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Fourier-integral-operator approximation of solutions to first-order hyperbolic pseudodifferential equations II: Microlocal analysis

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Résumé

Un Ansatz approximant l'opérateur solution, $U(z', z)$, d'une équation hyperbolique pseudodifférentielle du premier ordre, $\partial_z + a(z, x, D_x)$, avec $\mathrm{Re}(a) \geqslant 0$, est construit comme composition d'opérateurs intégraux de Fourier globaux à phase complexe. On étudie la propagation des singularités pour cet Ansatz et on montre une convergence microlocale : on démontre que le front d'onde de la solution approchée converge vers celui de la solution loin de la région où la phase est complexe.

Abstract

An approximation Ansatz for the solution operator, $U(z', z)$, of a hyperbolic first-order pseudodifferential equation, $\partial_z + a(z, x, D_x)$ with $\mathrm{Re}(a) \geqslant 0$, is constructed as the composition of global Fourier integral operators with complex phases. We investigate the propagation of singularities for this Ansatz and prove microlocal convergence: the wavefront set of the approximated solution is shown to converge to that of the exact solution away from the region where the phase is complex.

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0. Introduction

We consider the Cauchy problem:

$$\partial_z u + a(z, x, D_x)u = 0, \quad 0 < z \leqslant Z, \tag{0.1}$$

$$u|_{z=0} = u_0, \tag{0.2}$$

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with $Z > 0$ and $a(z, x, \xi) \in \mathcal{C}([0, Z], S^1(\mathbb{R}^n \times \mathbb{R}^n))$, with the usual notation $D = \frac{1}{\mathrm{i}}\partial$. Further assumptions will be made on the symbol $a(z, x, \xi)$. When $a(z, x, \xi)$ is independent of x and z it is natural to treat such a problem by means of Fourier transformation:

$$u(z, x') = \iint \exp\bigl[\mathrm{i}\langle x' - x|\xi\rangle - za(\xi)\bigr]u_0(x)\,đ\xi\,\mathrm{d}x,$$

where $đ\xi := \mathrm{d}\xi/(2\pi)^n$. For this to be well defined for all $u_0 \in \mathcal{S}(\mathbb{R}^n)$ we shall impose the real part of the principal symbol of a to be non-negative. When the symbol a depends on both x and z we can naively expect that

$$u(z, x') \approx u_1(z, x') := \iint \exp\bigl[\mathrm{i}\langle x' - x|\xi\rangle - za(0, x', \xi)\bigr]u_0(x)\,đ\xi\,\mathrm{d}x,$$

for z small, and hence approximately solve the Cauchy problem (0.1)–(0.2) for $z \in [0, z^{(1)}]$ with $z^{(1)}$ small. If we want to progress in the z direction we have to solve the Cauchy problem:

$$\begin{aligned}&\partial_z u + a(z, x, D_x)u = 0, \quad z^{(1)} < z \leqslant Z,\\ &u(z, .)|_{z=z^{(1)}} = u_1\bigl(z^{(1)}, .\bigr),\end{aligned}$$

which we again approximatively solve by

$$u(z, x') \approx u_2(z, x') := \iint \exp\bigl[\mathrm{i}\langle x' - x|\xi\rangle - \bigl(z - z^{(1)}\bigr)a\bigl(z^{(1)}, x', \xi\bigr)\bigr]u_1\bigl(z^{(1)}, x\bigr)\,đ\xi\,\mathrm{d}x.$$

This procedure can be iterated until we reach $z = Z$.

If we denote by $\mathcal{G}_{(z',z)}$ the operator with kernel,

$$G_{(z',z)}(x', x) = \int \exp\bigl[\mathrm{i}\langle x' - x|\xi\rangle\bigr]\exp\bigl[-(z' - z)a(z, x', \xi)\bigr]\,đ\xi,$$

then combining all iteration steps above involves composition of such operators: let $0 \leqslant z^{(1)} \leqslant \cdots \leqslant z^{(k)} \leqslant Z$, we then have:

$$u_{k+1}(z, x) = \mathcal{G}_{(z,z^{(k)})} \circ \mathcal{G}_{(z^{(k)},z^{(k-1)})} \circ \cdots \circ \mathcal{G}_{(z^{(1)},0)}(u_0)(x),$$

if $z \geqslant z^{(k)}$. We then define the operator $\mathcal{W}_{\mathfrak{P},z}$ for a subdivision $\mathfrak{P}$ of $[0, Z]$, $\mathfrak{P} = \{z^{(0)}, z^{(1)}, \ldots, z^{(N)}\}$ with $0 = z^{(0)} < z^{(1)} < \cdots < z^{(N)} = Z$,

$$\mathcal{W}_{\mathfrak{P},z} := \begin{cases}\mathcal{G}_{(z,0)} & \text{if } 0 \leqslant z \leqslant z^{(1)},\\ \mathcal{G}_{(z,z^{(k)})}\prod_{i=k}^{1}\mathcal{G}_{(z^{(i)},z^{(i-1)})} & \text{if } z^{(k)} \leqslant z \leqslant z^{(k+1)}.\end{cases}$$

According to the procedure described above $\mathcal{W}_{\mathfrak{P},z}(u_0)$ yields an approximation Ansatz for the solution to the Cauchy problem (0.1)–(0.2) with step size $\Delta_{\mathfrak{P}} = \sup_{i=1,\ldots,N}(z_i - z_{i-1})$. The operator $\mathcal{G}_{(z',z)}$ is often referred to as the *thin-slab propagator* (see e.g. [3,2]).

The approximation Ansatz proposed here is a tool to compute approximations of the exact solution to the Cauchy problem (0.1)–(0.2). Such computations in application to geophysical problems have been used in [3]. In exploration seismology one is confronted with solving equations of the type,

$$\bigl(\partial_z - \mathrm{i}b(z, x, D_t, D_x) + c(z, x, D_t, D_x)\bigr)v = 0, \tag{0.3}$$

$$v(0, .) = v_0(.), \tag{0.4}$$

where t is time, z is the vertical coordinate and x is the lateral or transverse coordinate; b and c are first-order pseudodifferential operators, with real principal parts b_1 and c_1, where $c_1(z, x, \tau, \xi)$ is non-negative. Note that the Cauchy problem (0.1)–(0.2) studied here is more general. The problem (0.3)–(0.4) is obtained in geophysics by a (microlocal) decoupling of the up-going and down-going wavefields in the acoustic wave equation (see Appendix A in [15] and [21] for details). In practice, the proposed Ansatz can then be a tool to approximate the exact solution for the purpose of imaging the Earth's interior [3,2]. As explained in [15, Appendix A] the operator c acts as a damping term that suppresses singularities in the microlocal region where its symbol does not vanishes. We show that this effect is recovered in the proposed Ansatz.

Seismic imaging aims at recovering the singularities in the subsurface (see for instance [23,1]). Thus, seismologists are not only interested in the convergence of this Ansatz to the exact solution of the Cauchy problem (0.3)–(0.4) but they also expect the wavefront set of the approximate solution to be close, in some sense, to that of the exact solution. Therefore, we investigate the microlocal properties of the proposed Ansatz and show how the results presented here and those of [15] can be applied to seismic imaging.

In the present paper, the operators $\mathcal{G}_{(z',z)}$ and $\mathcal{W}_{\mathfrak{P},z}$ are frequently considered as Fourier integral operators (FIO) with complex phase. They could be considered as FIO with real phase but with amplitude of type $\frac{1}{2}$ (see [15] and below). However, the wavefront set and the damping effect of the real part of the principal part of $a(x,\xi)$ would not be recovered in the same way. We follow here the terminology introduced in [10, Sections 25.4-25.5] for FIOs with complex phases.

We state our main results which are proved in the subsequent sections.

Theorem 1. *Let $z^{(N)} \geqslant z^{(N-1)} \geqslant \cdots \geqslant z^{(0)} \in [0, Z]$. If*

$$\Delta = \max_{0 \leqslant i \leqslant N-1} \bigl(z^{(i+1)} - z^{(i)}\bigr)$$

is sufficiently small then $\mathcal{G}_{(z^{(N)},\dots,z^{(0)})} := \mathcal{G}_{(z^{(N)},z^{(N-1)})} \circ \cdots \circ \mathcal{G}_{(z^{(1)},z^{(0)})}$ is a global Fourier integral operator of order 0. It can be globally parameterized by the non-degenerate phase function of positive type,

$$\begin{aligned}
&\phi_{(z^{(N)},\dots,z^{(0)})}\bigl(x^{(N)}, x^{(0)}, \xi^{(N-1)}, x^{(N-1)}, \dots, \xi^{(1)}, x^{(1)}, \xi^{(0)}\bigr)\\
&\quad := \sum_{i=1}^{N} \phi_{(z^{(i)},z^{(i-1)})}\bigl(x^{(i)}, x^{(i-1)}, \xi^{(i-1)}\bigr)\\
&\quad = \sum_{i=1}^{N} \bigl\langle x^{(i)} - x^{(i-1)} \big| \xi^{(i-1)} \bigr\rangle + \bigl(z^{(i)} - z^{(i-1)}\bigr) a_1\bigl(z^{(i-1)}, x^{(i)}, \xi^{(i-1)}\bigr),
\end{aligned}$$

where a_1 is the principal symbol of a.

Corollary 2. *For $\Delta_{\mathfrak{P}}$ sufficiently small, the operator $\mathcal{W}_{\mathfrak{P},z}$ ($z \in [0, Z]$) is a global Fourier integral operator of order 0 with complex phase.*

In Section 3, we shall denote by χ_z the bicharacteristic flow associated to $-b_1(x,\xi) = \mathrm{Im}(a_1(x,\xi))$.

Theorem 3. *Let $u_0(.) \in H^{(-\infty)}(\mathbb{R}^n)$ and $u(z, .)$, $z \in [0, Z]$, be the solution to the Cauchy problem (0.1)–(0.2). Let $Z' \in [0, Z]$ and K be a compact set in $T^*(\mathbb{R}^n)$ such that for all $\gamma^{(0)} = (x^0, \xi^0) \in K \setminus 0$ the bicharacteristics $\chi_z(\gamma^{(0)})$ associated to $-b_1$ originating from $\gamma^{(0)}$ at $z = 0$ remains away from the region where $c_1 > 0$ for all $z \in [0, Z']$. Then $\gamma^{(0)} \in K \cap \mathrm{WF}(u_0)$ implies $\chi_{Z'}(\gamma^{(0)}) \in \mathrm{WF}(u(Z', .))$. For a subdivision $\mathfrak{P}$ of $[0, Z]$, with $\Delta_{\mathfrak{P}}$ sufficiently small, we then have*:

$$\mathrm{dist}\bigl(\chi_z\bigl(\gamma^{(0)}\bigr), \mathrm{WF}\bigl(\mathcal{W}_{\mathfrak{P},z}(u_0)\bigr)\bigr) \to 0, \quad \text{as } \Delta_{\mathfrak{P}} \to 0,$$

uniformly w.r.t. $\gamma^{(0)} \in K \cap \mathrm{WF}(u_0)$ and $z \in [0, Z']$. Furthermore, the convergence is of order α, $0 < \alpha \leqslant 1$, if $b(z, .)$ is in $\mathcal{C}^{0,\alpha}([0, Z], S^1(\mathbb{R}^n, \mathbb{R}^n))$, in the sense that

$$b(z', x, \xi) - b(z, x, \xi) = (z' - z)^\alpha \tilde{b}(z', z, x, \xi), \quad 0 \leqslant z \leqslant z' \leqslant Z,$$

where $\tilde{b}(z', z, x, \xi)$ is bounded w.r.t. z' and z with values in $S^1(\mathbb{R}^n \times \mathbb{R}^n)$.

In [15], a different approximation Ansatz, $\widetilde{\mathcal{W}}_{\mathfrak{P},z}$, was introduced for which the convergence rate for the Sobolev norm was improved with less regularity of the symbol $a_z(x,\xi)$ w.r.t. z. Here, we also show that this phenomenon occurs and that the continuity of $a_z(x,\xi)$ w.r.t. z implies the convergence of order 1 of the wavefront set of $\widetilde{\mathcal{W}}_{\mathfrak{P},z}(u_0)$ to that of the solution of the Cauchy problem (0.1)–(0.2) (in the sense given in the previous theorem—see Theorem 3.12).

In Section 1, we briefly recall some of the set-up and assumptions of [15] which will be used here. In Section 2 we present the geometrical properties of the Ansatz $\mathcal{W}_{\mathfrak{P},z}$ and prove that it is a global FIO with complex phase. In Section 3 we show microlocal convergence of $\mathcal{W}_{\mathfrak{P},z}(u_0)$ to the exact solution of the Cauchy problem (0.1)–(0.2). In Section 4 we show how the analysis made in this paper and [15] can be applied to seismic imaging theory via the so-called 'double-square-root' equation. Appendix A is dedicated to some general results on FIOs with complex phases.

In the present paper we shall generally write $X, X', X'', X^{(1)}, \dots, X^{(N)}$ for $\mathbb{R}^n$, according to variables, e.g., $x, x', \dots, x^{(N)}$.

Throughout the paper, we use spaces of global symbols: a function $a \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^p)$ is in $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^p)$, $0 < \rho \leqslant 1$, $0 \leqslant \delta < 1$, if for all multi-indices α, β there exists $C_{\alpha\beta} > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leqslant C_{\alpha\beta}\big(1+|\xi|\big)^{m-\rho|\beta|+\delta|\alpha|}, \quad x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^p.$$

The best possible constants $C_{\alpha\beta}$, i.e.,

$$p_{\alpha\beta}(a) := \sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^p} \big(1+|\xi|\big)^{-m+\rho|\beta|-\delta|\alpha|}\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right|,$$

define seminorms for a Fréchet space structure on $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^p)$. As usual we write $S^m_\rho(\mathbb{R}^n \times \mathbb{R}^p)$ in the case $\rho = 1-\delta$, $\frac{1}{2} \leqslant \rho < 1$, and $S^m(\mathbb{R}^n \times \mathbb{R}^p)$ in the case $\rho = 1$, $\delta = 0$.

1. Assumptions and previous results

The symbol $a(z,x,\xi)$ is assumed to satisfy:

Assumption 1.1.

$$a_z(x,\xi) = a(z,x,\xi) = -\mathrm{i}b(z,x,\xi) + c(z,x,\xi),$$

where $b, c \in \mathcal{C}^0([0,Z], S^1(\mathbb{R}^n \times \mathbb{R}^n))$; b has real principal symbol b_1 and c has non-negative principal symbol c_1. The principal symbols b_1 and c_1 are homogeneous of degree 1 for $|\xi| \geqslant 1$.

We denote by $a_1 = -\mathrm{i}b_1 + c_1$ the principal symbol of a and write $b = b_1 + b_0$ and $c = c_1 + c_0$ with $b_0, c_0 \in \mathcal{C}^0([0,Z], S^0(\mathbb{R}^n \times \mathbb{R}^n))$. Assumption 1.1 ensures that the hypotheses (i)–(iii) of Theorem 23.1.2 in [11] are satisfied. Then there exists a unique solution in $\mathcal{C}^0([0,Z], H^{(s+1)}(\mathbb{R}^n)) \cap \mathcal{C}^1([0,Z], H^{(s)}(\mathbb{R}^n))$ to the Cauchy problem (0.1)–(0.2) if $u_0 \in H^{(s+1)}(\mathbb{R}^n)$.

By Proposition 9.3 in [5, Chapter VI] the family of operators $(a_z)_{z\in[0,Z]}$ generates a strongly continuous evolution system, $U(z',z)$, on the Sobolev space $H^{(s+1)}(\mathbb{R}^n)$, $s \in \mathbb{R}$,

$$U(z'',z') \circ U(z',z) = U(z'',z), \quad Z \geqslant z'' \geqslant z' \geqslant z \geqslant 0,$$

and

$$\begin{aligned} &\partial_z U(z,z_0)u_0 + a(z,x,D_x)U(z,z_0)u_0 = 0, \quad 0 \leqslant z_0 < z \leqslant Z, \\ &U(z_0,z_0)u_0 = u_0 \in H^{(s+1)}\big(\mathbb{R}^n\big), \end{aligned}$$

while $U(z,z_0)u_0 \in H^{(s+1)}(\mathbb{R}^n)$ for all $z \in [z_0,Z]$.

We now recall some results obtained in [15]. Let $z', z \in [0,Z]$ with $z' \geqslant z$ and let $\Delta := z' - z$. Define $\phi_{(z',z)} \in \mathcal{C}^\infty(X' \times X \times \mathbb{R}^n)$ by:

$$\phi_{(z',z)}(x',x,\xi) := \langle x'-x|\xi\rangle + \mathrm{i}\Delta a_1(z,x',\xi) = \langle x'-x|\xi\rangle + \Delta b_1(z,x',\xi) + \mathrm{i}\Delta c_1(z,x',\xi). \tag{1.1}$$

Lemma 1.2. *$\phi_{(z',z)}$ is a non-degenerate complex phase function of positive type (at any point (x_0', x_0, ξ_0) where $\partial_\xi \phi_{(z',z)} = 0$).*

We put

$$g_{(z',z)}(x,\xi) := \exp\bigl[-\Delta a_0(z,x,\xi)\bigr] \in S^0\bigl(X \times \mathbb{R}^n\bigr), \tag{1.2}$$

and define a distribution kernel $G_{(z',z)}(x',x) \in \mathcal{D}'(X' \times X)$ by the oscillatory integral:

$$G_{(z',z)}(x',x) = \int \exp\bigl[\mathrm{i}\langle x'-x|\xi\rangle\bigr]\exp\bigl[-\Delta a(z,x',\xi)\bigr]\,đ\xi = \int \exp\bigl[\mathrm{i}\phi_{(z',z)}(x',x,\xi)\bigr]g_{(z',z)}(x',\xi)\,đ\xi.$$

We denote the associated operator by $\mathcal{G}_{(z',z)}$. (This corresponds to the thin-slab propagator (see e.g. [3,2]).)

Let $J_{(z',z)}$ be the canonical ideal locally generated by the phase function $\phi_{(z',z)}$.

Proposition 1.3. *There exists $\Delta_1 > 0$, such that, for all $z', z \in [0, Z]$, with $z' > z$ and $\Delta = z' - z \leqslant \Delta_1$, the phase function $\phi_{(z',z)}$ globally generates the canonical ideal $J_{(z',z)}$. Alternatively, it is also generated by the functions*:

$$v_{\xi_j}(x',x,\xi',\xi) = \partial_{x'_j}\phi_{(z',z)}(x',x,\xi) - \xi'_j = \xi_j - \xi'_j + \mathrm{i}\Delta\partial_{x_j}a_1(z,x',\xi), \tag{1.3}$$

$$v_{x_j}(x',x,\xi',\xi) = \partial_{\xi_j}\phi_{(z',z)}(x',x,\xi) = x'_j - x_j + \mathrm{i}\Delta\partial_{\xi_j}a_1(z,x',\xi), \tag{1.4}$$

$j = 1, \ldots, n$.

Proposition 1.4. *If $0 \leqslant \Delta = z' - z \leqslant \Delta_1$ then the operator $\mathcal{G}_{(z',z)}$ is a global Fourier integral operator with complex phase and kernel $G_{(z',z)} \in I^0(X' \times X, (J_{(z',z)})', \Omega^{1/2}_{X' \times X})$.*

We denote the half density bundle on $X' \times X$ by $\Omega^{1/2}_{X' \times X}$ and note that $(J_{(z',z)})'$ stands for the twisted canonical ideal, i.e. a Lagrangian ideal (see Section 25.5 in [10]).

Proposition 1.5. *Let $s \in \mathbb{R}$. There exists $\Delta_2 > 0$ such that if $z', z \in [0, Z]$ with $0 \leqslant \Delta := z' - z \leqslant \Delta_2$ then $\mathcal{G}_{(z',z)}$ continuously maps $\mathcal{S}$ into $\mathcal{S}$, $\mathcal{S}'$ into $\mathcal{S}'$, and $H^{(s)}(\mathbb{R}^n)$ into $H^{(s)}(\mathbb{R}^n)$.*

The approximation Ansatz is defined by:

Definition 1.6. Let $\mathfrak{P} = \{z^{(0)}, z^{(1)}, \ldots, z^{(N)}\}$ be a subdivision of $[0, Z]$ with $0 = z^{(0)} < z^{(1)} < \cdots < z^{(N)} = Z$ such that $z^{(i+1)} - z^{(i)} = \Delta_{\mathfrak{P}}$. The operator $\mathcal{W}_{\mathfrak{P},z}$ is defined as

$$\mathcal{W}_{\mathfrak{P},z} := \begin{cases} \mathcal{G}_{(z,0)} & \text{if } 0 \leqslant z \leqslant z^{(1)}, \\ \mathcal{G}_{(z,z^{(k)})}\prod_{i=k}^{1}\mathcal{G}_{(z^{(i)},z^{(i-1)})} & \text{if } z^{(k)} \leqslant z \leqslant z^{(k+1)}. \end{cases}$$

In the sequel we shall need the following lemma [15].

Lemma 1.7. *Consider $H : \mathbb{R}^n \to \mathbb{R}^n$, $H_{(\Delta,z,x',x)}(\xi) = \xi + \Delta h(z,x',x,\xi)$, where h is continuous w.r.t. z with values in $S^1(\mathbb{R}^{2n} \times \mathbb{R}^n)$. If Δ is sufficiently small, uniformly w.r.t. $z \in [0, Z]$, then $H_{(\Delta,z,x',x)}$ is a global diffeomorphism. Furthermore, $\tilde{\xi}(\Delta,z,x',x,\xi) = H^{-1}_{(\Delta,z,x',x)}(\xi)$ is homogeneous of degree 1 in ξ, for $|\xi| \geqslant 1$, continuous w.r.t. z, and $\mathcal{C}^\infty$ w.r.t. Δ with values in $S^1(\mathbb{R}^{2n} \times \mathbb{R}^n)$, when Δ is sufficiently small, i.e.,*

$$\exists \Delta_3 > 0, \quad \tilde{\xi} \in \mathcal{C}^0\bigl([0, Z], \mathcal{C}^\infty\bigl([0, \Delta_3], S^1\bigl(\mathbb{R}^{2n} \times \mathbb{R}^n\bigr)\bigr)\bigr).$$

Recall that the smoothness (or differentiability) of a map with values in a Fréchet space is to be understood in the sense of Definition 40.2 in [25].

In the applications we have in mind, the principal part of the damping term, c_1, will affect only certain parts of phase-space (see Appendix A in [15]). In this paper, where the propagation of singularities is analyzed, we shall make the additional assumption

Assumption 1.8. The open set $\Omega = ([0, Z] \times (T^*(\mathbb{R}^n) \setminus 0)) \setminus \mathrm{supp}(c_1)$ is not empty.

2. Geometrical and FIO properties of $\mathcal{W}_{\mathfrak{P}}$

In this section we investigate the microlocal properties of $\mathcal{W}_{\mathfrak{P}}$. To do so we need to analyze how the product,

$$\mathcal{W}_{\mathfrak{P},z} = \mathcal{G}_{(z,z^{(k)})} \prod_{i=k}^{1} \mathcal{G}_{(z^{(i)},z^{(i-1)})},$$

for $z^{(k)} \leqslant z \leqslant z^{(k+1)}$, $k \geqslant 1$, can be understood as a composition of FIOs and yields in turn an FIO. Let $z', z \in [0, Z]$ with $z' \geqslant z$ and put $\Delta = z' - z$. We recall that the global phase function of $\mathcal{G}_{(z',z)}$ is given by (1.1). As in [10, Sections 25.4 and 25.5], if I is an ideal of complex valued functions on $T^*(\mathbb{R}^n)$, we denote by $I_{\mathbb{R}}$ the subset of $T^*(\mathbb{R}^n)$ where all the functions in I vanish. By Lemma 1.3 the following holds *globally*:

$$\begin{aligned} J_{(z',z)\mathbb{R}} = \big\{ &(x', \partial_{x'}\phi_{(z',z)}(x',x,\xi), x, \xi) \mid \partial_\xi \phi_{(z',z)}(x',x,\xi) = 0, \\ &(x',x,\xi) \in X' \times X \times (\mathbb{R}^n \setminus 0)\big\} \subset T^*(X' \times X) \setminus 0. \end{aligned} \tag{2.1}$$

Remark 2.1.

(i) The phase function $\phi_{(z',z)}$ is homogeneous of degree 1 for $|\xi| \geqslant 1$. With a cut-off function $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\psi(\xi) = 1$ when $|\xi| \leqslant 1$ and $\psi(\xi) = 0$ when $|\xi| \geqslant 2$ we can write $\mathcal{G}_{(z',z)} = \mathcal{G}^{(1)}_{(z',z)} + \mathcal{G}^{(2)}_{(z',z)}$ with respective amplitudes $\psi(\xi) g_{(z',z)}(x',\xi)$ and $(1-\psi(\xi)) g_{(z',z)}(x',\xi)$. We can now assume that $\phi_{(z',z)}$ is homogeneous of degree 1 in the expression of the kernel of $\mathcal{G}^{(2)}_{(z',z)}$ and $\mathcal{G}^{(1)}_{(z',z)}$ is a regularizing operator. For the study of the microlocal properties of $\mathcal{G}_{(z',z)}$, and $\mathcal{W}_{\mathfrak{P},z}$, we may thus consider $\mathcal{G}^{(2)}_{(z',z)}$ in place of $\mathcal{G}_{(z',z)}$. Note that $\mathcal{G}^{(2)}_{(z',z)}$ maps $\mathcal{S}$ into $\mathcal{S}$ and $\mathcal{S}'$ into $\mathcal{S}'$, $H^{(s)}(\mathbb{R}^n)$ into $H^{(s)}(\mathbb{R}^n)$, for any $s \in \mathbb{R}$, continuously, as does $\mathcal{G}_{(z',z)}$. In the sequel, we may therefore assume that $\phi_{(z',z)}$, b_1 and c_1 are homogeneous of degree 1 in ξ.

(ii) Observe that the composition of the two FIOs $\mathcal{G}_{(z'',z')}$ and $\mathcal{G}_{(z',z)}$ is natural as operators on $\mathcal{S}$, $\mathcal{S}'$, or $H^{(s)}(\mathbb{R}^n)$, without further requirement such as having the operators properly supported.

(iii) If $\partial_\xi \phi_{(z',z)} = 0$ then $\partial_\xi c_1(z,x',\xi) = 0$. Since c_1 is homogeneous of degree 1 in ξ, Euler's identity then yields $c_1(z,x',\xi) = 0$. Conversely, since $c_1(z,x',\xi)$ is non-negative, $c_1(z,x',\xi) = 0$ implies $\partial_x c_1(z,x',\xi) = 0$ and $\partial_\xi c_1(z,x',\xi) = 0$. Thus if $(x',\xi',x,\xi) \in J_{(z',z)\mathbb{R}}$ then $\partial_x c(z,x',\xi) = 0$ and $\partial_\xi c(z,x',\xi) = 0$ which is equivalent to having $c_1(z,x',\xi) = 0$. Observe that $\partial_{x'}\phi_{(z',z)}(x',x,\xi)$ is thus real in (2.1).

Lemma 2.2. *There exists $\Delta_4 > 0$ such that for all $z', z \in [0, Z]$ with $\Delta = z' - z \in [0, \Delta_4]$ we have $J_{(z',z)\mathbb{R}} \subset T^*(X') \setminus 0 \times T^*(X) \setminus 0$.*

Proof. Let $(x',\xi',x,\xi) \in J_{(z',z)\mathbb{R}}$. Then by Proposition 1.3 we have:

$$\xi - \xi' + \mathrm{i}\Delta \partial_x a_1(z,x',\xi) = 0, \qquad x' - x + \mathrm{i}\Delta \partial_\xi a_1(z,x',\xi) = 0. \tag{2.2}$$

Remark 2.1-(iii) (or only considering the real part in (2.2)) yields

$$\xi - \xi' + \Delta \partial_x b_1(z,x',\xi) = 0, \qquad x' - x + \Delta \partial_\xi b_1(z,x',\xi) = 0.$$

By Lemma 1.7 the map $\xi \mapsto \xi + \Delta \partial_x b_1(z,x',\xi)$ is a global diffeomorphism for Δ sufficiently small and its inverse map is also homogeneous of degree 1. We thus obtain that $\xi = 0 \Leftrightarrow \xi' = 0$. Since $J_{(z',z)\mathbb{R}} \subset T^*(X' \times X) \setminus 0$ the result follows. □

Let $z^{(N)} \geqslant z^{(N-1)} \cdots \geqslant z^{(0)} \in [0, Z]$. We define:

$$\widetilde{J}_{(z^{(N)},\dots,z^{(0)})\mathbb{R}} := J_{(z^{(N)},z^{(N-1)})\mathbb{R}} \circ \cdots \circ J_{(z^{(1)},z^{(0)})\mathbb{R}}. \tag{2.3}$$

By induction on N one proves the following lemma.

Lemma 2.3. *For all $z^{(N)} \geqslant z^{(N-1)} \geqslant \cdots \geqslant z^{(0)} \in [0, Z]$, with $z^{(i+1)} - z^{(i)} \leqslant \Delta_4$, $i = 0, \dots, N-1$, we have*:

$$\widetilde{J}_{(z^{(N)},\dots,z^{(0)})\mathbb{R}} \subset T^*\big(X^{(N)}\big) \setminus 0 \times T^*\big(X^{(0)}\big) \setminus 0.$$

Lemma 2.4. *There exists $\Delta_5 > 0$ such that with $z'' \geqslant z' \geqslant z \in [0, Z]$ the map,*

$$\pi : J_{(z'',z')\mathbb{R}} \times J_{(z',z)\mathbb{R}} \cap T^*(X'') \times \operatorname{diag}\big(T^*(X')\big) \times T^*(X) \to T^*(X'' \times X) \setminus 0,$$
$$(x'', \xi'', x', \xi', x', \xi', x, \xi) \mapsto (x'', \xi'', x, \xi),$$

is injective and proper if $\max(z'' - z', z' - z) \leqslant \Delta_5$.

We write $\operatorname{diag}(T^*(X'))$ for the diagonal of $T^*(X') \times T^*(X')$. Here we give a direct proof of the lemma but it follows in fact from results on the real part of the phase function (see (2.13) and Remark 2.13(i) below) and Proposition 3.13 in [13, Chapter 10].

Proof. Let $\gamma = (x'', \xi'', x, \xi)$ be in the range of π, that is in $J_{(z'',z')\mathbb{R}} \circ J_{(z',z)\mathbb{R}}$. With Lemma 1.3 (use Remark 2.1(iii)) we have:

$$\xi - \xi' + \Delta \partial_x b_1(z, x', \xi) = 0, \tag{2.4}$$
$$x' - x + \Delta \partial_\xi b_1(z, x', \xi) = 0, \tag{2.5}$$
$$\xi' - \xi'' + \Delta' \partial_x b_1(z', x'', \xi') = 0, \tag{2.6}$$
$$x'' - x' + \Delta' \partial_\xi b_1(z', x'', \xi') = 0, \tag{2.7}$$

where $\Delta := z' - z$ and $\Delta' := z'' - z'$. Define $F(\xi') := \xi'' - \Delta' \partial_x b_1(z', x'', \xi')$. It follows that

$$\big|F(\xi') - F(\tilde{\xi}')\big| \leqslant \Delta' \sup\big(\big|\partial_{\xi_i} \partial_{x_j} b_1(z', x'', \xi')\big|\big) |\xi' - \tilde{\xi}'|,$$

where the supremum is taken over $z \in [0, Z]$, $x'' \in \mathbb{R}^n$, $\xi' \in \mathbb{R}^n$ and $1 \leqslant i, j \leqslant n$. As $b_1 \in \mathcal{C}^0([0, Z], S^1(X \times \mathbb{R}^n))$ it follows that $\partial_{\xi_i} \partial_{x_j} b_1(z', x'', \xi')$ is globally bounded. Thus, for Δ' sufficiently small the map F is a contraction and ξ' in (2.6) is uniquely defined by the fixed point theorem. Eq. (2.7) then shows that x' is uniquely defined by the above identities if Δ' is sufficiently small. Hence the map π is injective. (Notice that we only need either Δ or Δ' to be sufficiently small to reach the conclusion; in fact we could form $G(x') = x - \Delta \partial_\xi b_1(z, x', \xi)$ and prove that it is contracting for sufficiently small Δ.)

Let now $K \subset T^*(X'' \times X) \setminus 0$ be a compact set. As $\pi^{-1}(K)$ is closed we just have to prove that it is bounded. Note that the equations above give $x' = x + \Delta \partial_\xi b_1(z, x', \xi)$ and since $\partial_\xi b_1 \in \mathcal{C}^0([0, Z], S^0(X \times \mathbb{R}^n))$, it is globally bounded. Assume then that $\gamma \in K$. Then x stays in a bounded set and so does x'. We also have $\xi' = \xi + \Delta \partial_x b_1(z, x', \xi)$. As x' and ξ stay in a bounded domain so does ξ' by (2.4). Therefore, π is a proper map. □

Lemma 2.5. *There exists $\Delta_6 > 0$ such that if $z'' \geqslant z' \geqslant z \in [0, Z]$ with $z'' - z' \leqslant \Delta_1$ and $z' - z \leqslant \Delta_6$, then*

$$\phi_{(z'',z',z)}(x'', x, \xi', x', \xi) := \phi_{(z'',z')}(x'', x', \xi') + \phi_{(z',z)}(x', x, \xi)$$

is a non-degenerate phase function of positive type in $X'' \times X \times (\mathbb{R}^{3n} \setminus 0)$.

This follows from a more general result which will be of use in the sequel as well:

Lemma 2.6. *There exists $\Delta_6 > 0$ such that if $z^{(N)} \geqslant z^{(N-1)} \geqslant \cdots \geqslant z^{(0)} \in [0, Z]$ with $z^{(i)} - z^{(i-1)} \leqslant \Delta_6$, $i = 1, \ldots, N$, then*

$$\begin{aligned} &\phi_{(z^{(N)},\ldots,z^{(0)})}\big(x^{(N)}, x^{(0)}, \xi^{(N-1)}, x^{(N-1)}, \ldots, \xi^{(1)}, x^{(1)}, \xi^{(0)}\big) \\ &\quad := \sum_{i=1}^{N} \phi_{(z^{(i)},z^{(i-1)})}\big(x^{(i)}, x^{(i-1)}, \xi^{(i-1)}\big) \\ &\quad = \sum_{i=1}^{N} \big\langle x^{(i)} - x^{(i-1)} \big| \xi^{(i-1)} \big\rangle + \big(z^{(i)} - z^{(i-1)}\big) a_1\big(z^{(i-1)}, x^{(i)}, \xi^{(i-1)}\big) \end{aligned} \tag{2.8}$$

is a phase function of positive type in $X^{(N)} \times X^{(0)} \times (\mathbb{R}^{n(2N-1)} \setminus 0)$ which is non-degenerate.

We collect the phase variables of $\phi_{(z^{(N)},\dots,z^{(0)})}$ as

$$\theta_{N-1} := \big(\xi^{(N-1)}, x^{(N-1)}, \dots, \xi^{(1)}, x^{(1)}, \xi^{(0)}\big) \in \mathbb{R}^{n(2N-1)}. \tag{2.9}$$

The function ϕ is homogeneous of degree 1 in

$$\tilde{\theta}_{N-1} := \big(\xi^{(N-1)}, \lambda x^{(N-1)}, \dots, \xi^{(1)}, \lambda x^{(1)}, \xi^{(0)}\big) \in \mathbb{R}^{n(2N-1)} \setminus 0,$$

where $\lambda := |(\xi^{(N-1)}, \dots, \xi^{(0)})|$. Apart from the reasoning immediately below we shall, as usual, omit this scaling. Yet one should keep in mind that $\tilde{\theta}_{N-1}$ is the actual phase variable for $\phi_{(z^{(N)},\dots,z^{(0)})}$. When we write that the phase variable belongs to $(\mathbb{R}^{n(2N-1)}) \setminus 0$ in the statement of the lemma, it is meant in the sense that $\tilde{\theta}_{N-1} \in (\mathbb{R}^{n(2N-1)}) \setminus 0$.

Proof. For simplicity, we write ϕ instead of $\phi_{(z^{(N)},\dots,z^{(0)})}$. Suppose $\mathrm{d}\phi = 0$, then

$$\partial_{x^{(0)}}\phi = \dots = \partial_{x^{(N-1)}}\phi = 0$$

yield $\xi^{(0)} = \dots = \xi^{(N-1)} = 0$ and with the scaling by λ we have $\tilde{\theta}_{N-1} = 0$. Thus $\mathrm{d}\phi \neq 0$ in $X^{(N)} \times X^{(0)} \times (\mathbb{R}^{n(2N-1)} \setminus 0)$. Clearly $\operatorname{Im}\phi \geqslant 0$. It remains to show that the differentials $\mathrm{d}(\partial_{x^{(i)}}\phi)$, $i = 1, \dots, N-1$, and $\mathrm{d}(\partial_{\xi^{(j)}}\phi)$, $j = 0, \dots, N-1$ are linearly independent. We observe that

$$\begin{aligned}
\partial_{x^{(i)}}\phi &= \xi^{(i-1)} - \xi^{(i)} + \mathrm{i}\Delta^{(i-1)}\partial_x a_{z^{(i-1)}}\big(x^{(i)}, \xi^{(i-1)}\big), \quad i = 1, \dots, N-1,\\
\partial_{\xi^{(j)}}\phi &= x^{(j+1)} - x^{(j)} + \mathrm{i}\Delta^{(j)}\partial_\xi a_{z^{(j)}}\big(x^{(j+1)}, \xi^{(j)}\big), \quad j = 0, \dots, N-1,
\end{aligned}$$

where $\Delta^{(i)} := z^{(i+1)} - z^{(i)}$. The structure of the partial differentials $\partial(\partial_{x^{(i)}}\phi)$, $i = 1, \dots, N-1$, and $\partial(\partial_{\xi^{(j)}}\phi)$, $j = 0, \dots, N-1$, w.r.t. $x^{(0)}, \dots, x^{(N)}$ and $\xi^{(0)}, \dots, \xi^{(N-1)}$ can be summarized as follows:

$$\begin{array}{c|ccccccc}
 & \partial(\phi'_{\xi^{(0)}}) & \partial(\phi'_{x^{(1)}}) & \partial(\phi'_{\xi^{(1)}}) & \partial(\phi'_{x^{(2)}}) & \dots & \partial(\phi'_{x^{(N-1)}}) & \partial(\phi'_{\xi^{(N-1)}})\\
\hline
x^{(0)} & -I & 0 & 0 & 0 & \dots & 0 & 0\\
\xi^{(0)} & \square & \blacksquare & 0 & 0 & \dots & 0 & 0\\
x^{(1)} & \square & \square & -I & 0 & \dots & 0 & 0\\
\xi^{(1)} & 0 & -I & \square & \blacksquare & \dots & 0 & 0\\
x^{(2)} & 0 & 0 & \square & \square & \dots & 0 & 0\\
\xi^{(2)} & 0 & 0 & 0 & -I & \dots & 0 & 0\\
\vdots & \vdots & & & & \ddots & & \vdots\\
\xi^{(N-2)} & 0 & 0 & 0 & 0 & \dots & \blacksquare & 0\\
x^{(N-1)} & 0 & 0 & 0 & 0 & \dots & \square & -I\\
\xi^{(N-1)} & 0 & 0 & 0 & 0 & \dots & -I & \square\\
x^{(N)} & 0 & 0 & 0 & 0 & \dots & 0 & \square
\end{array},$$

where $\square$ is some $n \times n$ matrix and $\blacksquare$ is a $n \times n$ matrix of the form $I + \mathrm{i}\Delta^{(j)}\partial_x\partial_\xi a_{z^{(j)}}$ for some $0 \leqslant j \leqslant N-1$. As $\partial_{x_k}\partial_{\xi_l} a_{z^{(j)}} \in S^0(X \times \mathbb{R}^n)$ continuously w.r.t. $z^{(j)}$, it is globally bounded. Thus for $\Delta^{(j)}$ sufficiently small every matrix $\blacksquare$ is invertible. The partial differentials of interest are thus of maximal rank. $\square$

Definition 2.7. For $z'' \geqslant z' \geqslant z \in [0, Z]$ we write $\mathcal{G}_{(z'',z',z)} := \mathcal{G}_{(z'',z')} \circ \mathcal{G}_{(z',z)}$ and more generally for $z^{(N)} \geqslant z^{(N-1)} \geqslant \dots \geqslant z^{(0)} \in [0, Z]$ we write $\mathcal{G}_{(z^{(N)},\dots,z^{(0)})} := \mathcal{G}_{(z^{(N)},z^{(N-1)})} \circ \dots \circ \mathcal{G}_{(z^{(1)},z^{(0)})}$.

Proposition 2.8. *Let $z'' \geqslant z' \geqslant z \in [0, Z]$. The operator $\mathcal{G}_{(z'',z',z)}$ is a global Fourier integral operator if $z'' - z' \leqslant \min(\Delta_4, \Delta_5, \Delta_6)$ and $z' - z \leqslant \Delta_5$. Its kernel $G_{(z'',z',z)}$ is in $I^0(X'' \times X, (J_{(z'',z',z)})', \Omega^{1/2}_{X'' \times X})$ where the canonical ideal is given by $J_{(z'',z',z)} := J_{(z'',z')} \circ J_{(z',z)}$ with transversal composition. $J_{(z'',z',z)}$ is globally parameterized by the non-degenerate phase function of positive type $\phi_{(z'',z',z)}$.*

$I^0(X'' \times X, (J_{(z'',z',z)})', \Omega^{1/2}_{X'' \times X})$ is the set of Lagrangian-distribution half-densities on $X'' \times X$ of order 0 associated to the Lagrangian ideal $(J_{(z'',z',z)})'$ (see [10, Definition 25.4.9]).

Proof. We apply Theorem 25.5.5 in [10] and we use Lemmas 2.2, and 2.4. Lemma 2.5 and Proposition A.4 yield transversal composition for the two canonical ideals $J_{(z'',z')}$ and $J_{(z',z)}$. Observe that $J_{(z'',z',z)\mathbb{R}} = J_{(z'',z')\mathbb{R}} \circ J_{(z',z)\mathbb{R}}$ by Proposition A.3. At every point of $J_{(z'',z')\mathbb{R}} \circ J_{(z',z)\mathbb{R}}$ the non-degenerate phase function $\phi_{(z'',z',z)}$ locally defines $J_{(z'',z')} \circ J_{(z',z)}$ by Proposition 25.5.4 in [10] hence we obtain that $\phi_{(z'',z',z)}$ is a global phase function for $J_{(z'',z',z)}$ and consequently for $\mathcal{G}_{(z'',z',z)}$. The order of $\mathcal{G}_{(z'',z',z)}$ follows since both kernels $G_{(z'',z')}$ and $G_{(z',z)}$ are in I^0. □

Theorem 2.9. *Let $z^{(N)} \geqslant z^{(N-1)} \geqslant \ldots \geqslant z^{(0)} \in [0, Z]$ with $\Delta^{(i)} := z^{(i+1)} - z^{(i)} \leqslant \min(\Delta_4, \Delta_5, \Delta_6)$, for all $i = 0, \ldots, N-1$. Then $\mathcal{G}_{(z^{(N)},\ldots,z^{(0)})}$ is a global Fourier integral operator with complex phase and with distribution kernel*:

$$G_{(z^{(N)},\ldots,z^{(0)})} \in I^0\big(X^{(N)} \times X^{(0)}, \big(J_{(z^{(N)},\ldots,z^{(0)})}\big)', \Omega^{1/2}_{X^{(N)}\times X^{(0)}}\big),$$

where $J_{(z^{(N)},\ldots,z^{(0)})} := J_{(z^{(N)},z^{(N-1)})} \circ \cdots \circ J_{(z^{(1)},z^{(0)})}$ with transversal compositions. $J_{(z^{(N)},\ldots,z^{(0)})}$ is globally parameterized by the non-degenerate phase function of positive type $\phi_{(z^{(N)},\ldots,z^{(0)})}$. We have $J_{(z^{(N)},\ldots,z^{(0)})\mathbb{R}} = J_{(z^{(N)},z^{(N-1)})\mathbb{R}} \circ \cdots \circ J_{(z^{(1)},z^{(0)})\mathbb{R}}$.

Proof. We proceed by induction assuming the result is true for $\mathcal{G}_{(z^{(N)},\ldots,z^{(0)})}$ and $J_{(z^{(N)},\ldots,z^{(0)})}$. By Lemma 2.6 and Proposition A.4 we see that $J_{(z^{(N+1)},z^{(N)})}$ and $J_{(z^{(N)},\ldots,z^{(0)})}$ compose transversally. Lemma 2.3 shows that $J_{(z^{(N+1)},z^{(N)})\mathbb{R}} \subset T^*(X^{(N+1)}) \setminus 0 \times T^*(X^{(N)}) \setminus 0$. In the induction we assume that $J_{(z^{(N)},\ldots,z^{(0)})\mathbb{R}} = \widetilde{J}_{(z^{(N)},\ldots,z^{(0)})\mathbb{R}}$ (see (2.3)) thus $J_{(z^{(N)},\ldots,z^{(0)})\mathbb{R}} \subset T^*(X^{(N)}) \setminus 0 \times T^*(X^{(0)}) \setminus 0$. At this point we claim:

Lemma 2.10. *The map*

$$\pi_N : J_{(z^{(N+1)},z^{(N)})\mathbb{R}} \times J_{(z^{(N)},\ldots,z^{(0)})\mathbb{R}} \cap T^*\big(X^{(N+1)}\big) \times \Delta T^*\big(X^{(N)}\big) \times T^*\big(X^{(0)}\big) \to T^*\big(X^{(N+1)} \times X^{(0)}\big) \setminus 0,$$
$$\big(x^{(N+1)}, \xi^{(N+1)}, x^{(N)}, \xi^{(N)}, x^{(N)}, \xi^{(N)}, x^{(0)}, \xi^{(0)}\big) \mapsto \big(x^{(N+1)}, \xi^{(N+1)}, x^{(0)}, \xi^{(0)}\big)$$

is injective and proper if $\Delta^{(i)} = z^{(i+1)} - z^{(i)} \leqslant \Delta_5$, $i = 0, \ldots, N-1$.

The proof of this lemma can be copied to a large extent from that of Lemma 2.4 (with an induction). (This lemma also follows directly from (2.13) and Remark 2.13(i) below.)

With the above observations we can apply Theorem 25.5.5 in [10], which yields the first part of the result. Now, Lemma A.3 yields,

$$J_{(z^{(N+1)},\ldots,z^{(0)})\mathbb{R}} = J_{(z^{(N+1)},z^{(N)})\mathbb{R}} \circ J_{(z^{(N)},\ldots,z^{(0)})\mathbb{R}} = \widetilde{J}_{(z^{(N+1)},\ldots,z^{(0)})\mathbb{R}},$$

which completes the induction. □

Corollary 2.11. *Let $\mathfrak{P} = \{z^{(0)}, z^{(1)}, \ldots, z^{(N)}\}$ be a subdivision of $[0, Z]$ with $0 = z^{(0)} < z^{(1)} < \cdots < z^{(N)} = Z$ such that $z^{(i+1)} - z^{(i)} \leqslant \Delta_{\mathfrak{P}}$. Let $z \in [0, Z]$. Then the operator $\mathcal{W}_{\mathfrak{P},z}$ given in Definition 1.6 is a global Fourier integral operator of order 0 if $\Delta_{\mathfrak{P}} < \min(\Delta_4, \Delta_5, \Delta_6)$.*

Let $z^{(N)} \geqslant z^{(N-1)} \geqslant \cdots \geqslant z^{(0)} \in [0, Z]$. Note that

$$\big(x^{(N)}, \xi^{(N)}, x^{(0)}, \xi^{(0)}\big) \in J_{(z^{(N)},\ldots,z^{(0)})\mathbb{R}}$$

if and only if there exists $\theta_{N-1} \in \mathbb{R}^{n(2N-1)} \setminus 0$ as defined in (2.9) such that

$$\xi^{(j)} - \xi^{(j+1)} + \Delta^{(j)} \partial_x b_1\big(z^{(j)}, x^{(j+1)}, \xi^{(j)}\big) = 0, \tag{2.10}$$
$$x^{(j+1)} - x^{(j)} + \Delta^{(j)} \partial_\xi b_1\big(z^{(j)}, x^{(j+1)}, \xi^{(j)}\big) = 0, \tag{2.11}$$

for $j = 0, \ldots, N$, and

$$c_1\big(z^{(j)}, x^{(j+1)}, \xi^{(j)}\big) = 0, \quad j = 0, \ldots, N-1 \tag{2.12}$$

(see Remark 2.1(iii)).

Let $z^{(N)} \geqslant z^{(N-1)} \cdots \geqslant z^{(0)} \in [0, Z]$, we define:

$$\mathcal{J}_{(z^{(N)},\dots,z^{(0)})} = \bigl\{\bigl(x^{(N)}, \xi^{(N)}, x^{(0)}, \xi^{(0)}\bigr) \mid \exists \theta_{N-1} \in \mathbb{R}^{n(2N-1)} \setminus 0, \\ \text{as defined in (2.9) such that (2.10)–(2.11) are satisfied}\bigr\}. \tag{2.13}$$

Note that $J_{(z',z)\mathbb{R}} = \mathcal{J}_{(z',z)} \cap \{(x', \xi', x, \xi) \mid c_1(z, x', \xi) = 0\}$ and

$$J_{(z^{(N)},\dots,z^{(0)})\mathbb{R}} = \bigl\{\bigl(x^{(N)}, \xi^{(N)}, x^{(0)}, \xi^{(0)}\bigr) \in \mathcal{J}_{(z^{(N)},\dots,z^{(0)})} \mid \text{for } \theta_{N-1} \in \mathbb{R}^{n(2N-1)} \setminus 0, \text{ defined above} \\ c_1\bigl(z^{(j)}, x^{(j+1)}, \xi^{(j)}\bigr) = 0, \text{ for } j = 0, \dots, N-1\bigr\}. \tag{2.14}$$

Note also that $\mathcal{J}_{(z^{(N)},\dots,z^{(0)})}$ is locally a canonical relation from $T^*(X^{(0)}) \setminus 0$ into $T^*(X^{(N)}) \setminus 0$: simply apply the classical results for real phase functions [4,10] to the non-degenerate phase function $\varphi_{(z^{(N)},\dots,z^{(0)})} = \operatorname{Re} \phi_{(z^{(N)},\dots,z^{(0)})}$, that is,

$$\begin{aligned} &\varphi_{(z^{(N)},\dots,z^{(0)})}\bigl(x^{(N)}, x^{(0)}, \xi^{(N-1)}, x^{(N-1)}, \dots, \xi^{(1)}, x^{(1)}, \xi^{(0)}\bigr) \\ &\quad := \operatorname{Re} \sum_{i=1}^{N} \phi_{(z^{(i)},z^{(i-1)})}\bigl(x^{(i)}, x^{(i-1)}, \xi^{(i-1)}\bigr) \\ &\quad = \sum_{i=1}^{N} \bigl\langle x^{(i)} - x^{(i-1)} \big| \xi^{(i-1)} \bigr\rangle + \bigl(z^{(i)} - z^{(i-1)}\bigr) b_1\bigl(z^{(i-1)}, x^{(i)}, \xi^{(i-1)}\bigr). \end{aligned} \tag{2.15}$$

Proposition 1.3, in the case of a real phase function, yields that $\mathcal{J}_{(z^{(N)},\dots,z^{(0)})}$ is a canonical relation globally defined by $\varphi_{(z^{(N)},\dots,z^{(0)})}$. We can actually say more about $\mathcal{J}_{(z^{(N)},\dots,z^{(0)})}$:

Lemma 2.12. *There exists $\Delta_7 > 0$ such that if $z^{(N)} \geqslant z^{(N-1)} \cdots \geqslant z^{(0)} \in [0, Z]$ with $z^{(i)} - z^{(i-1)} \leqslant \Delta_7$ then $\mathcal{J}_{(z^{(N)},\dots,z^{(0)})}$ is a one-to-one canonical transformation from $T^*(X^{(0)}) \setminus 0$ onto $T^*(X^{(N)}) \setminus 0$.*

Proof. It suffices to prove the result for $\mathcal{J}_{(z',z)}$, with $z' - z$ sufficiently small, as

$$\mathcal{J}_{(z^{(N)},\dots,z^{(0)})} = \mathcal{J}_{(z^{(N)},z^{(N-1)})} \circ \cdots \circ \mathcal{J}_{(z^{(1)},z^{(0)})}.$$

The canonical relation $\mathcal{J}_{(z',z)}$ is *globally* generated by the non-degenerate real phase function $\varphi_{(z',z)} = \langle x' - x | \xi\rangle + (z' - z) b_1(z, x', \xi)$. For Δ sufficiently small we see that $\varphi_{(z',z)} - \langle x | \xi\rangle$ satisfies Definition 1.2 in [13, Chapter 10]. Then Proposition 3.13 in [13, Chapter 10] applies. □

Remark 2.13.

(i) With the results obtained so far we immediately deduce that the projection,

$$\tilde\pi : \mathcal{J}_{(z'',z')} \times \mathcal{J}_{(z',z)} \cap T^*(X'') \times \operatorname{diag}\bigl(T^*(X')\bigr) \times T^*(X) \to T^*(X'' \times X) \setminus 0, \\ (x'', \xi'', x', \xi', x', \xi', x, \xi) \mapsto (x'', \xi'', x, \xi),$$

and also

$$\tilde\pi_N : \mathcal{J}_{(z^{(N+1)},z^{(N)})} \times \mathcal{J}_{(z^{(N)},\dots,z^{(0)})} \cap T^*\bigl(X^{(N+1)}\bigr) \times \Delta T^*\bigl(X^{(N)}\bigr) \times T^*\bigl(X^{(0)}\bigr) \to T^*\bigl(X^{(N+1)} \times X^{(0)}\bigr) \setminus 0, \\ \bigl(x^{(N+1)}, \xi^{(N+1)}, x^{(N)}, \xi^{(N)}, x^{(N)}, \xi^{(N)}, x^{(0)}, \xi^{(0)}\bigr) \mapsto \bigl(x^{(N+1)}, \xi^{(N+1)}, x^{(0)}, \xi^{(0)}\bigr),$$

are injective and proper (see [9, pp. 174–175]). This alternatively yields the results of Lemma 2.4 and Lemma 2.10 as $J_{(z^{(N)},\dots,z^{(0)})\mathbb{R}}$ is closed in $\mathcal{J}_{(z^{(N)},\dots,z^{(0)})}$.

(ii) Since $\mathcal{J}_{(z'',z')}$ and $\mathcal{J}_{(z',z)}$, $z'' \geqslant z' \geqslant z$ are canonical transformations, they compose transversally (see [9, pp. 174–175]). However, this does not apply to $J_{(z'',z')}$ and $J_{(z',z)}$ since their tangent spaces, in the complexification of the tangent space of $T^*(X'' \times X' \setminus 0)$ and $T^*(X' \times X \setminus 0)$ at $\gamma' = (x'', \xi'', x', \xi')$ and $\gamma = (x', \xi', x, \xi)$, may differ from those of $\mathcal{J}_{(z'',z')}$ and $\mathcal{J}_{(z',z)}$. In fact $T_\gamma(J_{(z',z)})$ is defined by $\mathrm{d}v_{\xi_j} = 0$, $\mathrm{d}v_{x_j} = 0$, $j = 1, \dots, n$ by Proposition 1.3, while $T_\gamma(\mathcal{J}_{(z',z)})$ is defined by $\mathrm{d}\tilde v_{\xi_j} = 0$, $\mathrm{d}\tilde v_{x_j} = 0$, $j = 1, \dots, n$ with

$$\tilde{v}_{\xi_j}(x',x,\xi',\xi) = \xi_j - \xi'_j + \mathrm{i}\Delta\partial_{x_j} b_1(z,x',\xi), \quad j=1,\dots,n,$$
$$\tilde{v}_{x_j}(x',x,\xi',\xi) = x'_j - x_j + \mathrm{i}\Delta\partial_{\xi_j} b_1(z,x',\xi), \quad j=1,\dots,n.$$

Note that $\mathcal{J}_{(z^{(N)},\dots,z^{(0)})}(x^{(0)},\xi^{(0)})$ now means the image of $(x^{(0)},\xi^{(0)})$ under the map defined according to Lemma 2.12. In a similar fashion, we shall write $J_{(z^{(N)},\dots,z^{(0)})\mathbb{R}}(x^{(0)},\xi^{(0)})$ as the image, *if it exists*, of $(x^{(0)},\xi^{(0)})$ under the relation $J_{(z^{(N)},\dots,z^{(0)})\mathbb{R}}$.

3. Convergence of the wavefront set of $\mathcal{W}_{\mathfrak{P}}(u_0)$

Consider the Hamilton system (associated to $-b_1 = \mathrm{Im}(a_1)$):

$$\frac{\mathrm{d}x}{\mathrm{d}z} = -\partial_\xi b_1(z,x,\xi), \tag{3.1}$$
$$\frac{\mathrm{d}\xi}{\mathrm{d}z} = \partial_x b_1(z,x,\xi). \tag{3.2}$$

We denote its flow by χ_z: for initial conditions $x(0) = x^{(0)}$, and $\xi(0) = \xi^{(0)}$ we write the solution to this system as $(x(z),\xi(z)) := \chi_z(x^{(0)},\xi^{(0)})$. Observe that Assumption 1.1 ensures unique solutions to system (2.10)–(2.11) for $z \in [0,Z]$.

We note that Eqs. (2.10)–(2.11) form a one-step discrete scheme for this Hamilton system. The scheme is explicit for ξ, while implicit for x. Standard numerical analysis results (see e.g. [6,8]) show that such a scheme converges uniformly[3] w.r.t. initial conditions $(x^{(0)},\xi^{(0)})$ in a compact domain K of $T^*(\mathbb{R}^n)$. The consistency order is then equal to the Hölder exponent of $(-\partial_\xi b_1, \partial_x b_1)$ w.r.t. z. We thus have the following basic convergence result.

Lemma 3.1. *Let $\mathfrak{P} = \{z^{(0)}, z^{(1)},\dots,z^{(N)}\}$ be a subdivision of $[0,Z]$ with $0 = z^{(0)} < z^{(1)} < \cdots < z^{(N)} = Z$ such that $z^{(i+1)} - z^{(i)} \leqslant \Delta_{\mathfrak{P}}$. Let $\varepsilon > 0$ and let K be a compact set in $T^*(\mathbb{R}^n)$. There exists $d > 0$ such that for $\Delta_{\mathfrak{P}} \leqslant d$ and $(x^{(0)},\xi^{(0)}) \in K \setminus 0$, we have*:

$$\left|\left(x^{(j)},\xi^{(j)}\right) - \chi_{z^{(j)}}\left(x^{(0)},\xi^{(0)}\right)\right| \leqslant \varepsilon, \tag{3.3}$$

where $j = 0,\dots,N$ and $(x^{(i)},\xi^{(i)})$, $i = 1,\dots,N$, are solutions to

$$\xi^{(j)} - \xi^{(j+1)} + \Delta^{(j)}\partial_x b_1\left(z^{(j)}, x^{(j+1)}, \xi^{(j)}\right) = 0, \tag{3.4}$$
$$x^{(j+1)} - x^{(j)} + \Delta^{(j)}\partial_\xi b_1\left(z^{(j)}, x^{(j+1)}, \xi^{(j)}\right) = 0. \tag{3.5}$$

Furthermore, if the map

$$z \mapsto \left(-\partial_\xi b_1(z,.,.), \partial_x b_1(z,.,.)\right) \in \mathcal{C}^{0,\alpha}\left([0,Z], \left(\mathcal{C}^\infty\left(\mathbb{R}^n \times \mathbb{R}^n\right)\right)^2\right), \quad 0 < \alpha \leqslant 1,$$

i.e. is Hölder continuous of order α w.r.t. to z, then the convergence rate is of order α.

Remark 3.2.

(i) By homogeneity of b_1 w.r.t. ξ it suffices that the initial condition $x^{(0)}$ stays in a compact domain. Then (3.3) may be replaced by $|x^{(j)} - x(z^{(j)})| \leqslant \varepsilon$, and $|\xi^{(j)} - \xi(z^{(j)})| \leqslant \varepsilon|\xi^{(0)}|$.

(ii) Such a numerical scheme is often referred to as a symplectic Euler method [7]. It exhibits the interesting property of preserving the symplectic form at each step of the integration process hence preserving volume in the cotangent bundle $T^*(\mathbb{R}^n) \setminus 0$.

(iii) The Hölder continuity condition above can be fulfilled by assuming as in [15, Theorem 3.11] that $b(z,.)$ is in $\mathcal{C}^{0,\alpha}([0,Z], S^1(\mathbb{R}^n,\mathbb{R}^n))$, that is,

$$b(z',x,\xi) - b(z,x,\xi) = (z'-z)^\alpha \tilde{b}(z',z,x,\xi), \quad 0 \leqslant z \leqslant z' \leqslant Z,$$

with $\tilde{b}(z',z,x,\xi)$ bounded w.r.t. z' and z with values in $S^1(\mathbb{R}^n \times \mathbb{R}^n)$.

[3] The convergence proofs in [6,8] can be adapted here to be uniform w.r.t. initial conditions varying in a bounded domain.

We now apply the results of [26, Sections XI.1, XI.2] to problem (0.1)–(0.2). In Treves' notation $a^+ = b_1$ and $a^- = c_1$. Theorem 2.2 in [26] yields[4] that singularities at z can only propagate along bicharacteristics associated to $-b_1$ along which c_1 vanishes in the interval $[0, z]$. Let us consider the following example:

Example 3.3. Assume here that $b(z,x,\xi) = 0$ and $c(z,x,\xi) = c_1(z,x,\xi) = |\xi|$ and assume $n = 1$. The Cauchy problem (0.1)–(0.2) then becomes:

$$\partial_z u + |D_x| u = 0, \quad 0 < z \leqslant Z,$$
$$u|_{z=0} = u_0 \in H^{(s+1)}(\mathbb{R}^n).$$

Consider the case $u_0 = \delta_0$, the Dirac measure. We can solve this problem explicitly by mean of a Fourier transformation in x:

$$u(z,x) = \int \exp\bigl[\mathrm{i}x\xi - z|\xi|\bigr]\,đ\xi = \int_0^\infty \exp[\mathrm{i}x\xi - z\xi]\,đ\xi + \int_0^\infty \exp[-\mathrm{i}x\xi - z\xi]\,đ\xi = \frac{1}{2\pi}\left(\frac{1}{z+\mathrm{i}x} - \frac{1}{-z+\mathrm{i}x}\right),$$

which is smooth when $z > 0$. (Note that Example 3.1.13 in [12] yields that the initial condition is indeed satisfied.) In this example, where the support of c_1 is $[0, Z] \times (T^*(\mathbb{R}^n)$ we observe that the initial-value singularity does not propagate.

Thus the region where $c_1 = 0$ must not be too 'small' to allow for propagation of singularities in the solution to the Cauchy problem (0.1)–(0.2). We set:

$$\Omega := \bigl([0,Z] \times \bigl(T^*(\mathbb{R}^n) \setminus 0\bigr)\bigr) \setminus \operatorname{supp}(c_1).$$

The previous example motivates the following basic assumption, which is essential in Lemma 3.5 below:

Assumption 3.4. Ω is not empty.

We shall see that this ensures the propagation of some singularities. Theorem 2.2 in [26] states that singularities only propagate in $\overline{\Omega}$. We can actually make this result more precise.

Lemma 3.5. *Define $\Omega_z := \{(x,\xi) \in T^*(\mathbb{R}^n) \setminus 0 \mid (z,x,\xi) \in \Omega\}$. Let $u(z)$, $z \in [0,Z]$ be the solution of problem* (0.1)–(0.2). *Let $Z' \in [0,Z]$ and assume $\gamma_0 = (x^{(0)}, \xi^{(0)})$ is such that $\chi_z(\gamma_0) \in \Omega_z$ for all $z \in [0,Z']$. Then $\gamma_0 \in \mathrm{WF}(u_0)$ if and only if $\chi_{Z'}(\gamma_0) \in \mathrm{WF}(u(Z',.))$.*

Proof. The proof is along the lines of that of Theorem 23.1.4 in [11]. Suppose $(x^{(0)}, \xi^{(0)}) \in T^*(\mathbb{R}^n) \setminus 0$ and $(x^{(0)}, \xi^{(0)}) \notin \mathrm{WF}(u_0)$. Choose q_0 and $q \sim \sum_j q_j$ polyhomogeneous in $S^0(\mathbb{R}^n \times \mathbb{R}^n)$ such that $q_0(x^{(0)}, \xi^{(0)}) \neq 0$ and q vanish outside a small neighborhood of $(x^{(0)}, \xi^{(0)})$. We can choose $A = \overline{\bigcup_j \operatorname{supp}(q_j)}$ sufficiently small such that

$$\chi_z(A) \cap \operatorname{supp}\bigl(c_1(z,.)\bigr) = \emptyset, \quad z \in [0,Z'].$$

We then define:

$$U = \bigl\{(z,x,\xi) \mid z \in [0,Z'] \text{ and } (x,\xi) = \chi_z(x_A,\xi_A) \text{ for some } (x_A,\xi_A) \in A\bigr\}.$$

We now design an 0-order ψdo, with symbol $Q \sim \sum_{j \geqslant 0} Q_j$ such that $[\partial_z - \mathrm{i}b(z,x,D_x) + c_0(z,x,D_x), Q(z,x,D)]$ is regularizing. The principal part of the commutator is given by $\{\zeta - b_1, Q_0\} = (\partial_z - H_{b_1})Q_0$, where $\{.,.\}$ denotes

[4] In [26] the case where the symbol a is independent of the evolution parameter is addressed. The proof given in Section XI.2 of [26] can however be adapted to the present case.

the Poisson bracket of two functions. H_{b_1} is the Hamiltonian vector field associated to b_1,

$$H_{b_1} = \sum_{1 \leqslant i \leqslant n} (\partial_{\xi_i} b_1)\partial_{x_i} - (\partial_{x_i} b_1)\partial_{\xi_i}.$$

The term of order $-j$ in the commutator, $j \geqslant 1$, is given by $(\partial_z - H_{b_1})Q_j + R_j$ where R_j is determined by $Q_0, \ldots, Q_{j-1}$. We recursively set these terms to zero. This yields $Q_0(z, x, \xi) := q_0(\chi_z^{-1}(x, \xi))$. Then, $\mathrm{supp}(Q_0) \cap \mathrm{supp}(c_1) = \emptyset$. We then find $\mathrm{supp}(R_1) \subset U$. With Q_1 following as

$$Q_1\bigl(z, \chi_z(y, \eta)\bigr) = q_1(y, \eta) - \int_0^z R_1\bigl(s, \chi_s(y, \eta)\bigr)\,\mathrm{d}s,$$

we obtain $\mathrm{supp}(Q_1) \subset U$ (use that if $(z, \chi_z(y, \eta)) \notin U$ then $(s, \chi_s(y, \eta)) \notin U$ for all $s \in [0, z]$). The construction thus gives $\mathrm{supp}(Q) \subset U$ and the operator $Q(z, x, D_x)$ commutes with $\partial_z - \mathrm{i}b(z, x, D_x) + c_0(z, x, D_x)$ up to a regularizing operator. Now $u(z, .)$ satisfies

$$\partial_z u(z, .) + \bigl(c_0(z, x, D_x) - \mathrm{i}b(z, x, D_x)\bigr)u(z, .) = -c_1(z, x, D_x)u(z, .), \quad z \in]0, Z].$$

Observe that $Q(z, x, D_x) \circ c_1(z, x, D_x)$ is a regularizing operator by construction, because of the support of $Q \sim \sum_{j \geqslant 0} Q_j$. We then obtain that

$$\bigl(\partial_z + c_0(z, x, D_x) - \mathrm{i}b(z, x, D_x)\bigr)Q(z, x, D_x)u(z, .) \in \mathcal{C}^\infty\bigl(\mathbb{R}^n\bigr) \tag{3.6}$$

for all $z \in [0, Z']$. As $Q(z, x, D_x)u(0, .)$ is smooth by the choice we made for q, application of Theorem 23.1.2 in [11] for all $s \in \mathbb{R}$ thus proves that $Q(z, x, D_x)u(z, .)$ is smooth for all $z \in [0, Z']$. Since $Q_0(Z', x, \xi) \neq 0$ for $(x, \xi) = \chi_{Z'}(x^{(0)}, \xi^{(0)})$ we have $(x, \xi) \notin \mathrm{WF}(u(Z', .))$. We can now reverse the evolution parameter z in (3.6), since the principal symbol of $c_0(z, x, D_x) - \mathrm{i}b(z, x, D_x))$ is pure imaginary, and the proof is complete. □

Remark 3.6. The proof of Theorem 23.1.4 in [11] makes use of the homogeneity of the principal part b_1. In the Cauchy problem (0.1)–(0.2), b_1 is only homogeneous of degree 1 in ξ for $|\xi| \geqslant 1$. For the wavefront set we are only interested in the direction of ξ. We can thus assume that along the flow χ_z we remain in the region where b_1 is homogeneous by taking ξ^0 sufficiently large.

We now naturally focus on initial conditions $\gamma^{(0)} = (x^{(0)}, \xi^{(0)}) = (x(0), \xi(0))$ such that $\gamma^{(0)} \in \Omega_0$, i.e. away from the support of $c_1(0, ., .)$.

Let $Z' > 0$ and $\gamma^{(0)} \in \Omega_0$ such that $\chi_z(\gamma^{(0)}) \in \Omega_z$ for all $z \in [0, Z']$. Lemma 3.1 shows that there exists $d > 0$ such that for any subdivision $\mathfrak{P}$ of $[0, Z]$, with $\Delta_{\mathfrak{P}} \leqslant d$, if $z^{(k)} \leqslant Z' < z^{(k+1)}$, then $(x^{(j)}, \xi^{(j)}) \in \Omega_{z^{(j)}}$ for $j = 1, \ldots, k$. This can be done uniformly w.r.t. $\gamma^{(0)}$ in a compact domain K of $T^*(\mathbb{R}^n)$. We have thus proved the following convergence result (illustrated in Fig. 1).

Proposition 3.7. *Let K be a compact set in $T^*(\mathbb{R}^n)$, $K \subset \Omega_0$, and $Z' \in [0, Z]$ such that $\chi_z(\gamma^{(0)}) \in \Omega_z$ for all $z \in [0, Z']$ and for all $\gamma^{(0)} \in K \setminus 0$. Let $\varepsilon > 0$ be such that*

$$\varepsilon < \mathrm{dist}\bigl(\bigl\{\bigl(z, \chi_z\bigl(\gamma^{(0)}\bigr)\bigr) \bigm| z \in [0, Z'],\ \gamma^{(0)} \in K \setminus 0\bigr\}, \mathrm{supp}(c_1)\bigr).$$

There exists $d > 0$ such that for any subdivision $\mathfrak{P} = \{z^{(0)}, z^{(1)}, \ldots, z^{(N)}\}$ of $[0, Z]$ with $\Delta_{\mathfrak{P}} \leqslant d$ and all $(x^{(0)}, \xi^{(0)}) \in K \setminus 0$ the following holds:

$$\gamma^{(j)} := \bigl(x^{(j)}, \xi^{(j)}\bigr) = \mathcal{J}_{(z^{(j)}, \ldots, z^{(0)})}\bigl(\gamma^{(0)}\bigr), \quad j = 1, \ldots, N,$$

is such that $\gamma^{(j)} \in \Omega_{z^{(j)}}$ for $j = 1, \ldots, k$, that is $\gamma^{(j)} = J_{(z^{(j)}, \ldots, z^{(0)})\mathbb{R}}(\gamma^{(0)})$ and

$$\bigl|\gamma^{(j)} - \chi_{z^{(j)}}\bigl(\gamma^{(0)}\bigr)\bigr| \leqslant \varepsilon, \quad j = 1, \ldots, k,$$

where k is defined by $z^{(k)} \leqslant Z' < z^{(k+1)}$.

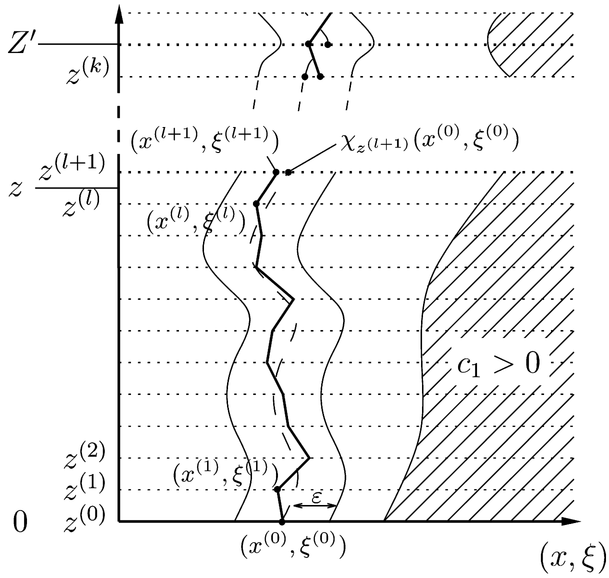

Fig. 1. Convergence of the discrete Hamiltonian flow away from the region where c_1 is positive.

Since $\mathcal{W}_{\mathfrak{P},z}$ is an FIO with complex phase, one can estimate the wave front set of $\mathcal{W}_{\mathfrak{P},z}(u_0)$ if $u_0 \in \mathcal{E}'(\mathbb{R}^n)$,

$$\mathrm{WF}\big(\mathcal{W}_{\mathfrak{P},z}(u_0)\big) \subset J_{(z,z^{(l)},\ldots,z^{(0)})\mathbb{R}}\big(\mathrm{WF}(u_0)\big),$$

when $z^{(l)} \leqslant z \leqslant z^{(l+1)}$. Kumano-go [13, Theorem 3.14, Chapter 10] proves such an estimate of the wavefront set in the case of particular real phase functions for $u_0 \in H^{(-\infty)}(\mathbb{R}^n) = \bigcup_{s\in\mathbb{R}} H^{(s)}(\mathbb{R}^n)$. For $z' - z$ sufficiently small, we can apply Kumano-go's result to the real phase function $\varphi_{(z',z)}$ and obtain that

$$\mathrm{WF}\big(\mathcal{G}_{(z',z)}(u_0)\big) \subset \mathcal{J}_{(z',z)}\big(\mathrm{WF}(u_0)\big),$$

considering $\mathcal{G}_{(z',z)}$ as an FIO with real phase and amplitude of type (ρ,δ) with $\delta = 1-\rho$ and $0 \leqslant \rho \leqslant \frac{1}{2}$ (cf. [15]). By induction, for $\Delta_{\mathfrak{P}}$ sufficiently small, this result applies to the real phase function $\varphi_{(z,z^{(l)},\ldots,z^{(0)})}$ and thus yields propagation of singularities along $\mathcal{J}_{(z,z^{(l)},\ldots,z^{(0)})}$ for the operator $\mathcal{W}_{\mathfrak{P},z}$ for $u_0 \in H^{(-\infty)}(\mathbb{R}^n)$. However, the upper-bound given by map $\mathcal{J}_{(z,z^{(l)},\ldots,z^{(0)})}$ is too large and we would like to obtain a similar result with $J_{(z,z^{(l)},\ldots,z^{(0)})\mathbb{R}}$ instead; that is, considering $\mathcal{W}_{\mathfrak{P},z}$ as an FIO with complex phase. We therefore aim at a result similar to that of Kumano-go, in the case of a complex phase function, which yields the expected propagation of singularities along $J_{(z,z^{(l)},\ldots,z^{(0)})\mathbb{R}}$ for $\mathcal{W}_{\mathfrak{P},z}$. We also allow for u_0 to be in $\mathcal{S}'(\mathbb{R}^n)$ since the considered operators map $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$ for $\Delta_{\mathfrak{P}}$ sufficiently small.

Proposition 3.8. *Let $Z \geqslant z^{(N)} \geqslant z^{(n-1)} \geqslant \cdots \geqslant z^{(0)} \geqslant 0$ with $z^{(j+1)} - z^{(j)} \leqslant \Delta$, $j = 0,\ldots,N-1$. Let $\mathcal{A}_{(z^{(j)},z^{(j-1)})}$, $j = 1,\ldots,N$ be (global) Fourier integral operators with complex phase functions $\phi_{(z^{(j)},z^{(j-1)})}$ (Δ chosen sufficiently small). Then for all $u_0 \in \mathcal{S}'(\mathbb{R}^n)$*

$$\mathrm{WF}\big(\mathcal{A}_{(z^{(N)},z^{(N-1)})} \circ \cdots \circ \mathcal{A}_{(z^{(1)},z^{(0)})}(u_0)\big) \subset J_{(z^{(N)},\ldots,z^{(0)})\mathbb{R}}\big(\mathrm{WF}(u_0)\big).$$

Proof. It suffices to prove the result for $\mathcal{A}_{(z^{(1)},z^{(0)})}$. In the proof we shall alternatively consider $\mathcal{A}_{(z^{(1)},z^{(0)})}$ as an FIO with complex phase and symbol of type $(1,0)$, or as an FIO with real phase and symbol of type (ρ,δ) with $\delta = 1-\rho$ and $0 \leqslant \rho \leqslant \frac{1}{2}$; the values of ρ and δ depends on the symbol $c_1(x,\xi)$ (see [15]).

Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Choose $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that χ is 1 on

$$K = \pi_x\big((\pi_x \circ \mathcal{J}_{(z^{(1)},z^{(0)})})^{-1}\big(\mathrm{supp}(\psi)\big)\big),$$

where $\pi_x(x,\xi) = x$ is the natural projection of $T^*(\mathbb{R}^n)$ onto $\mathbb{R}^n$. Note that K is compact. We then have:

$$\psi(x')\mathcal{A}_{(z^{(1)},z^{(0)})}(u_0)(x') = \psi(x')\mathcal{A}_{(z^{(1)},z^{(0)})}(\chi u_0)(x') + \psi(x')\mathcal{A}_{(z^{(1)},z^{(0)})}\big((1-\chi)u_0\big)(x'), \quad u_0 \in \mathcal{S}'\big(\mathbb{R}^n\big).$$

Considering $\mathcal{A}_{(z^{(1)},z^{(0)})}$ as an FIO with real phase and symbol of type (ρ,δ), $\rho \geqslant \delta$, and denoting its distribution kernel by $A_{(z^{(1)},z^{(0)})}(x',x)$, we can apply the asymptotic formula (3.42) of [13, Section 10.3] to the symbol $\sigma(x',\xi)$ of the Lagrangian distribution $\psi(x')A_{(z^{(1)},z^{(0)})}(x',x)(1-\chi(x))$,

$$\psi(x')A_{(z^{(1)},z^{(0)})}(x',x)\big(1-\chi(x)\big) = \int \exp\big[\mathrm{i}\varphi_{(z^{(1)},z^{(0)})}(x',x,\xi)\big]\sigma(x',\xi)\,đ\xi,$$

which is then in $\Psi^{-\infty}_{\rho,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$. (For the case $\rho = \frac{1}{2}$, see the bottom of p. 310 in [13].)

If $u_0 \in \mathcal{S}(\mathbb{R}^n)$ we can write

$$D^\alpha_{x'}\big(\psi(x')\mathcal{A}_{(z^{(1)},z^{(0)})}\big((1-\chi)u_0\big)\big)(x') = \big\langle \widehat{u_0}(\xi), D^\alpha_{x'}\big(\exp\big[\mathrm{i}\langle x'|\xi\rangle + \mathrm{i}\big(z^{(1)}-z^{(0)}\big)b_1\big(z^{(0)},x',\xi\big)\big]\sigma(x',\xi)\big)\big\rangle.$$

Since the right-hand side is continuous from $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{C}^0(\mathbb{R}^n)$ we obtain that $\psi(x')\mathcal{A}_{(z^{(1)},z^{(0)})}((1-\chi)u_0)(x') \in \mathcal{C}^\infty(\mathbb{R}^n)$ if $u_0 \in \mathcal{S}'(\mathbb{R}^n)$. This yields:

$$\begin{aligned}\mathrm{WF}\big(\psi\mathcal{A}_{(z^{(1)},z^{(0)})}(u_0)\big) &= \mathrm{WF}\big(\psi\mathcal{A}_{(z^{(1)},z^{(0)})}(\chi u_0)\big) \subset \mathrm{WF}\big(\mathcal{A}_{(z^{(1)},z^{(0)})}(\chi u_0)\big)\\ &\subset J_{(z^{(1)},z^{(0)})\mathbb{R}}\big(\mathrm{WF}(\chi u_0)\big) \subset J_{(z^{(1)},z^{(0)})\mathbb{R}}\big(\mathrm{WF}(u_0)\big),\end{aligned}$$

since $\chi u_0 \in \mathcal{E}'(\mathbb{R}^n)$ when we use $\mathcal{A}_{(z^{(1)},z^{(0)})}$ as an FIO with complex phase. □

Corollary 3.9. *Let $\mathfrak{P} = z^{(0)},\dots,z^{(N)}$ be a subdivision of $[0,Z]$ and let $u_0 \in \mathcal{S}'(\mathbb{R}^n)$. Then*

$$\begin{aligned}&\mathrm{WF}\big(\mathcal{G}_{(z^{(j)},\dots,z^{(0)})}(u_0)\big) \subset J_{(z^{(j)},\dots,z^{(0)})\mathbb{R}}\mathrm{WF}(u_0), \quad j=1,\dots,N,\\ &\mathrm{WF}\big(\mathcal{W}_{\mathfrak{P},z}(u_0)\big) \subset J_{(z,z^{(l)},\dots,z^{(0)})\mathbb{R}}\big(\mathrm{WF}(u_0)\big),\end{aligned}$$

if $z^{(l)} \leqslant z \leqslant z^{(l+1)}$.

In the following statement, we give a sharper estimate of the wavefront set of $\mathcal{W}_{\mathfrak{P},z}(u_0)$.

Proposition 3.10. *Let K be a compact set in Ω_0 and $Z' \in \,]0,Z]$ be such that every bicharacteristics $\chi_z(\gamma^{(0)})$ associated to $-b_1 = \mathrm{Im}(a_1)$ originating from $\gamma^{(0)} \in K\setminus 0$ at $z=0$ satisfies $\chi_z(\gamma^{(0)}) \in \Omega_z$ for all $z \in [0,Z']$. Then there exists $d>0$ such that if $\Delta_{\mathfrak{P}} \leqslant d$ and $z \in [0,Z']$, with $z^{(l)} \leqslant z < z^{(l+1)}$, then $K\setminus 0$ is in the domain of the relation $J_{(z,z^{(l)},\dots,z^{(0)})\mathbb{R}}$ and*

$$J_{(z,z^{(l)},\dots,z^{(0)})\mathbb{R}}\big(\gamma^{(0)}\big) \in \mathrm{WF}\big(\mathcal{W}_{\mathfrak{P},z}(u_0)\big)$$

for all $u_0 \in \mathcal{S}'(\mathbb{R}^n)$ and $\gamma^{(0)} \in \mathrm{WF}(u_0)\cap K$. Moreover $J_{(z,z^{(l)},\dots,z^{(0)})\mathbb{R}}(\gamma^{(0)}) \in \mathrm{WF}(\mathcal{W}_{\mathfrak{P},z}(u_0))$ implies $\gamma^{(0)} \in \mathrm{WF}(u_0)$.

Proof. Since here $J_{(z,z^{(l)},\dots,z^{(0)})\mathbb{R}}(\gamma^{(0)}) = \mathcal{J}_{(z,z^{(l)},\dots,z^{(0)})}(\gamma^{(0)})$ by Proposition 3.7, the last statement follows from Corollary 3.9 and Lemma 2.12.

Let $\varepsilon > 0$ be such that

$$\varepsilon < \mathrm{dist}\big(\big\{\big(z,\chi_z\big(\gamma^{(0)}\big)\big) \,\big|\, z\in[0,Z'], \gamma^{(0)} \in K\setminus 0\big\}, \mathrm{supp}(c_1)\big).$$

Choose $d>0$, according to Proposition 3.7, such that

$$\big|\gamma^{(j)} - \chi_{z^{(j)}}\big(\gamma^{(0)}\big)\big| \leqslant \frac{\varepsilon}{2}, \quad j=1,\dots,k,$$

where k is defined by $z^{(k)} \leqslant Z' < z^{(k+1)}$, and $\gamma^{(0)} \in K\setminus 0$ (see Fig. 1 with ε replaced by $\varepsilon/2$). Assume further that d is sufficiently small such that

$$d \times \big|\partial_\xi b_1(z,x,\xi)\big| \leqslant \frac{\varepsilon}{2}, \quad \forall z\in[0,Z],\ \forall x\in\mathbb{R}^n,\ \forall \xi \in \mathbb{R}^n. \tag{3.7}$$

We then choose a subdivision $\mathfrak{P}$ of $[0,Z]$ satisfying $\Delta_{\mathfrak{P}} \leqslant d$.

Let $\gamma^{(0)} = (x^{(0)}, \gamma^{(0)}) \in K \setminus 0$ and $0 \leqslant j < k$. Then $(x^{(j+1)}, \xi^{(j)}) \in \Omega_{z^{(j)}}$ as $|(x^{(j+1)}, \xi^{(j)}) - \gamma^{(j)})| \leqslant \frac{\varepsilon}{2}$ by (3.7) and (3.5). Thus for all j, $0 \leqslant j < k$, $c_1(z^{(j)}, x^{(j+1)}, \xi^{(j)}) = 0$ which implies that $\gamma^{(0)}$ is in the domain of the relation $J_{(z^{(j)},\dots,z^{(0)})\mathbb{R}}$, for all j, $0 \leqslant j \leqslant k$, by (2.14).

Now let $z \in [0, Z']$ with $z^{(l)} \leqslant z < z^{(l+1)}$. We have:

$$\left(x^z, \xi^z\right) := J_{(z,z^{(l)},\dots,z^{(0)})\mathbb{R}}\left(\gamma^{(0)}\right) = \mathcal{J}_{z,(z^{(l)},\dots,z^{(0)})}\left(\gamma^{(0)}\right) \in \Omega_z.$$

Let $\psi_0(x, \xi) \in S^0(\mathbb{R}^n \times \mathbb{R}^n)$, e.g. homogeneous of degree 0 w.r.t. ξ, be equal to 1 in a conic neighborhood of $(x^{(1)}, \xi^{(0)})$ with support in $\Omega_{z^{(0)}}$. Define the operator $\mathcal{G}^{\psi_0}_{(z^{(1)},z^{(0)})}$ with distribution kernel:

$$G^{\psi_0}_{(z^{(1)},z^{(0)})}(x', x) := \int \exp\left[i\phi_{(z',z)}(x', x, \xi)\right]\psi_0(x', \xi) g_{(z',z)}(x', \xi)\, đ\xi.$$

On the support of ψ_0 the phase function is real and thus $\mathcal{G}^{\psi_0}_{(z^{(1)},z^{(0)})}$ is a global FIO associated with the real phase $\varphi_{(z^{(1)},z^{(0)})}$ defined in (2.15) with an amplitude of type $(1, 0)$. Furthermore because of the choice of the support of ψ_0 we have:

$$\left(x^{(1)}, \xi^{(1)}, x^{(0)}, -\xi^{(0)}\right) \notin \mathrm{WF}\left(G^{\psi_0}_{(z^{(1)},z^{(0)})} - G_{(z^{(1)},z^{(0)})}\right), \tag{3.8}$$

by Theorem 8.1.9 in [12]. Note that the operator $\mathcal{G}^{\psi_0}_{(z^{(1)},z^{(0)})}$ is non-characteristic at $(x^{(1)}, \xi^{(1)}, x^{(0)}, \xi^{(0)})$ because of the forms of ψ_0 and $g_{(z^{(1)},z^{(0)})}$ (see Definition 25.3.4 in [10]). It then follows that $(\mathcal{G}^{\psi_0}_{(z^{(1)},z^{(0)})})^* \circ \mathcal{G}^{\psi_0}_{(z^{(1)},z^{(0)})} \in \Psi^0(\mathbb{R}^n)$ and is non-characteristic at $(x^{(0)}, \xi^{(0)})$. If $\gamma^{(0)} \in \mathrm{WF}(u_0)$, then

$$\gamma^{(0)} \in \mathrm{WF}\left(\left(\mathcal{G}^{\psi_0}_{(z^{(1)},z^{(0)})}\right)^* \circ \mathcal{G}^{\psi_0}_{(z^{(1)},z^{(0)})}(u_0)\right),$$

and thus $\gamma^{(1)} \in \mathrm{WF}(\mathcal{G}^{\psi_0}_{(z^{(1)},z^{(0)})}(u_0))$ as the canonical transformation $\mathcal{J}^{-1}_{(z^{(1)},z^{(0)})}$ associated to $(\mathcal{G}^{\psi_0}_{(z^{(1)},z^{(0)})})^*$ is bijective. In turn, by (3.8), we obtain:

$$\gamma^{(1)} \in \mathrm{WF}\left(\mathcal{G}_{(z^{(1)},z^{(0)})}(u_0)\right). \tag{3.9}$$

Inspecting the proof of Theorem 2.22 in [15] one finds that, while the operator $(\mathcal{G}_{(z^{(1)},z^{(0)})})^* \circ \mathcal{G}_{(z^{(1)},z^{(0)})} \in \Psi^0_{\frac{1}{2}}(\mathbb{R}^n)$, it is non-characteristic and in Ψ^0 in a conic neighborhood of $(x^{(0)}, \xi^{(0)})$. This alternatively yields (3.9).

By induction we prove that $\gamma^{(j)} \in \mathrm{WF}(\mathcal{G}_{(z^{(j)},\dots,z^{(0)})}(u_0))$, $j = 1, \dots, l$, and that $(x^z, \xi^z) \in \mathrm{WF}(\mathcal{W}_{\mathfrak{P},z}(u_0))$. □

With the previous results we have thus obtained the following microlocal convergence result of the wavefront set:

Theorem 3.11. *Let $u_0(.) \in H^{(-\infty)}(\mathbb{R}^n)$ and $u(z, .)$, $z \in [0, Z]$, be the solution to the Cauchy problem* (0.1)–(0.2). *Let $Z' \in [0, Z]$ and K be a compact set in $T^*(\mathbb{R}^n)$ such that for all $\gamma^{(0)} = (x^{(0)}, \xi^{(0)}) \in K \setminus 0$ the bicharacteristics $\chi_z(\gamma^{(0)})$ associated to $-b_1$ originating from $\gamma^{(0)}$ at $z = 0$ satisfies $\chi_z(\gamma^{(0)}) \in \Omega_z$ for all $z \in [0, Z']$. Then if $\gamma^{(0)} \in K \cap \mathrm{WF}(u_0)$ we have $\chi_{Z'}(\gamma^{(0)}) \in \mathrm{WF}(u(Z', .))$. For a subdivision $\mathfrak{P}$ of $[0, Z]$, with $\Delta_{\mathfrak{P}}$ sufficiently small, we then have*:

$$\mathrm{dist}\left(\chi_z\left(\gamma^{(0)}\right), \mathrm{WF}\left(\mathcal{W}_{\mathfrak{P},z}(u_0)\right)\right) \to 0, \quad \text{as } \Delta_{\mathfrak{P}} \to 0,$$

uniformly w.r.t. $\gamma^{(0)} \in K \cap \mathrm{WF}(u_0)$ and $z \in [0, Z']$. Furthermore, the convergence is of order α, $0 < \alpha \leqslant 1$, if $b_1(z, .)$ is in $\mathcal{C}^{0,\alpha}([0, Z], S^1(\mathbb{R}^n, \mathbb{R}^n))$, in the sense that

$$b_1(z', x, \xi) - b_1(z, x, \xi) = (z' - z)^\alpha \tilde{b}_1(z', z, x, \xi), \quad 0 \leqslant z \leqslant z' \leqslant Z,$$

with $\tilde{b}_1(z', z, x, \xi)$ bounded w.r.t. z' and z with values in $S^1(\mathbb{R}^n \times \mathbb{R}^n)$.

As in [15] we introduce the second following Ansatz by modifying the thin-slab propagator. For a symbol $q(z, y, \eta) \in \mathcal{C}^0([0, Z], S^m(\mathbb{R}^p \times \mathbb{R}^r))$ we define $\hat{q}_{(z',z)}(y, \eta) \in \mathcal{C}^0([0, Z]^2, S^m(\mathbb{R}^p \times \mathbb{R}^r))$:

$$\hat{q}_{(z',z)}(y,\eta) := \frac{1}{z'-z}\int_z^{z'} q(s,y,\eta)\,\mathrm{d}s.$$

Then we set:

$$\hat{\phi}_{(z',z)}(x',x,\xi) := \langle x'-x|\xi\rangle + \mathrm{i}\Delta\hat{a}_{1(z',z)}(x',\xi) = \langle x'-x|\xi\rangle + \Delta\hat{b}_{1(z',z)}(x',\xi) + \mathrm{i}\Delta\hat{c}_{1(z',z)}(x',\xi), \tag{3.10}$$

and

$$\hat{g}_{(z',z)}(x,\xi) := \exp\bigl[-\Delta\hat{a}_{0(z',z)}(x,\xi)\bigr]. \tag{3.11}$$

Finally, following [14], we denote by $\widehat{\mathcal{G}}_{(z',z)}$ the FIO with distribution kernel:

$$\widehat{G}_{(z',z)}(x',x) = \int \exp\bigl[\mathrm{i}\langle x'-x|\xi\rangle\bigr]\exp\bigl[-\Delta\hat{a}_{(z',z)}(x',\xi)\bigr]\,\mathrm{đ}\xi = \int \exp\bigl[\mathrm{i}\hat{\phi}_{(z',z)}(x',x,\xi)\bigr]\hat{g}_{(z',z)}(x',\xi)\,\mathrm{đ}\xi.$$

The corresponding approximation Ansatz is as follows: let $\mathfrak{P}$ be a subdivision of $[0,Z]$, $\mathfrak{P} = \{z^{(0)}, z^{(1)}, \ldots, z^{(N)}\}$ with $0 = z^{(0)} < z^{(1)} < \cdots < z^{(N)} = Z$ such that $z^{(i+1)} - z^{(i)} = \Delta_{\mathfrak{P}}$. The operator $\widehat{\mathcal{W}}_{\mathfrak{P},z}$ is defined by:

$$\widehat{\mathcal{W}}_{\mathfrak{P},z} := \begin{cases} \widehat{\mathcal{G}}_{(z,0)} & \text{if } 0 \leqslant z \leqslant z^{(1)},\\ \widehat{\mathcal{G}}_{(z,z^{(k)})}\prod_{i=k}^{1}\widehat{\mathcal{G}}_{(z^{(i)},z^{(i-1)})} & \text{if } z^{(k)} \leqslant z \leqslant z^{(k+1)}. \end{cases}$$

Results of Section 2 apply to this Ansatz[5] as well. We define the set (which turns out to be a graph) $\widehat{\mathcal{J}}_{(z^{(N)},\ldots,z^{(0)})}$ by the following equations (compare with (2.10)–(2.11)):

$$\bigl(x^{(N)},\xi^{(N)},x^{(0)},\xi^{(0)}\bigr) \in \widehat{\mathcal{J}}_{(z^{(N)},\ldots,z^{(0)})}$$

if there exists $\theta_{N-1} \in \mathbb{R}^{n(2N-1)} \setminus 0$ as defined in (2.9) such that

$$\xi^{(j)} - \xi^{(j+1)} + \Delta^{(j)}\partial_x\hat{b}_{1(z^{(j+1)},z^{(j)})}\bigl(x^{(j+1)},\xi^{(j)}\bigr) = 0, \tag{3.12}$$

$$x^{(j+1)} - x^{(j)} + \Delta^{(j)}\partial_\xi\hat{b}_{1(z^{(j+1)},z^{(j)})}\bigl(x^{(j+1)},\xi^{(j)}\bigr) = 0. \tag{3.13}$$

The numerical scheme (3.12)–(3.13) is of the same nature as (3.4)–(3.5). Here however the order of consistency is 1 even if $b_1(z,.,.)$ is only continuous w.r.t. z. We thus observe convergence of order 1. As all the results in this section apply to the second Ansatz we obtain

Theorem 3.12. *Let $u_0(.) \in H^{(-\infty)}(\mathbb{R}^n)$ and $u(z,.)$, $z \in [0,Z]$, be the solution to the Cauchy problem* (0.1)–(0.2)*. Let $Z' \in [0,Z]$ and K be a compact set in $T^*(\mathbb{R}^n)$ such that for all $\gamma^{(0)} = (x^{(0)},\xi^{(0)}) \in K \setminus 0$ the bicharacteristics $\chi_z(\gamma^{(0)})$ associated to $-b_1$ originating from $\gamma^{(0)}$ at $z=0$ satisfies $\chi_z(\gamma^{(0)}) \in \Omega_z$ for all $z \in [0,Z']$. Then if $\gamma^{(0)} \in K \cap \mathrm{WF}(u_0)$ we have $\chi_{Z'}(\gamma^{(0)}) \in \mathrm{WF}(u(Z',.))$. For $\mathfrak{P}$ a subdivision of* $[0,Z]$*, with $\Delta_{\mathfrak{P}}$ sufficiently small, we then have*:

$$\operatorname{dist}\bigl(\chi_z\bigl(\gamma^{(0)}\bigr), \mathrm{WF}\bigl(\widehat{\mathcal{W}}_{\mathfrak{P},z}(u_0)\bigr)\bigr) \to 0, \quad \text{as } \Delta_{\mathfrak{P}} \to 0,$$

uniformly w.r.t. $\gamma^{(0)} \in K \cap \mathrm{WF}(u_0)$ and $z \in [0,Z']$ with a convergence rate of order 1.

4. Application to the 'double-square-root' equation and imaging

In [21], it was shown, following [24], that the acoustic wave field can be microlocally decomposed into up-going and down-going components (see also [15, Appendix A]). Each component is then the solution to a 'one-way' wave equation:

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[5] Some small modifications are required, e.g. in the proof of Lemma 2.2: there, Lemma 1.7 is used but in the present case smoothness of the change of variables w.r.t. Δ is lost; only continuity w.r.t. Δ remains. This modification is of no consequence here.

$$\big(\partial_z + a(z, x, D_t, D_x)\big)v(z, t, x) = 0,$$
$$v(0, .) = v_0(.),$$

with $a(z, x, D_t, D_x) = -\mathrm{i}b(z, x, D_t, D_x) + c(z, x, D_t, D_x)$, to which the results of [15] and the present paper apply: the symbols of the operators are assumed to be continuous w.r.t. z and the principal symbol c_1 is assumed non-negative.

The so-called 'downward continuation' operator for seismic data is actually the solution operator $H(z', z)$ to

$$\big(\partial_z + a(z, s, D_t, D_s) + a(z, r, D_t, D_r)\big)v(z, t, s, r) = 0, \tag{4.1}$$
$$v(0, .) = v(0, .), \tag{4.2}$$

(see [19]). Here $r \in \mathbb{R}^d$ and $s \in \mathbb{R}^d$ are the source and receiver coordinates respectively and $t \in \mathbb{R}$ is time. Eq. (4.1) is the so-called 'double-square equation' (DSR), since in the 'propagating regime', $c_1 = 0$, the principal symbol of the operator $b(z, x, D_t, D_x)$ is the square root of the symbol of an operator [21].

From $H(z', z)$ one can generate a linear[6] seismic modeling operator F (see [19]),

$$F(\delta v)(s, r, t) := Q_r^*(0)Q_s^*(0)\int_0^Z \big(H(0, z)Q_r(z)Q_s(z)g(z, .)\big)\,\mathrm{d}z, \quad \delta v \in \mathcal{E}'\big(\mathbb{R}^{d+1}\big),$$

where $Q_r(z)$ and $Q_s(z)$ are two families of properly chosen pseudodifferential operators and $g(z, .)$ is given by:

$$g(z, s, r, t) := \delta(t)\delta(s - r)\left(\frac{\delta v}{v_0}\right)\left(z, \frac{s + r}{2}\right).$$

The compactly supported distribution δv represents the singular component of the wavespeed responsible for the scattering of the incoming wave in a seismic experiment and is thus responsible for the data recorded at the surface. The total wavespeed is given by $v_0 + \delta v$ and v_0 is in fact used in the computation of the symbol $b(z, x, \tau, \xi)$.

With the results of [22] and [19], we observe that for the solution u to the DSR equation we have:

$$\mathrm{WF}\big(u(z)\big) \subset \big\{(t, s, r, \tau, \sigma, \rho);\ |\sigma| \leqslant K(z, s)\tau,\ |\rho| \leqslant K(z, r)\tau\big\}.$$

The constants $K(z, s)$ and $K(z, r)$ are given by the choices made for the damping terms in the DSR equation, i.e. the real part of the symbol a_z (see [21,20]).

Define $\psi_z(s, r, \tau, \sigma, \rho)$ to be a z-parametrized family of symbols in $S^0(\mathbb{R}^{2d+1} \times \mathbb{R}^{2d+1})$ such that

$$\begin{aligned} \psi_z(s, r, \tau, \sigma, \rho) = 0 \quad & \text{if } |\sigma| \geqslant 2K(z, s)\tau \text{ and } \sigma \geqslant 1, \\ & \text{or } |\rho| \geqslant 2K(z, s)\tau \text{ and } \rho \geqslant 1; \\ \psi_z(s, r, \tau, \sigma, \rho) = 1 \quad & \text{if } |\sigma| \leqslant K(z, s)\tau \text{ and } |\sigma| \leqslant K(z, s)\tau. \end{aligned}$$

With such a pseudodifferential cut-off we define:

$$\underline{a}_z(s, r, D_t, D_s, D_r) := \psi_z(s, r, D_t, D_s, D_r) \circ \big(a_z(s, D_t, D_s) + a_z(r, D_t, D_r)\big),$$

and observe with Theorem 18.1.35 in [11] that $\underline{a}_z$ is a z-parametrized family of pseudodifferential operators in t, s and r. The solutions of,

$$\big(\partial_z + \underline{a}_z(s, r, D_t, D_s, D_r)\big)v(z, t, s, r) = 0, \tag{4.3}$$
$$v(0, .) = v_0(.), \tag{4.4}$$

are microlocally equal to that of (4.1)–(4.2) while the z-parametrized family of operators $\underline{a}_z$ falls into the class of operators studied in the present paper and in [15]. We thus obtain approximations for the 'downward continuation' operator $H(z', z)$ with Sobolev and microlocal convergence of the wavefront set, i.e. Theorems 3.18 in [15] can be applied as well as Theorems 3.11 and 3.12 proved here.

Noting that the operator $H(z', z)^*$ is at the heart of the seismic imaging operator [18], we thus obtain approximations for this imaging operator.

[6] The scattering problem is linearized by mean of the Born approximation.

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Appendix A. Some results on Lagrangian (and canonical) ideals

In this section we give some results on Lagrangian (or canonical) ideals of positive type and on related complex phase functions of positive type. We also provide simple criteria for transversal composition of FIOs. These results are to be used in Section 2. We follow the notations of [10, Sections 25.4, 25.5]. The following proposition can be viewed as a complement to Proposition 25.4.4 in [10].

In this paper we do not rely on the techniques of almost analytic continuation developed in [16,17] but we use the techniques of Lagrangian ideals as developed in [10, Sections 25.4, 25.5]. The following two propositions have their counterpart in [16], namely Theorem 3.6, p. 167, for Proposition A.1 and Proposition 7.1, p. 204, for Proposition A.3, but it seems that they are not included in [10].

Proposition A.1. *Let X be a $\mathcal{C}^\infty$ manifold of dimension n, $\Gamma \subset X \times (\mathbb{R}^N \backslash 0)$ an open conic neighborhood of (x^0, θ^0) and let $\phi \in \mathcal{C}^\infty(\Gamma)$ be a non-degenerate phase function of positive type at (x^0, θ^0). Let $\xi^0 = \phi'_x(x^0, \theta^0) \neq 0$ and let J be the Lagrangian ideal defined by ϕ in a conic neighborhood of (x^0, ξ^0). Then locally $J_\mathbb{R} = \{(x, \xi) \mid (x, \theta) \in \Gamma$, $\xi = \phi'_x(x, \theta)$ with $\phi'_\theta(x, \theta) = 0\}$.*

Proof. We have $\{(x, \xi) \mid (x, \theta) \in \Gamma,\ \xi = \phi'_x(x, \theta)$ with $\phi'_\theta(x, \theta) = 0\} \subset J_\mathbb{R}$. We follow the proof of Proposition 25.4.4 in [10] and use its notations. The ideal $\hat{J}$ is locally generated by $\partial_{\theta_j}\phi(x, \theta) - \xi_k$, $j = 1, \dots, N$ and $\partial_{x_k}\phi(x, \theta)$, $k = 1, \dots, n$ ($x_1, \dots, x_n$ are local coordinates on X and $\xi_1, \dots, \xi_n$ are the corresponding coordinates in $T^*(X)$). One can choose the local coordinates on X such that $\hat{J}$ is actually generated by functions of the form:

$$x_k - X_k(\xi), \qquad \theta_j - \Theta_j(\xi), \quad k = 1, \dots, n,\ j = 1, \dots, N,$$

where the X_k are homogeneous of degree 0 and the Θ_j are homogeneous of degree 1. A function $f^0(\xi)$ can be chosen homogeneous of degree 1 such that

$$\operatorname{Im} f^0(\xi) \geqslant C\left(\left|\operatorname{Im} X(\xi)\right|^2 + \left|\operatorname{Im} \Theta(\xi)\right|^2\right)$$

in a conic neighborhood of (x^0, ξ^0) and such that J is generated by $x_k + \partial_{\xi_k} f^0(\xi)$, $k = 1, \dots, n$. Let us now take $(x^1, \xi^1) \in J_\mathbb{R}$ in the considered neighborhood of (x^0, ξ^0). We then have $x^1_k + \partial_{\xi_k} f^0(\xi^1) = 0$, $k = 1, \dots, n$, which gives $\partial_{\xi_k} f^0(\xi^1) \in \mathbb{R}$. Euler's identity yields $f^0(\xi^1) \in \mathbb{R}$ from which we find $\operatorname{Im} X_k(\xi^1) = 0$, $k = 1, \dots, n$, and $\operatorname{Im} \Theta_j(\xi^1) = 0$, $j = 1, \dots, N$. Define $\theta^1_j = \Theta^1_j(\xi^1) \in \mathbb{R}$, $j = 1, \dots, N$. As $0 \neq \theta^0_j = \Theta^1_j(\xi^0)$ we can shrink the conic neighborhood of (x^0, ξ^0) so that $\theta^1_j \neq 0$. Then the generators of $\hat{J}$ vanish at (x^1, ξ^1, θ^1) (in fact note that the function $x_k - X_k(\xi)$, $k = 1, \dots, n$, of $\hat{J}$ are independent of θ and hence belong to J and thus vanish at (x^1, ξ^1)). We therefore obtain $\partial_{\theta_j}\phi(x^1, \theta^1) = 0$, $j = 1, \dots, N$, and $\partial_{x_k}\phi(x^1, \theta^1) - \xi^1_k = 0$, $k = 1, \dots, n$. In other words,

$$\left(x^1, \xi^1\right) \in \left\{(x, \xi) \mid (x, \theta) \in \Gamma, \xi = \phi'_x(x, \theta) \text{ with } \phi'_\theta(x, \theta) = 0\right\},$$

which completes the proof. □

Remark A.2. Note that if a Lagrangian ideal J is globally parameterized by a phase function $\phi(x, \theta)$ then globally we have $J_\mathbb{R} = \{(x, \xi) \mid (x, \theta) \in \Gamma, \xi = \phi'_x(x, \theta)$ with $\phi'_\theta(x, \theta) = 0\}$.

To understand the propagation of singularities when composing two FIOs with respective canonical ideals J_1 in $T^*(X \times Y)\backslash 0$ and J_2 in $T^*(Y \times Z)\backslash 0$ we need to keep track of the set $J_\mathbb{R} = (J_1 \circ J_2)_\mathbb{R}$. Characterizing the set $J_\mathbb{R}$ is of importance to further compose the resulting FIO with other FIOs (see [10, Theorem 25.5.5]). This is the subject of the following proposition. This point is of importance here as in Sections 2 and 3 we compose FIOs with the number of factors tending to ∞.

Proposition A.3. *Let J_1 and J_2 be two positive conic canonical ideals in $T^*(X\times Y)\backslash 0$ and $T^*(Y\times Z)\backslash 0$ respectively such that*

1. $J_{1\mathbb{R}} \subset (T^*(X)\backslash 0)\times (T^*(Y)\backslash 0)$, $J_{2\mathbb{R}} \subset (T^*(Y)\backslash 0)\times (T^*(Z)\backslash 0)$,
2. *The composition is transversal at each point*
$$\big(x^0,\xi^0,y^0,\eta^0\big)\in J_{1\mathbb{R}},\qquad \big(y^0,\eta^0,z^0,\zeta^0\big)\in J_{2\mathbb{R}},$$
3. *The projection*
$$\pi : J_{1\mathbb{R}}\times J_{2\mathbb{R}}\cap \big(T^*(X)\big)\times \operatorname{diag}\big(T^*(X)\big)\times\big(T^*(Z)\big)\to T^*(X\times Z)\backslash 0,$$
$$(x,\xi,y,\eta,y,\eta,z,\zeta)\mapsto (x,\xi,z,\zeta)$$
is injective and proper.

Then $(J_1\circ J_2)_{\mathbb{R}} = J_{1\mathbb{R}}\circ J_{2\mathbb{R}}$.

Proof. By Theorem 25.5.5 in [10] $J_1\circ J_2$ is locally defined in neighborhoods of points in $J_{1\mathbb{R}}\circ J_{2\mathbb{R}}$ (away from such neighborhoods $J_1\circ J_2$ is locally the trivial algebra, i.e. the whole set of $\mathcal{C}^\infty$ functions and there $(J_1\circ J_2)_{\mathbb{R}}=\emptyset$). The definition $J_1\circ J_2$ naturally yields $J_{1\mathbb{R}}\circ J_{2\mathbb{R}}\subset (J_1\circ J_2)_{\mathbb{R}}$ (see Proposition 25.5.3 in [10] and its proof).

Let $(x^0,\xi^0,z^0,\zeta^0)\in J_{1\mathbb{R}}\circ J_{2\mathbb{R}}$. Then there exists $(y^0,\eta^0)\in T^*(Y)\backslash 0$ such that $(x^0,\xi^0,y^0,\eta^0)\in J_{1\mathbb{R}}$ and $(y^0,\eta^0,z^0,\zeta^0)\in J_{2\mathbb{R}}$. In a neighborhood of (x^0,ξ^0,y^0,η^0), J_1 is defined by a non-degenerate phase function $\phi_1(x,y,\theta)\in\mathcal{C}^\infty(X\times Y\times(\mathbb{R}^{N_\theta}\backslash 0))$. In a neighborhood of (y^0,η^0,z^0,ζ^0), J_2 is defined by a non-degenerate phase function $\phi_2(y,z,\tau)\in\mathcal{C}^\infty(X\times Y\times(\mathbb{R}^{N_\tau}\backslash 0))$. This also means that there exists $\theta^0\in\mathbb{R}^{N_\theta}\backslash 0$ and $\tau^0\in\mathbb{R}^{N_\tau}\backslash 0$ such that

$$\begin{aligned}
&\partial_\theta\phi_1\big(x^0,y^0,\theta^0\big)=0, &&\xi^0=\partial_x\phi_1\big(x^0,y^0,\theta^0\big), &&\eta^0=-\partial_y\phi_1\big(x^0,y^0,\theta^0\big),\\
&\partial_\tau\phi_2\big(y^0,z^0,\tau^0\big)=0, &&\eta^0=\partial_y\phi_2\big(y^0,z^0,\tau^0\big), &&\zeta^0=-\partial_z\phi_2\big(y^0,z^0,\tau^0\big).
\end{aligned}$$

By Proposition 25.5.4 in [10] $\phi_1(x,y,\theta)+\phi_2(y,z,\tau)$ defines $J_1\circ J_2$ in a neighborhood U of (x^0,ξ^0,z^0,ζ^0). By Proposition A.1, we have that in U (or possibly a smaller neighborhood):

$$\begin{aligned}
(J_1\circ J_2)_{\mathbb{R}} = \big\{&(x,\xi,z,\zeta)\mid \partial_\theta\phi_1(x,y,\theta)=0,\ \partial_\tau\phi_2(y,z,\tau)=0,\\
&\partial_y\big(\phi_1(x,y,\theta)+\phi_2(y,z,\tau)\big)=0,\ \xi=\partial_x\phi_1(x,y,\theta),\ \zeta=-\partial_z\phi_2(y,z,\tau)\big\}.
\end{aligned}$$

Let $(x^1,\xi^1,z^1,\zeta^1)\in(J_1\circ J_2)_{\mathbb{R}}\cap U$. Let θ^1,τ^1,y^1 be such that

$$\begin{aligned}
&\partial_\theta\phi_1\big(x^1,y^1,\theta^1\big)=0,\qquad \partial_\tau\phi_2\big(y^1,z^1,\tau^1\big)=0,\\
&\partial_y\big(\phi_1\big(x^1,y^1,\theta^1\big)+\phi_2\big(y^1,z^1,\tau^1\big)\big)=0,\\
&\xi^1=\partial_x\phi_1\big(x^1,y^1,\theta^1\big),\qquad \zeta^1=-\partial_z\phi_2\big(y^1,z^1,\tau^1\big).
\end{aligned}$$

Noting that $\eta^1:=-\partial_y\phi_1(x^1,y^1,\theta^1)=\partial_y\phi_2(y^1,z^1,\tau^1)$ is real we see that ϕ_1 is of positive type at (x^1,y^1,θ^1) and hence $(x^1,\xi^1,y^1,\eta^1)\in J_{1\mathbb{R}}$ and similarly $(y^1,\eta^1,z^1,\zeta^1)\in J_{2\mathbb{R}}$. We thus find that locally $(J_1\circ J_2)_{\mathbb{R}}\subset J_{1\mathbb{R}}\circ J_{2\mathbb{R}}$. □

The following proposition gives easy-to-check criteria to ensure transversality in the composition of FIOs. It is a converse to Proposition 25.5.4 in [10]. In the case of non-degenerate real phase functions this result is implicit in [9, pp. 175–176]. In the case of clean real phase functions it is clear that the proof of Proposition 21.2.19 in [11] yields the converse of its statement. Here we only treat the case of non-degenerate complex phase functions and use the techniques of canonical ideals.

Let X, Y, Z be three $\mathcal{C}^\infty$ manifolds of dimension n_x,n_y and n_z. Let $x_1,\dots,x_{n_x}$, $y_1,\dots,y_{n_y}$ and $z_1,\dots,z_{n_z}$ be local coordinates for X, Y and Z with $\xi_1,\dots,\xi_{n_x}$, $\eta_1,\dots,\eta_{n_y}$ and $\zeta_1,\dots,\zeta_{n_z}$ the corresponding coordinates in $T^*(X)$, $T^*(Y)$ and $T^*(Z)$.

Proposition A.4. *Let $\phi_1(x, y, \theta)$ and $\phi_2(y, z, \tau)$ be two non-degenerate complex phase functions of positive type at $(x^0, y^0, \theta^0) \in X \times Y \times (\mathbb{R}^{N_\theta} \backslash 0)$ and $(y^0, z^0, \tau^0) \in Y \times Z \times (\mathbb{R}^{N_\tau} \backslash 0)$ respectively such that*

$$\eta^0 := -\partial_y \phi_1\big(x^0, y^0, \theta^0\big) = \partial_y \phi_2\big(y^0, z^0, \tau^0\big) \in \mathbb{R}^{n_y} \backslash 0.$$

Define $\xi^0 := \partial_x \phi_1(x^0, y^0, \theta^0) \in \mathbb{R}^{n_x} \backslash 0$ and $\zeta^0 = -\partial_z \phi_2(y^0, z^0, \tau^0) \in \mathbb{R}^{n_z} \backslash 0$. Let J_1 and J_2 be the canonical ideals parameterized by ϕ_1 and ϕ_2 in neighborhoods of $(x^0, \xi^0, y^0, \eta^0)$ and $(y^0, \eta^0, z^0, \zeta^0)$ respectively. J_1 and J_2 compose transversally at these points if and only if $\phi_1 + \phi_2$ is non degenerate at $(x^0, z^0, \theta^0, \tau^0, y^0)$ (where y is considered as a phase variable).

Proof. Throughout the proof everything is done locally; neighborhoods are refined if necessary without mentioning it. $\hat{J}_1$ is the ideal generated by the functions $\partial_{\theta_j} \phi_1(x, y, \theta)$, $\partial_{x_i} \phi_1(x, y, \theta) - \xi_i$ and $\partial_{y_k} \phi_1(x, y, \theta) + \eta_k$, $j = 1, \dots, N_\theta$, $i = 1, \dots, n_x$, $k = 1, \dots, n_y$. We call these functions $U_i(x, y, \theta, \xi, \eta)$, $i = 1, \dots, N_\theta + n_x + n_y$. The differentials $\mathrm{d}U_i$, $i = 1, \dots, N_\theta + n_x + n_y$, are linearly independent (use Definition 25.4.3 in [10]). The canonical ideal J_1 is locally the set of functions of $\hat{J}_1$ that are independent of θ. There are $n_x + n_y$ generators of J_1 in the considered neighborhood. We denote them by $u_i(x, y, \xi, \eta)$, $i = 1, \dots, n_x + n_y$. The differentials $\mathrm{d}u_i$, $i = 1, \dots, n_x + n_y$, are linearly independent. We can thus choose $n_{x'}$ x coordinates, $n_{y'}$ y coordinates, $n_{\xi'}$ ξ coordinates and $n_{\eta'}$ η coordinates such that (after reordering the coordinates) $x = (x', x'')$, $y = (y', y'')$, $\xi = (\xi', \xi'')$, $\eta = (\eta', \eta'')$ with $n_{x'} + n_{y'} + n_{\xi'} + n_{\eta'} = n_x + n_y$ and the partial differentials of the functions u_i, $i = 1, \dots, n_x + n_y$, w.r.t. x', y', ξ', η' are linearly independent. (Note that $n_{x'}$ may differ from $n_{\xi'}$ and so on.) We denote $n_{x''} = n_x - n_{x'}$, $n_{y''} = n_y - n_{y'}$, $n_{\xi''} = n_x - n_{\xi'}$, $n_{\eta''} = n_y - n_{\eta'}$. Theorem 7.5.7 in [12] gives the existence of some functions $g_{ij}(x, y, \xi, \eta, \theta)$, and $R_i(\theta, x'', y'', \xi'', \eta'')$, $i = 1, \dots, n_x + n_y + N_\theta$, $j = 1, \dots, n_x + n_y$, such that

$$U_i(x, y, \theta, \xi, \eta) = \sum_{j=1}^{n_x+n_y} g_{ij}(x, y, \xi, \eta, \theta) u_j(x, y, \xi, \eta) + R_i(\theta, x'', y'', \xi'', \eta''), \quad i = 1, \dots, n_x + n_y + N_\theta, \tag{A.1}$$

in a neighborhood of $(x^0, y^0, \theta^0, \xi^0, \eta^0)$. As the U_i and u_j are in $\hat{J}_1$, so are the functions R_i. With (A.1) we see that the functions R_i, $i = 1, \dots, n_x + n_y + N_\theta$, and the functions u_j, $j = 1, \dots, n_x + n_y$, generate $\hat{J}_1$. Yet, their differentials are not linearly independent but they are of rank $n_x + n_y + N_\theta$ like the differentials $\mathrm{d}(U_i)$, $i = 1, \dots, n_x + n_y + N_\theta$. The partial differentials of u_j, $j = 1, \dots, n_x + n_y$, w.r.t. x', y', ξ' and η' are of rank $n_x + n_y$ while the functions R_i, $i = 1, \dots, n_x + n_y + N_\theta$, are independent of these variables. From this we obtain that the differentials $\mathrm{d}(R_i)$, $i = 1, \dots, n_x + n_y + N_\theta$, are of rank N_θ. We can therefore select $R_1, \dots, R_{N_\theta}$ (after some reordering) such that the functions u_j, R_i, $j = 1, \dots, n_x + n_y$, $i = 1, \dots, N_\theta$, have their differentials linearly independent and generate $\hat{J}_1$ (use Lemma 7.5.8 in [12]).

To carry on with the proof of Proposition A.4 we need the following lemma.

Lemma A.5. *The matrix $\mathcal{R} := (\partial_{\theta_j} R_i)_{\substack{1 \leqslant i \leqslant N_\theta \\ 1 \leqslant j \leqslant N_\theta}}$ is of rank N_θ at the point $(x^0, y^0, \xi^0, \eta^0, \theta^0)$.*

Proof. Let us first consider the system $\sum_{j=0}^{N_\theta} \partial_{\theta_j} R_i \, \mathrm{d}\theta_j = 0$ at the point[7] $(x^0, y^0, \xi^0, \eta^0, \theta^0)$. For the θ variables it is equivalent to the system:

$$\begin{cases} \mathrm{d}u_1 = \cdots = \mathrm{d}u_{n_x+n_y} = 0, \\ \mathrm{d}R_1 = \cdots = \mathrm{d}R_{N_\theta} = 0, \\ \mathrm{d}x = 0, \ \mathrm{d}y = 0, \ \mathrm{d}\xi = 0, \ \mathrm{d}\eta = 0, \end{cases}$$

as the functions u_i, $i = 1, \dots, n_x + n_y$, solely depend on x', y', ξ', η'. Now this system is equivalent to (see the proof of Lemma 7.5.8 in [12]):

$$\begin{cases} \mathrm{d}U_1 = \cdots = \mathrm{d}U_{n_x+n_y+N_\theta} = 0, \\ \mathrm{d}x = 0, \ \mathrm{d}y = 0, \ \mathrm{d}\xi = 0, \ \mathrm{d}\eta = 0, \end{cases}$$

[7] Functions are to be evaluated at this point in the proof of Lemma A.5.

which, using the expressions of the functions U_i, can be written as

$$\begin{cases} \partial^2_{\theta x}\phi_1\,\mathrm{d}x + \partial^2_{\theta y}\phi_1\,\mathrm{d}y + \partial^2_{\theta\theta}\phi_1\,\mathrm{d}\theta = 0,\\ \partial^2_{xx}\phi_1\,\mathrm{d}x + \partial^2_{xy}\phi_1\,\mathrm{d}y + \partial^2_{x\theta}\phi_1\,\mathrm{d}\theta - \mathrm{d}\xi = 0,\\ \partial^2_{yx}\phi_1\,\mathrm{d}x + \partial^2_{yy}\phi_1\,\mathrm{d}y + \partial^2_{y\theta}\phi_1\,\mathrm{d}\theta + \mathrm{d}\eta = 0,\\ \mathrm{d}x = 0,\ \mathrm{d}y = 0,\ \mathrm{d}\xi = 0,\ \mathrm{d}\eta = 0. \end{cases}$$

This implies the system

$$\partial^2_{\theta\theta}\phi_1\,\mathrm{d}\theta = 0, \qquad \partial^2_{x\theta}\phi_1\,\mathrm{d}\theta = 0, \qquad \partial^2_{y\theta}\phi_1\,\mathrm{d}\theta = 0,$$

which in turn yields $d\theta = 0$ as the phase function ϕ_1 is non-degenerate. The matrix $(\partial_{\theta_j} R_i)_{\substack{1\leqslant i\leqslant N_\theta\\ 1\leqslant j\leqslant N_\theta}}$ is thus of rank N_θ. $\square$

End of the proof of Proposition A.4: We now perform the same analysis on the phase function $\phi_2(y,z,\tau)$. Denote by $\widehat{J_2}$ the ideal locally generated by $\partial_{\tau_j}\phi_2(y,z,\tau)$, $\partial_{y_i}\phi_2(y,z,\tau) - \eta_i$ and $\partial_{z_k}\phi_2(y,z,\tau) + \zeta_k$, $j = 1,\dots,N_\tau$, $i = 1,\dots,n_y$, $k = 1,\dots,n_z$. We call these functions $V_i(y,z,\tau,\eta,\zeta)$, $i = 1,\dots,N_\tau + n_y + n_z$. The ideal $\widehat{J_2}$ is also generated by some functions $v_1,\dots,v_{n_y+n_z}$, $S_1,\dots,S_{N_\tau}$, where the functions v_j, $j = 1,\dots,n_y+n_z$, solely depend on y, z, η, and ζ and locally generate J_2 while the matrix $\mathcal{S} := (\partial_{\tau_j} S_i)_{\substack{1\leqslant i\leqslant N_\tau\\ 1\leqslant j\leqslant N_\tau}}$ is of rank N_τ by the previous lemma.

The tangent planes $T_{\gamma^0}(J_1)$ and $T_{\varepsilon^0}(J_2)$ at $\gamma^0 = (x^0, y^0, \xi^0, \eta^0)$ and $\varepsilon^0 = (y^0, z^0, \eta^0, \zeta^0)$ are respectively defined by the equations (in the complexification of $T_{\gamma^0}(T^*(X) \times T^*(Y))$ and $T_{\varepsilon^0}(T^*(Y) \times T^*(Z))$):

$$\mathrm{d}u_1 = \cdots = \mathrm{d}u_{n_x+n_y} = 0,$$

and

$$\mathrm{d}v_1 = \cdots = \mathrm{d}v_{n_y+n_z} = 0,$$

evaluated[8] at $(x^0, y^0, \xi^0, \eta^0)$ and $(y^0, z^0, \eta^0, \zeta^0)$ respectively. The ideals J_1 and J_2 compose transversally at γ^0 and ε^0 if and only if $T_{\gamma^0}(J_1) \times T_{\varepsilon^0}(J_2)$ intersects transversally with

$$\widetilde{\Delta} = T_{x^0,\xi^0}\big(T^*(X)\big) \times \mathrm{diag}\big(T_{y^0,\eta^0}\big(T^*(Y)\big)\big) \times T_{z^0,\zeta^0}\big(T^*(Z)\big),$$

that is

$$T_{\gamma^0}(J_1) \times T_{\varepsilon^0}(J_2) \cap \{0\} \times \mathrm{diag}\big(T_{y^0,\eta^0}\big(T^*(Y)\big)\big) \times \{0\} = \{0\}, \tag{A.2}$$

as $T_{\gamma^0}(J_1) \times T_{\varepsilon^0}(J_2)$ is Lagrangian for the symplectic form $\sigma = \sigma_X - \sigma_Y + \sigma_{\tilde{Y}} - \sigma_Z$ and $\widetilde{\Delta}^\sigma = \{0\} \times \mathrm{diag}(T_{y^0,\eta^0}(T^*(Y))) \times \{0\}$. We have denoted by $\sigma_{\tilde{Y}}$ the symplectic form on the second copy of $T^*(Y)$. Eq. (A.2) in turn is equivalent to:

$$\begin{cases} \mathrm{d}u_1 = \cdots = \mathrm{d}u_{n_x+n_y} = 0,\\ \mathrm{d}v_1 = \cdots = \mathrm{d}v_{n_y+n_z} = 0,\\ \mathrm{d}x = 0,\ \mathrm{d}\xi = 0,\ \mathrm{d}z = 0,\ \mathrm{d}\zeta = 0,\\ \mathrm{d}\tilde{y} = \mathrm{d}y,\ \mathrm{d}\tilde{\eta} = \mathrm{d}\eta, \end{cases} \quad \Rightarrow \quad \mathrm{d}y = 0,\ \mathrm{d}\eta = 0, \tag{A.3}$$

where $(\tilde{y}, \tilde{\eta})$ are the coordinates in the second copy of $T^*(Y)$. The previous statement is equivalent to

$$\begin{cases} \mathrm{d}u_1 = \cdots = \mathrm{d}u_{n_x+n_y} = 0,\\ \mathrm{d}v_1 = \cdots = \mathrm{d}v_{n_y+n_z} = 0,\\ \mathrm{d}x = 0,\ \mathrm{d}\xi = 0,\ \mathrm{d}z = 0,\ \mathrm{d}\zeta = 0,\\ \mathrm{d}\tilde{y} = \mathrm{d}y,\ \mathrm{d}\tilde{\eta} = \mathrm{d}\eta,\\ \mathrm{d}R_1 = \cdots = \mathrm{d}R_{N_\theta} = 0,\ \mathrm{d}S_1 = \cdots = \mathrm{d}S_{N_\tau} = 0, \end{cases} \quad \Rightarrow \quad \begin{array}{l} \mathrm{d}y = 0,\ \mathrm{d}\eta = 0,\\ \mathrm{d}\theta = 0,\ \mathrm{d}\tau = 0. \end{array} \tag{A.4}$$

[8] In the remaining of the proof functions are to be evaluated at $(x^0, y^0, z^0, \xi^0, \eta^0, \zeta^0, \theta^0, \tau^0)$.

In fact (A.3) states that the rank of the first system is $2n_x + 4n_y + 2n_z$ and (A.4) states that the rank of the second system is $2n_x + 4n_y + 2n_z + N_\theta + N_\tau$. As the matrices $\mathcal{R}$ and $\mathcal{S}$ are of full rank, by Lemma A.5, the statements are equivalent. Using some argument in the proof of Lemma A.5 we obtain that (A.4) is equivalent to:

$$\begin{cases} \mathrm{d}U_1 = \cdots = \mathrm{d}U_{n_x+n_y+N_\theta} = 0, \\ \mathrm{d}V_1 = \cdots = \mathrm{d}V_{n_y+n_z+N_\tau} = 0, \\ \mathrm{d}x = 0,\ \mathrm{d}\xi = 0,\ \mathrm{d}z = 0,\ \mathrm{d}\zeta = 0, \\ \mathrm{d}\tilde{y} = \mathrm{d}y,\ \mathrm{d}\tilde{\eta} = \mathrm{d}\eta, \end{cases} \quad \Rightarrow \quad \begin{matrix} \mathrm{d}y = 0,\ \mathrm{d}\eta = 0, \\ \mathrm{d}\theta = 0,\ \mathrm{d}\tau = 0. \end{matrix}$$

Because of the forms of the U_i and V_j, the previous statement is equivalent to:

$$\begin{cases} \partial^2_{x\theta}\phi_1\,\mathrm{d}\theta & & +\,\partial^2_{xy}\phi_1\,\mathrm{d}y & & = 0, \\ & \partial^2_{z\tau}\phi_2\,\mathrm{d}\tau & +\,\partial^2_{zy}\phi_2\,\mathrm{d}y & & = 0, \\ \partial^2_{\theta\theta}\phi_1\,\mathrm{d}\theta & & +\,\partial^2_{\theta y}\phi_1\,\mathrm{d}y & & = 0, \\ & \partial^2_{\tau\tau}\phi_2\,\mathrm{d}\tau & +\,\partial^2_{\tau y}\phi_2\,\mathrm{d}y & & = 0, \\ \partial^2_{y\theta}\phi_1\,\mathrm{d}\theta & +\,\partial^2_{y\tau}\phi_2\,\mathrm{d}\tau & +\,(\partial^2_{yy}\phi_1 + \partial^2_{yy}\phi_2)\,\mathrm{d}y & & = 0, \\ \partial^2_{y\theta}\phi_1\,\mathrm{d}\theta & & +\,\partial^2_{yy}\phi_1\,\mathrm{d}y & +\mathrm{d}\eta & = 0, \end{cases} \quad \Rightarrow \quad \begin{cases} \mathrm{d}y = 0, \\ \mathrm{d}\eta = 0, \\ \mathrm{d}\theta = 0, \\ \mathrm{d}\tau = 0. \end{cases}$$

This now yields that the matrix,

$$\begin{pmatrix} \partial^2_{x\theta}\phi_1 & 0 & \partial^2_{xy}\phi_1 \\ 0 & \partial^2_{z\tau}\phi_2 & \partial^2_{zy}\phi_2 \\ \partial^2_{\theta\theta}\phi_1 & 0 & \partial^2_{\theta y}\phi_1 \\ 0 & \partial^2_{\tau\tau}\phi_2 & \partial^2_{\tau y}\phi_2 \\ \partial^2_{y\theta}\phi_1 & \partial^2_{y\tau}\phi_2 & (\partial^2_{yy}\phi_1 + \partial^2_{yy}\phi_2) \end{pmatrix},$$

evaluated at $(x^0, z^0, \theta^0, \tau^0, y^0)$ is of rank $N_\theta + N_\tau + n_y$. If we now write $\Phi(x, z, \theta, \tau, y) = \phi_1(x, y, \theta) + \phi_2(y, z, \tau)$ we see that the previous statement means that the differentials $\mathrm{d}(\partial_\theta \Phi)$, $\mathrm{d}(\partial_\tau \Phi)$ and $\mathrm{d}(\partial_y \Phi)$ are linearly independent at $(x^0, z^0, \theta^0, \tau^0, y^0)$. □

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