot t \text{ is ingressing to } B\};\\
b^- &= \{p \in b \ : \text{integral curve } p \cdot t \text{ is egressing from } B\};\\
\tau &= \{p \in b \ : X(p) \text{ is tangent to } b\};\\
a^+ &= \{p \in b^+ : p \cdot t \text{ never exits } B \text{ again}\};\\
a^- &= \{p \in b^- : p \cdot t \text{ exits from } B \text{ for the first time}\}.
\end{aligned}$$

Notice that in general $b^+ \cup b^- \cup \tau = b$.

DEFINITION 2. We say that B as above is an *isolating block* (for X) if $b^+ \cap b^- = \tau$, and τ is a codimension 1 submanifold of b. In particular, b^+ and b^- must be submanifolds of b with common boundary τ.

The following property connects the two objects, above:

PROPOSITION 1. *If f is a Lyapunov function for X such that* 0 *is a regular value for f and* $(0, 0)$ *is a regular value for $G = (f, \dot{f}) : M - S \to \mathbb{R}^2$, then $B = \{p \in M - S; f(p) \leqslant 0\}$ is an isolating block for X.*

Proof. From the definition of Lyapunov functions, it is clear that

$$\begin{aligned}
b &= \{p : f(p) = 0\},\\
b^+ &= \{p : f(p) = 0, \dot{f}(p) \leqslant 0\},\\
b^- &= \{p : f(p) = 0, \dot{f}(p) \geqslant 0\},\\
\tau &= \{p : f(p) = 0, \dot{f}(p) = 0\},
\end{aligned}$$

so that $b^+ \cap b^- = \tau$. The fact that b and τ are submanifolds of the right dimension is clear from the hypothesis on f and G. Q.E.D.

The property $b^+ \cap b^- = \tau$ can be interpreted by saying that any trajectory which becomes tangent to $b = \partial B$ must "bounce off" b from outside the block. This is implicit in the definition of Lyapunov function, as Proposition 1 shows.

Define the map Π: $b^+ - a^+ \to b^- - a^-$ by setting $\Pi\ (p) = p \cdot \sigma$, where $\sigma = \sup\{t > 0 : p \cdot t \in B\}$.

It is known [2], that Π is a diffeomorphism.

DEFINITION 3. The singularity S is *regularizable* [4] if there exists an isolating block B of X such that $B \subset M - S$, with the properties: (a) Any integral curve approaching S as time goes to some (right or left) limiting value, must eventually enter and stay in B. (b) The map Π admits a unique extension as a diffeomorphism Π: $b^+ \to b^-$.

DEFINITION 4. Suppose the singularity S is regularizable. We define the *regularized manifold* M' as follows:

Let $\sim$ be the equivalence relation in $M - S - \text{int}\, B$ such that $x \sim y$ if $x = y$, or if $x = \Pi(y)$, or if $y = \Pi(x)$, and then define $M' = (M - S - \text{int}\, B)/\sim$ as a topological space with the standard quotient topology. In fact, M' can be given a C^∞ manifold structure.

DEFINITION 5. Suppose the singularity S is regularizable. Then we define in M' the *regularized* (continuous) *vector field* X', induced from X in $M - S$. Roughly speaking, its integral curves coincide with those of X for points in $M - S - B$, they cross B in zero time via the map Π when hitting points of b, and continue as before along the appropriate integral curves of X.

3. THE RESTRICTED THREE-BODY PROBLEM

As it is well known, the restricted three-body problem consists in studying the motion of a particle of zero mass (planetoid) subject to gravitational attraction from two masses of positive mass (primaries) revolving in circles around each other in the same plane [6].

Using complex coordinates $z = x + iy$ for the plane, the differential equations governing the motion can be written in the form

$$\ddot{z} + 2i\dot{z} = \text{grad}_z\, U = \text{grad}_z\, \Phi, \tag{1}$$

where

$$\begin{aligned} U(z) &= \Phi(z) - (C/2), \\ \Phi(z) &= \tfrac{1}{2}\mid z \mid^2 + \tfrac{1}{2}\mu(1-\mu) + V(z), \qquad 0 < \mu < 1, \end{aligned} \tag{2}$$

and

$$V(z) = [(1-\mu)/\rho_1] + (\mu/\rho_2), \qquad \rho_1 = |z+\mu|, \qquad \rho_2 = |z+\mu-1|, \tag{3}$$

and the grad_z indicates gradient with respect to the variables x, y interpreted as a complex number, keeping C constant. The C denotes a fixed value of the Jacobi integral of motion, defined by

$$J(z, \dot{z}) = 2\Phi(z) - |\dot{z}|^2. \tag{4}$$

Comparing (2) and (4), we see that

$$|\dot{z}|^2 = 2U(z) \tag{5}$$

is satisfied on integral curves where $J = C$.

There are exactly two singularities, corresponding to collision with each one of the primaries: $\rho_1 = 0$, and $\rho_2 = 0$. Since the situation is symmetrical, we will just describe the case $\rho_2 = 0$.

In what follows it will be convenient to use freely the complex and the vector structure of the plane, with its inner and cross product.

To define our isolating block, we start by defining a Lyapunov function:

$$f(z, \dot{z}) = \tfrac{1}{2}\rho_2{}^2 - (\mu^2/50)\,\phi(C)^2, \tag{6}$$

where $C = J(z, \dot{z})$ was found convenient to consider as constant for the purposes of computing $\dot{f}, \ddot{f}$ along integral curves. The ϕ is a sufficiently close C^∞, smoothing out of the following function at the corners:

$$\phi_1(t) = \begin{cases} \frac{1}{5}\,|t|^{-1}, & |t| > 8, \\ \frac{1}{40}, & |t| < 8, \end{cases} \tag{7}$$

in such a way that $0 < \phi \leqslant \phi_1$. Finally, the constant 40 has been chosen to satisfy the estimates of Lemma 2 and Proposition 6, below.

Let $B = \{(z, \dot{z}) \colon z \neq -\mu,\ 1-\mu\ \&\ f(z, \dot{z}) \leqslant 0\}$. That B is an isolating block will be immediate after the following two lemmas.

LEMMA 2. *The f defined by* (6) *is a Lyapunov function for the system* (1).

Proof. Some computation shows that

$$\begin{aligned} \dot{f}(z, \dot{z}) &= \dot{z}\cdot(z+\mu-1), \\ \ddot{f}(z, \dot{z}) &= \ddot{z}\cdot(z+\mu-1) + \dot{z}\cdot\dot{z} \\ &= |\dot{z}|^2 + 2(z+\mu-1)\times\dot{z} + \text{grad}_z\,\Phi\cdot(z+\mu-1), \end{aligned} \tag{8}$$

where we used (1) and the fact that $(ia)\cdot b = a \times b$ for a, b, $i \in \mathbf{C} = \mathbb{R}^2$.

From (3) and considering $\rho_i = (\rho_i^2)^{1/2}$ for $i = 1, 2$, we easily get

$$\text{grad}_z\, \Phi = z - [(1-\mu)/\rho_1^3](z+\mu) - (\mu/\rho_2^3)(z+\mu-1),$$

so that

$$\begin{aligned} \ddot{f}(z, \dot{z}) = {} & |\dot{z}|^2 + 2(z+\mu-1) \\ & \times \dot{z} - V(z) + |z|^2 + (1-\mu)x + [(1-\mu)/\rho_1^3](x+\mu). \end{aligned}$$

On the other hand, from (5), (2), and (3) we have

$$\tfrac{1}{2}|\dot{z}|^2 = V(z) + \tfrac{1}{2}|z|^2 + \tfrac{1}{2}\mu(1-\mu) - (C/2), \tag{9}$$

from which we can eliminate $V(z)$ into the above equation, giving

$$\begin{aligned} \ddot{f}(z, \dot{z}) = {} & \tfrac{1}{2}|\dot{z}|^2 - (C/2) + 2(z+\mu-1) \\ & \times \dot{z} - (1-\mu)x + \tfrac{3}{2}|z|^2 + [(1-\mu)/\rho_1^3](x+\mu). \end{aligned} \tag{10}$$

To check that f is Lyapunov function, it will be enough to prove that if $\dot{f} = 0$, then $\ddot{f} > 0$.

Indeed, in Eq. (10) when $\rho_2 = (\mu\phi/5)$ ($f = 0$) the last two terms are greater than zero, $(1-\mu)\,|x| < 1$, and if we can find positive bounds

$$\tfrac{1}{2}|z|^2 - (C/2) \geqslant K,$$
$$2\rho_2\,|\dot{z}| \leqslant L,$$

such that

$$K - L - 1 \geqslant 0 \tag{11}$$

it will follow $\ddot{f} > 0$ as required.

Since $\rho_2 = \mu\phi/5$, and therefore $\rho_1 \geqslant 1 - (\mu\phi/5) \geqslant 1 - \mu$, we have

$$5/\phi \leqslant V = (5/\phi) + [(1-\mu)/\rho_1] \leqslant (5/\phi) + 1. \tag{12}$$

Using (9) with the remark $|z| \leqslant 1$, $\mu(1-\mu) \leqslant 1$, we get

$$(5/\phi) - (C/2) \leqslant \tfrac{1}{2}|\dot{z}|^2 \leqslant (5/\phi) + 2 - (C/2), \tag{13}$$
$$(5/\phi) - C \leqslant \tfrac{1}{2}|\dot{z}|^2 - (C/2) \leqslant (5/\phi) - C + 2. \tag{14}$$

We consider now two cases, according to the values of C:

(a) If $|C| \geqslant 8$, then $\phi(C) = 1/(5\,|C|)$ and inequalities (13), (14) give

$$\tfrac{1}{2}|\dot{z}|^2 - (C/2) \geqslant 24\,|C| = K,$$
$$|\dot{z}|^2 \leqslant 52\,|C|,$$
$$2\rho_2\,|\dot{z}| \leqslant 2(52)^{1/2}\mu/(25\,|C|^{1/2}) \leqslant 1/|C|^{1/2} = L,$$

and clearly condition (11) is satisfied in this range.

(b) If $|C| < 8$, then $\phi(C) = \frac{1}{40}$ and in this case

$$\tfrac{1}{2}|\dot{z}|^2 - (C/2) \geqslant 200 - C \geqslant 192 = K,$$
$$|\dot{z}|^2 \leqslant 404 - C \leqslant 412,$$
$$2\rho_2 |\dot{z}| \leqslant (412)^{1/2}\mu/100 \leqslant 1 = L,$$

with condition (11) satisfied again.

This completes our proof. Q.E.D.

In the notation of Section 2, we have $b = \{f = 0\}$ and $\tau = \{f = 0, \dot{f} = 0\}$.

Lemma 3. *The sets b, τ are submanifolds of the four-dimensional phase space, with codimensions 1 and 2, respectively.*

Proof. It is enough to prove that f does not have critical points on b, and $G = (f, \dot{f})$ does not have critical points on τ, by Proposition 1.

We first remark that $\dot{z}$ is bounded away from zero in b, since (13) shows $|\dot{z}|^2 \geqslant 392$ for any C. In fact, we can slightly modify the argument to see that the same lower bound holds in B, so that there are no critical points of (1) inside the block, as it should be.

We find $\operatorname{grad} f = (z + \mu - 1 - (2\mu^2/25)\phi\phi' \operatorname{grad}_z \Phi, \ -(2\mu^2/25)\phi\phi'\dot{z})$ and

$$\operatorname{grad} \dot{f} = (\dot{z}, z + \mu - 1),$$

where ϕ' denotes derivative of ϕ with respect to its argument.

By examining the two cases $\phi' = 0$ and $\phi' \neq 0$ we see that $\operatorname{grad} f \neq 0$ since $\rho_2 > 0$. Considering the same two cases and recalling that $\dot{f} = 0$ is equivalent to $\dot{z}$ orthogonal to $z + \mu - 1$, we check that grad f and grad $\dot{f}$ are linearly independent on τ as required. Q.E.D.

Our two lemmas above and Proposition 1 immediately imply the following result.

Theorem 4. *The set B is an isolating block for the singularity $\rho_2 = 0$.*

We can now characterize topologically the block:

Proposition 5. *There is a diffeomorphism $\theta\colon B \to (-1, 0] \times \mathbb{R} \times S^1 \times S^1$, such that*

$$\theta(b) = 0 \times \mathbb{R} \times S^1 \times S^1,$$
$$\theta(b^+) = 0 \times \mathbb{R} \times \{(e^{i\theta}, e^{i\phi}) : e^{i\theta} \cdot e^{i\phi} \leqslant 0\},$$
$$\theta(b^-) = 0 \times \mathbb{R} \times \{(e^{i\theta}, e^{i\phi}) : e^{i\theta} \cdot e^{i\phi} \geqslant 0\},$$
$$\theta(\tau) = 0 \times \mathbb{R} \times \{(e^{i\theta}, e^{i\phi}) : e^{i\theta} \cdot e^{i\phi} = 0\},$$
$$\approx 0 \times \mathbb{R} \times S^0 \times S^1.$$

Proof. Define

$$\theta(z, \dot{z}) = (50f/(\mu^2\phi^2), J, (z + \mu - 1)/\rho_2, \dot{z}/|\dot{z}|).$$

This map is one to one, since a value for J and a value for f determine ρ_2. The direction of the third coordinate function determines z, and (5) together with the last direction determine $\dot{z}$. Moreover, θ is a diffeomorphism, since its Jacobian matrix is equivalent to the following nonsingular matrix:

$$\begin{bmatrix} z+\mu-1 & 0 \\ \operatorname{grad}_z \Phi & -\dot{z} \\ -i(z+\mu-1) & 0 \\ 0 & +i\dot{z} \end{bmatrix}.$$

The other assertions are easily verified with the characterization of the sets in Proposition 1. Q.E.D.

4. Regularization of the Problem

The flow in $\mathbf{C}^2 - (-\mu + i0) \times \mathbf{C} - (1 - \mu + i0) \times \mathbf{C}$ for the restricted three body defines a map Π across the block B (Section 2). We want to show that the singularity $S_2 = (1 - \mu + i0) \times \mathbf{C}$ is regularizable by surgery, which requires the proof that Π has a unique extension from b^+ onto b^-. We will perform the Levi-Civita regularizing transformation of the flow on Jacobi level surfaces [6] to obtain a nice vector field. We extend Π to all of b^+ by using this vector field.

The transformation consists of the map

$$F(w, w') = (1 - \mu + w^2, ww'/(2\,|\,w\,|^2)) = (z, \dot{z}) \tag{15}$$

and a new time variable s satisfying the condition

$$dt/ds = 4\,|\,w\,|^2.$$

The system (1) becomes

$$w'' + 8i\,|\,w\,|^2 w' = \operatorname{grad}_w(4\,|\,w\,|^2 U), \tag{16}$$

where

$$4|w|^2U = 2[(1-\mu)|w^2+1|^2 + \mu\,|w|^4 + 2(1-\mu)/|w^2+1| - 2C]\,|w|^2 + 4\mu \tag{17}$$

and grad_w is interpreted as before, keeping C constant. Notice that the right-hand side of (16) now does depend on C.

Equation (5) now becomes

$$|\,w'\,|^2 = 8\,|\,w\,|^2 U. \tag{18}$$

If we denote by X_z the vector field in phase space corresponding to (1), and by X_w the one corresponding to (16), they are related by the derivative of F on each C Jacobi surface,

$$DF(X_w) = 4 \mid w \mid^2 X_z . \tag{19}$$

From (16), X_w is explicitly defined by the equations:

$$\begin{aligned} dw/ds &= w', \\ dw'/ds &= 4w[(1-\mu)\,\rho_1{}^2 + (2-\mu)\rho_2{}^2] + 8w(1-\mu)[(1+w^2)/\rho_1{}^3] \\ &\quad - 8wC - 8i\rho_2 w' + 8\bar{w}(1-\mu)\rho_2 , \end{aligned} \tag{20}$$

where $\rho_1 = \mid 1 + w^2 \mid$, $\rho_2 = \mid w \mid^2$. Notice that $\rho_2 = 0$ is not a singularity any more, as expected.

Denote by $\tilde{B}$ and $\tilde{B}_c$ the isolation block in w coordinates and its intersection with the C Jacobi level $\{J \circ F = C\}$, respectively, i.e.,

$$\begin{aligned} \tilde{B} &= \{(w, w') : 0 < \mid w \mid^2 \leqslant \mu\phi/5\} = F^{-1}(B), \\ \tilde{B}_c &= \{(w, w') : 0 < \mid w \mid^2 \leqslant \mu\phi(C)/5, \mid w' \mid^2 = 8 \mid w \mid^2 U\}. \end{aligned}$$

Notice that the collision singularity S_2 has been removed, and Eqs. (17) and (18) show that it has been transformed under regularization into the circle

$$\tilde{S}_2 = \{(w, w') : w = 0, \mid w' \mid = (8\mu)^{1/2}\} \tag{21}$$

for each fixed C.

In fact, $\tilde{B}_c$ is an isolating block for X_w on the C Jacobi level, isolating $\tilde{S}_2$. Let $\tilde{\Pi}$ be the map across $\tilde{B}_c$ defined by X_w.

Before going to our main theorem, we need the following proposition, which roughly speaking states that any orbit in $\tilde{B}_c$ must eventually leave it out at its boundary in finite time.

PROPOSITION 6. *Given $(w_0, w_0') \in \tilde{B}_c$ we can find a time $\Sigma > 0$ such that if $\gamma(s) = (w(s), w'(s))$ is the integral curve through (w_0, w_0'), then it is defined in $[0, \Sigma]$ and $\mid w(\Sigma)\mid^2 > \mu\phi(C)/5$.*

For its proof, we need the following technical lemma, giving estimates from (18) and (20).

LEMMA 7. *The velocity and acceleration have the bounds.*

$$\begin{aligned} &7.5\mu \leqslant \mid w' \mid^2 \leqslant 8.5\mu, \\ &\mid w'' \mid \leqslant (20 + 8 \mid C \mid)\,(\mu\phi(C)/5)^{1/2}, \text{ in } \tilde{B}_c . \end{aligned}$$

Proof. For the first estimate, (17) and (18) show that it will be enough to show that

$$M = 4[(1-\mu)\rho_1^2 + \mu\rho_2^2 + 2(1-\mu)/\rho_1 - 2C]\rho_2$$

is bounded in absolute value by $\mu/2$. Since $\rho_2 \leqslant \mu\phi(C)/5$, clearly $\frac{1}{2} \leqslant \rho_1 \leqslant 2$ and using (7), we get

$$8 \mid C \mid \rho_2 \leqslant 8\mu \mid C \mid \phi(C)/5 \leqslant 8\mu/25,$$
$$0 \leqslant [(1-\mu)\rho_1^2 + \mu\rho_2^2 + 2(1-\mu)/\rho_1]\, 4\rho_2 \leqslant \mu/4.$$

Combining them, we have $-8\mu/25 \leqslant M \leqslant \mu/4 + 8\mu/25$, as required.

The estimate on $w'' = dw'/ds$ is easy to get using the above one, and estimating the right-hand side in the second equation of (20). Q.E.D.

Proof of Proposition. In the regularized system integral curves are defined even beyond collision, so that they are continued for any time t, at least as long as they do not reach the other collision outside $\tilde{B}_c$.

It will be enough to prove that if $(w_0, w_0') \in \tilde{B}_c$ and $\sigma = 2\mu^{-1/2} \mid w_0 \mid$, then $\mid w(\sigma)\mid \geqslant 2 \mid w_0 \mid$. Iterating the procedure as many times as necessary shows the proposition. To prove this equivalent claim estimate in absolute value the integral of w'' from 0 to $s(0 < s \leqslant \sigma)$ and apply the Fundamental Theorem of calculus. We get

$$\mid w'(s) - w_0' \mid < \mu^{1/2} \qquad \text{for} \quad 0 < s \leqslant \sigma, \tag{22}$$

using $\mid w_0 \mid^2 \leqslant \mu\phi/5$ and $(20 + 8 \mid C \mid)\,\phi(C)/5 \leqslant 21/50$ for all C.

We may assume w_0' is real and positive by rotating the axis in the w' plane, if necessary, so that (22) implies that w' must be bounded away from the imaginary axis; i.e., $\mathrm{Re}(w') > 3\mu^{1/2}/2$ to meet the lemma estimate $\mid w' \mid \geqslant (7.5\mu)^{1/2}$. Estimating the integral of w' from below and using the fundamental theorem again, we get $\mid w(\sigma)\mid + \mid w_0 \mid \geqslant 3 \mid w_0 \mid$, as required. Q.E.D.

THEOREM 8. *The singularity S_2 of the restricted three-body problem is regularizable by surgery.*

Proof. It is enough to work on the Jacobi levels of regularized coordinates and to check for $\tilde{B}_c$ conditions (a) and (b) of Definition 3. We denote by $\tilde{b}^{\pm}$, τ, $\tilde{a}^{\pm}$ and $b^{\pm}$, τ, $a^{\pm}$ the subsets of $\tilde{B}_c$ defined by X_w, and of $F(\tilde{B}_c) \subset B$ defined by X_z, respectively.

From the definition of $\tilde{B}_c$, it is clear that any orbit going to collision must eventually enter the block and remain in it until collision occurs. The same is true when we get back to unregularized coordinates. This shows (a).

Now, the map $\tilde{\Pi}$ is in fact a diffeomorphism from $\tilde{b}^+$ to $\tilde{b}^-$, since x_w defines a flow in the whole submanifold with boundary $\tilde{B}_c \cup \tilde{S}_2$, which actually crosses it because of Proposition 6 and the fact that any integral curve starting at $\tilde{S}_2$ must immediately leave it (see (20)).

Define $\hat{\Pi}: b^+ \to b^-$ by $\hat{\Pi} = F \circ \tilde{\Pi} \circ F^{-1}$. Since (19) holds, it follows that $\hat{\Pi} = \Pi$ in $b^+ - a^+$, so that $\hat{\Pi}$ is a diffeomorphic extension of Π to b^+. This extension $\hat{\Pi}$ is unique since $\tilde{a}^+$ is in the closure of $\tilde{b}^+ - \tilde{a}^+$, which implies that a^+ is in the closure of $b^+ - a^+$. This completes (b). Q.E.D.

Since we can change roles of the two collisions in the problem by interchanging μ and $1 - \mu$ everywhere, as already remarked, our same arguments show how to construct an isolating block B' disjoint from B, which regularizes the other collision. So, the problem is globally regularized.

5. Regularized Topology by Surgery

Rather than defining the regularized phase as in Definition 4, it turns out to be more interesting to consider the same definition applied to the *noncritical* Jacobi levels, getting the so called regularized levels. Since the essential point here is a variation of the topological arguments in [3], we will just give an informal discussion.

We finish this section by describing the cylinders of orbits leading to collision.

Fix C once and for all. Consider the Jacobi level

$$J_c = \{(z, \dot{z}) : |\dot{z}|^2 = 2U(z),\ z \neq -\mu,\ 1 - \mu\},$$

and the isolating blocks in J_c for each collision:

$$B_1 = B' \cap J_c\,, \qquad B_2 = B \cap J_c\,.$$

The regularized Jacobi level $\hat{J}_c$ may be obtained as

$$\hat{J}_c = [J_c - \operatorname{int}(B_1 \cup B_2)]/\sim, \tag{23}$$

where $\sim$ identifies points at $b_2 = \partial B_2$ via Π, according to Definition 4, and likewise for $b_1 = \partial B_1$.

We will now describe, by way of illustration, the $\hat{J}_c$ in two cases. It is a simple task to check that they agree with the regularized $\hat{J}_c$ obtained via Levi–Civita regularization in the author's forthcoming article [5]. Most of the following topological discussion is more detailed in said paper.

The space $J_c - (B_1 \cup B_2)$ is topologically the same as J_c, and from Eq. (4) it is a pinched circle bundle over $\Phi(z) \geqslant C/2$ in $\mathbf{C} - \{-\mu, 1 - \mu\}$, using Smale's terminology, the pinching occurring over the bounding zero velocity curve $\Phi(z) = C/2$.

Case I. Assume $C \gg 3$. By looking at Eqs. (1), (2), and (3), we would expect that close to either primary the system behaves as a planar two-body problem with negative energy. Here $\Phi \geqslant C/2$ consists of the three connected regions A_0, A_1, A_2 shown in Fig. 1, where the crosses mark collision points. Correspondingly, J_c has three connected components

$$J_c = I_0 \cup I_1 \cup I_2 ,$$

where $I_i = P(A_i)$ denotes pinched circle bundle over A_i.

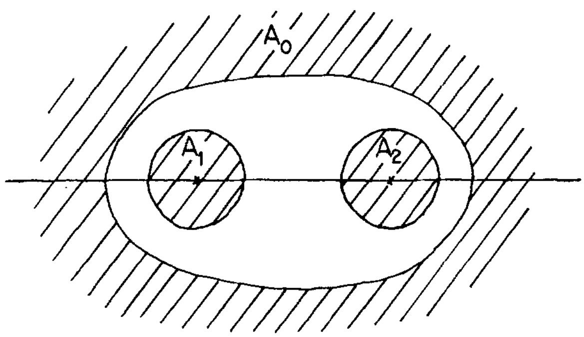

FIGURE 1

Since I_0 is not close to collision, it remains unaffected by regularization, and we easily check it is topologically a solid torus without boundary. Therefore (23) can be written as

$$\hat{J}_c = \hat{I}_1 \cup \hat{I}_2 \cup I_0 ,$$

where $\hat{I}_i = (I_i - \text{int } B_i)/\sim$.

On the other hand, $I_i - \text{int } B_i$ is a solid torus with boundary b_i for $i = 1, 2$, and it is shown in Fig. 2 (see Proposition 5), where the mapping Π is as in Easton [3], so that the quotient gives projective 3 space P^3, as described there. This topology is indeed as in central force problems.

Figure 2 without boundary is a topological representation of I_i before regularizing, and the whole effect of regularization may be thought of as a transformation of the boundary from a torus b_i into projective space P^2, represented as a sphere

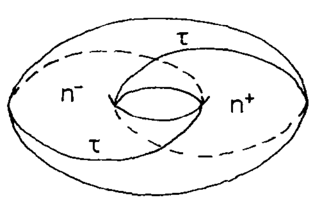

FIGURE 2

S^2 with antipodal points identified. Essentially this consideration helps to find the regularized topology in any other case.

Case II. Assume now $C < 3$. In this case $\Phi \geqslant C/2$ gives no restriction, the region in the plane looks like Fig. 3, and J_c has one connected component, which in fact is topologically the Cartesian product of the circle S^1, times the open region shown in Fig. 4 (no pinching in this case).

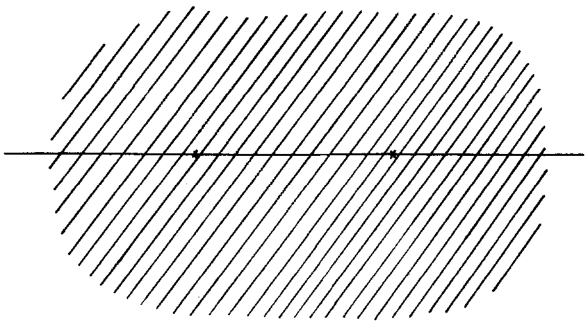

FIGURE 3

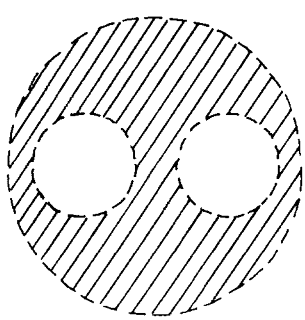

FIGURE 4

The inner tori correspond to points approaching collision, and for surgery regularization we have to replace their boundary tori b_1, b_2, and make the convenient identification on each one, as above.

Before identifying, we can give a homeomorphism inverting the inner torus b_1 with the outer one, by cutting and pasting. This gives something like Fig. 2, except that we have to drill the two concentric tori in it. By the remark at the end of Case I, surgery identification produces a closed 3 ball minus two unlinked solid tori. Finally, another inversion takes the S^2 boundary to the inside, and the torus b_2 to the outside, where we can proceed as before. The resulting $\hat{J}_c$ is described in Fig. 5, where the S^2 components of the "boundary" must be antipodally identified as P^2.

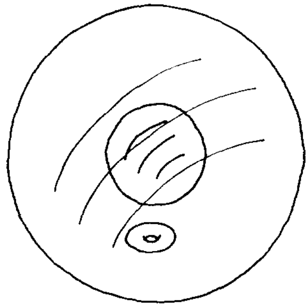

FIGURE 5

This sort of topological inversion [5, Sect. 3], would allow us to find the regularized topology for any other noncritical level, by considerations as above.

We will restrict the following final discussion to one collision, but the same applies to the other one.

Le us consider integral curves passing through collision on J_c. We have already seen that in regularized coordinates, collisions correspond to the circle $\tilde{S}_2$ defined by (22). Equations (20) show that the flow is transversal to $\tilde{S}_2$, hence its image under the flow for any time t will keep being transversal to the flow itself, thereby moving it diffeomorphically and generating an embedded cylinder $E = S^1 \times \mathbb{R}$. By Proposition 6 and the definition of isolating block, the integral curves contained in E do leave B_2 in $b^- - \tau$ and do enter in $b^+ - \tau$ as two disjoint circles. Also, E projects to configuration plane into a neighborhood of the collision.

It might be interesting to investigate how E is globally embedded into the level surface, and whether for varying C we generate something like $S^1 \times \mathbb{R} \times \mathbb{R}$, with C appearing as a parameter in the third coordinate.

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