Topology and its Applications 158 (2011) 708–717

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Some remarks on the global structure of proper Lie groupoids in low codimensions ☆

Giorgio Trentinaglia *

Courant Forschungszentrum Strukturen höherer Ordnung in der Mathematik, Georg-August-Universität Göttingen, Germany

ARTICLE INFO

Article history:
Received 24 February 2010
Received in revised form 7 January 2011
Accepted 17 January 2011

MSC:
58H05
22D35

Keywords:
Proper Lie groupoid
Vector bundle
Representation

ABSTRACT

We observe that any connected proper Lie groupoid whose orbits have codimension at most two admits a globally effective representation, i.e. one whose kernel consists only of ineffective arrows, on a smooth vector bundle. As an application, we deduce that any such groupoid can up to Morita equivalence be presented as an extension, by some bundle of compact Lie groups, of some action groupoid $G \ltimes X$ with G compact.

1. Introduction

Lie groupoids serve as models for certain geometric spaces which cannot be appropriately described in terms of classical notions like “topology” or “manifold”: orbifolds [1,2], spaces of leaves of foliations [3,4], differentiable stacks [5,6]. These “higher order” spaces have by now earned the status of basic objects in many areas of pure mathematics [7,8] and mathematical physics [9]. Valuable information about these (when considered from the traditional point of view “badly behaved”) spaces can be obtained by studying the Lie groupoids representing them. The latter have very nice properties and lend themselves to investigation by means of differential geometric or representation theoretic (algebraic) tools.

The main purpose of the present note is to draw attention on a few new open problems in the research field of Lie groupoids. These problems all concern the global topological structure (up to Morita equivalence) of proper Lie groupoids and may be placed among the so-called presentation (or normal form) problems for that kind of groupoids—such as, for example, the global quotient conjecture for noneffective orbifolds [1,2,10]. A secondary purpose is to supply evidence for a claim contained in [11, Example 4.13] but not proven therein, which concerns a special case of one of the problems we are going to propose. We decided to publish the proof of that claim separately in order to avoid getting into rather lengthy technical details for what was, after all, just a remark; besides, the proof was essentially self-contained.

Let us begin by discussing the above-mentioned problems and the reasons why they are of interest to us. The essential background on Lie groupoids which is necessary to follow the explanation below and the few notations which we shall be using without comment throughout the paper have been collected in the next section for the reader’s convenience.

First of all, we need some terminology.

☆ Supported by DFG (Deutsche Forschungsgemeinschaft) through the institutional development program of the University of Göttingen and, in part, by Fondazione Ing. Aldo Gini.

* Address for correspondence: Mathematisches Institut, Georg-August-Universität Göttingen, Bunsenstraße 3-5, D-37073 Göttingen, Germany. Tel.: +49 551 397817, fax: +49 551 392985.

E-mail address: gtrentin@uni-math.gwdg.de.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2011.01.019

Definition 1. We say that a proper Lie groupoid $\mathcal{G}$ *admits enough representations*, or that it is *reflexive*, if for each base point x there exists a representation $\mathcal{G} \to GL(E)$ of $\mathcal{G}$ on some smooth vector bundle E whose restriction $\mathcal{G}_x \to GL(E_x)$ to the isotropy group $\mathcal{G}_x = \mathcal{G}(x,x)$ is injective.

Definition 2. Call a representation $\mathcal{G} \to GL(E)$ of a Lie groupoid $\mathcal{G}$ on a smooth vector bundle E *effective at a base point x* if the kernel of its restriction $\mathcal{G}_x \to GL(E_x)$ to the isotropy group $\mathcal{G}_x$ is contained in the ineffective part[1] of $\mathcal{G}_x$. We say that a representation is (*globally*) *effective* if it is effective at each base point. We call a proper Lie groupoid *parareflexive* if it admits enough effective representations, i.e. for each base point x there exists some representation which is effective at x.

We ask the following questions:

(1) For an arbitrary connected proper Lie groupoid, does the property of reflexivity imply that the groupoid is Morita equivalent to the translation groupoid associated with a smooth, compact Lie group action?
(2) Is every proper Lie groupoid parareflexive? If not, is there a simple explicit characterization of parareflexivity for proper Lie groupoids?
(3) Does every parareflexive proper Lie groupoid admit a globally effective representation?

Our interest in these problems comes from our own investigations into the theory of representations of Lie groupoids on vector bundles [11]. The property of reflexivity is, for a proper Lie groupoid $\mathcal{G}$, equivalent to the property that the *Tannakian bidual* $\mathcal{T}(\mathcal{G})$ exists (as a Lie groupoid) and is isomorphic to $\mathcal{G}$; compare [11, Definition 2.6 and Theorem 2.9]. Parareflexive proper Lie groupoids, on the other hand, are precisely those proper Lie groupoids $\mathcal{G}$ whose Tannakian bidual $\mathcal{T}(\mathcal{G})$ exists (as a Lie groupoid) possibly without being isomorphic to $\mathcal{G}$; what one can say, in this case, is that there is a canonical surjective submersion of Lie groupoids from $\mathcal{G}$ onto $\mathcal{T}(\mathcal{G})$, and that $\mathcal{G}$ and $\mathcal{T}(\mathcal{G})$ have isomorphic categories of representations on smooth vector bundles [11, Theorem 4.11 and Proposition 3.2].

All examples of connected, reflexive, proper Lie groupoids known to us are, up to Morita equivalence, of the form $G \ltimes X$ for some smooth action of a compact Lie group G on a manifold X; conversely, any Lie groupoid of this form must be reflexive. We are led, therefore, to question (1). Any related progress would constitute an advance in the overall understanding of the duality theory of proper Lie groupoids.

Proper Lie groupoids need not be reflexive [11, Example 2.10]. This unfortunate circumstance makes parareflexivity a highly nontrivial property. One can show that all regular proper Lie groupoids are parareflexive [11, Corollary 4.12] although possibly not reflexive. We know at present of no example of a proper Lie groupoid that is not parareflexive. For this reason we ask question (2).

Finally, in the connected case, question (3) is essentially a reformulation of question (1). One easily sees that the existence of a globally effective representation implies, for an arbitrary (connected) proper Lie groupoid, that the groupoid is Morita equivalent to an extension of the form

$$1 \to \mathcal{B} \hookrightarrow \mathcal{G} \twoheadrightarrow G \ltimes X \to 1 \tag{1}$$

where $\mathcal{B}$ is a bundle of compact Lie groups and G is a compact Lie group acting smoothly on a manifold X. (More information can be found in the proof of Corollary 4 below.) Any progress with question (3) would lead to interesting results about the global topological structure of proper Lie groupoids.

We next proceed to describe the result contained in this note. Let us say that a proper Lie groupoid $\mathcal{G}$ has *orbit codimension at most n* if every $\mathcal{G}$-orbit is a submanifold (of the base manifold of $\mathcal{G}$) of codimension $\leqq n$.

Theorem 3. *Let $\mathcal{G}$ be a connected, proper Lie groupoid having orbit codimension two at most. Then $\mathcal{G}$ admits a globally effective representation. In particular, every proper Lie groupoid having orbit codimension at most two is parareflexive.*

We are not interested in claims to originality. This result simply represents a first step in the analysis of the problems proposed above. Even in such a simple case, there seem to be technical details which must be carried out, and we believe it useful for future investigations to write these down carefully. Note that the theorem implies that if a counterexample to parareflexivity is ever to be found, then it will involve a proper Lie groupoid whose base manifold will be three-dimensional at least.

Corollary 4. *Every Lie groupoid as in the theorem fits, possibly after replacement with a Morita equivalent Lie groupoid, in a short exact sequence of the form* (1)*.*

Proof. Let, more generally, $\mathcal{G}$ be an arbitrary connected proper Lie groupoid admitting a globally effective representation $\chi : \mathcal{G} \to GL(E)$.

[1] Roughly speaking, this is the normal subgroup of $\mathcal{G}_x$ formed by those arrows whose transversal infinitesimal action on the base manifold of $\mathcal{G}$ is trivial; the precise definition may be found at the end of Section 2.

A fortiori, existence of a globally effective representation implies parareflexivity. Hence, by [11, Definition 3.1, Theorem 4.11], the Tannakian bidual $\mathcal{T}(\mathcal{G})$ exists as a Lie groupoid and the canonical homomorphism $\pi : \mathcal{G} \to \mathcal{T}(\mathcal{G})$ is an epimorphism (that is, a surjective and submersive homomorphism) of Lie groupoids over the same base manifold. By the first homomorphism theorem for Lie groupoids and by the definition of π [11, 2.4], there exists a unique representation $\widetilde{\chi} : \mathcal{T}(\mathcal{G}) \to GL(E)$ such that $\widetilde{\chi} \circ \pi = \chi$.

We contend that $\widetilde{\chi}$ is also globally effective. This is essentially a direct consequence of the definitions (cf. Section 2) and of the fact that π induces the identity map on the level of base points. One easily checks that for every base point x, the restriction $\pi_x : \mathcal{G}_x \to \mathcal{T}(\mathcal{G})_x$ maps the ineffective part of $\mathcal{G}_x$ onto that of $\mathcal{T}(\mathcal{G})_x$, and the kernel of $\chi_x : \mathcal{G}_x \to GL(E_x)$ onto that of $\widetilde{\chi}_x$.

Now $\mathcal{T}(\mathcal{G})$ is, by its very definition [11, 1.5], a reflexive (proper) Lie groupoid. For any connected, reflexive proper Lie groupoid, the existence of a globally effective representation evidently implies also the existence of a (globally) faithful representation. By a well-known result (e.g., see [12, §5]), this implies in turn that the Tannakian bidual $\mathcal{T}(\mathcal{G})$ must be Morita equivalent to a translation groupoid $G \ltimes X$, with G compact. The rest of the proof is obvious. □

Structure of the paper. The paper is organized as follows. In the next section we recall the necessary background on Lie groupoids and fix some basic notations. Section 3 outlines the proof of the theorem. Technical details are worked out in Section 4 and in Appendix A. The numbering of definitions, lemmas, remarks, etc., is uniform throughout the paper and independent of the subdivision into sections.

2. Background and notations

This section contains a quick review of basic notions. For more information, we refer the reader to Chapters 5–6 of the classical textbook by Moerdijk and Mrčun [4].

Let $\mathcal{G} = \{G_1 \overset{\boldsymbol{s}}{\underset{\boldsymbol{t}}{\rightrightarrows}} G_0\}$ be a Lie groupoid. The manifold G_0 is called the base of $\mathcal{G}$. We let $\mathcal{G}(x, x')$ denote the set of all arrows whose source is the object x and whose target is the object x', and we use the abbreviation $\mathcal{G}_x$ for the isotropy group $\mathcal{G}(x, x)$. We will not make any distinction, notationally, between an object and the corresponding unit arrow. We write $g'g$ for the composition of two arrows, and g^{-1} for the inverse. One says that $\mathcal{G}$ is *compact* when G_1 is a compact Hausdorff manifold. However, the appropriate generalization to Lie groupoids of the notion of compactness is the following notion of properness: $\mathcal{G}$ is *proper* when G_1 is Hausdorff and the map $G_1 \to G_0 \times G_0$ which sends $g \mapsto (\boldsymbol{s}(g), \boldsymbol{t}(g))$ is proper in the usual sense.

Any Lie group G is an example of a Lie groupoid if we take $G_1 = G$ and $G_0 = \star$ (a single point). Any smooth manifold M can be viewed as a Lie groupoid by taking $G_1 = G_0 = M$. Less trivial examples, which play a central role in the present paper, are the so-called linear groupoids and the translation groupoids. Let E be a smooth, constant rank vector bundle over a smooth manifold M. The *linear groupoid* $GL(E)$ is the Lie groupoid with base M whose arrows from a base point x to another base point x' are the linear isomorphisms $E_x \xrightarrow{\sim} E_{x'}$ between the fibres of E corresponding to x and x'. Let G be a Lie group acting smoothly on a manifold M from the left. The *translation groupoid* $G \ltimes M$ is the Lie groupoid over M whose manifold of arrows is the Cartesian product $G \times M$, whose source and target are respectively the projection onto the second factor $(g, x) \mapsto x$ and the action $(g, x) \mapsto g \cdot x$, and whose composition law is $(g', x')(g, x) = (g'g, x)$. When G is compact, $G \ltimes M$ is proper.

A Lie groupoid $\mathcal{G}$ is said to be *regular*, when the anchor map $\rho : \mathfrak{g} \to \boldsymbol{T}M$ of the Lie algebroid of $\mathcal{G}$ has locally constant rank as a morphism of vector bundles over M. If $\mathcal{G}$ is regular then the image of the anchor map ρ is a subbundle of the tangent bundle of M which is also integrable and hence determines a foliation of M.

A homomorphism of Lie groupoids is a smooth functor. More precisely, a homomorphism $\phi : \mathcal{G} \to \mathcal{H}$ consists of a pair of smooth maps $\phi_0 : G_0 \to H_0$ and $\phi_1 : G_1 \to H_1$ compatible with the groupoid structures as in the usual definition of a functor. When $\mathcal{H} = GL(E)$ is the linear groupoid associated with a vector bundle E over G_0 and ϕ_0 is the identity map on G_0, we call ϕ a *representation* of $\mathcal{G}$. The intuition behind this is that ϕ represents each arrow $g \in \mathcal{G}(x, x')$ by a linear isomorphism $\phi(g) : E_x \to E_{x'}$ in a smooth, functorial way. When $\mathcal{G} = G$ is a Lie group, one recovers the usual notion of a representation on a finite-dimensional vector space.

For each arrow $g \in \mathcal{G}_x$ in the isotropy group of a Lie groupoid $\mathcal{G}$ at a base point x one obtains a well defined linear automorphism of the quotient tangent space $N_x := \boldsymbol{T}_x(X)/\boldsymbol{T}_x(O_x)$ (consisting of all tangent vectors "orthogonal" to the $\mathcal{G}$-orbit

$$O_x := \mathcal{G} \cdot x = \{\boldsymbol{t}(g) \mid g \in G_1 \text{ with } \boldsymbol{s}(g) = x\})$$

by first choosing an arbitrary local bisection $\sigma : U \hookrightarrow \mathcal{G}$, $\boldsymbol{s} \circ \sigma = id_U$, $\boldsymbol{t} \circ \sigma : U \xrightarrow{\sim} U'$ with $\sigma(x) = g$ and then taking the quotient linear map induced on N_x by the tangent map $\boldsymbol{T}_x(\boldsymbol{t} \circ \sigma)$. In fact, one obtains a continuous representation of the Lie group $\mathcal{G}_x$ on the vector space N_x, which we shall denote by μ_x. When g belongs to $\operatorname{Ker} \mu_x$ we say that the arrow g is *ineffective*. The subgroup of $\mathcal{G}_x$ formed by all the ineffective arrows will be called the *ineffective part* of $\mathcal{G}_x$.

3. Outline of the proof

Let $\mathcal{G} \rightrightarrows M$ be a (connected) proper Lie groupoid. (Recall that a Lie groupoid is said to be *connected* when its orbit space is a connected topological space.)

We shall let M^{pr} denote the set of all base points $x \in M$ such that the whole isotropy group of $\mathcal{G}$ at x is ineffective, i.e., such that $\ker \mu_x = \mathcal{G}_x$. From the explicit description of μ_x given above it follows immediately that M^{pr} is a $\mathcal{G}$-invariant subset of M. Moreover, by the properness of $\mathcal{G}$, M^{pr} must be open; this is an obvious consequence of the local linearizability theorem for proper Lie groupoids [8, Section 7.4, p. 218]. We shall denote by $\mathcal{G}^{\mathrm{pr}} \rightrightarrows M^{\mathrm{pr}}$ the Lie groupoid induced on M^{pr} by restriction, and call this the *principal part* of $\mathcal{G}$. Our terminology here is in agreement with Bredon's [13, §IV.3 and especially Theorem IV.3.2(iii)].

Incidentally, we observe that M^{pr} must be dense in M. This follows immediately from the local linearizability theorem and [13, Theorem IV.3.1]; for any linear slice $i : V \hookrightarrow M$, $M^{\mathrm{pr}} \cap i(V)$ must be relatively dense in $i(V)$.

Of course $\mathcal{G}^{\mathrm{pr}} \rightrightarrows M^{\mathrm{pr}}$ is a regular proper Lie groupoid. Hence the isotropy bundle $I(\mathcal{G}^{\mathrm{pr}})$ is an embedded submanifold of $\mathcal{G}^{\mathrm{pr}}$ and, in fact, a locally trivial bundle of compact Lie groups over M^{pr} [14]. Note that by the local linearizability theorem and by [13, Theorem IV.3.1], connectedness of $\mathcal{G}$ implies connectedness of $\mathcal{G}^{\mathrm{pr}}$, so the fibres of $I(\mathcal{G}^{\mathrm{pr}}) \to M^{\mathrm{pr}}$ are all isomorphic (as Lie groups).

5. *Component representation.* Let $\mathcal{K} \xrightarrow{s=t} M$ be any bundle of compact Lie groups. Suppose that all the isotropy groups of $\mathcal{K}$ are isomorphic to a fixed compact Lie group K. Let $\pi_0(K)$, and, for each $x \in X$, $\pi_0(\mathcal{K}_x)$, denote the finite group obtained by factoring out the identity connected component. Then there is a canonical representation of $\mathcal{K}$ on a vector bundle $R = R^{\mathcal{K}}$ over X, of rank equal to the order N of $\pi_0(K)$, to be called the *component representation* of $\mathcal{K}$ and to be denoted by $\varrho = \varrho^{\mathcal{K}}$, defined as follows.

For each base point $x \in X$, let

$$R_x := C^0\big(\pi_0(\mathcal{K}_x), \mathbb{C}\big) \approx \mathbb{C}^N \tag{2}$$

be the vector space of all complex functions on $\pi_0(\mathcal{K}_x)$. The local triviality of $\mathcal{K} \to X$ yields an evident smooth vector bundle structure on $R = \coprod R_x \to X$. Let $k \in \mathcal{K}_x$ act on R_x by right translation: for all $f : \pi_0(\mathcal{K}_x) \to \mathbb{C}$,

$$\varrho(k)(f) := f \circ \pi_0\big(\tau^k\big) \tag{3}$$

where $\pi_0(\tau^k)$ denotes the permutation on $\pi_0(\mathcal{K}_x)$ induced by the right translation diffeomorphism $\tau^k : h \mapsto hk$ of $\mathcal{K}_x$ onto itself.

Lemma 6. *Let $\mathcal{H} \rightrightarrows B$ be a connected, principal, proper Lie groupoid. Suppose that the orbit foliation of the base manifold B has codimension one. Then there exists a representation $\eta : \mathcal{H} \to GL(E)$ which induces on the isotropy bundle $I(\mathcal{H}) \to B$, locally over B and up to isomorphism, the component representation $\varrho^{I(\mathcal{H})}$.*

Proof. Since $\mathcal{H} \rightrightarrows B$ is a principal Lie groupoid, the orbit space $X = B/\mathcal{H}$, endowed with the evident functional structure [13, §VI.1], is a smooth manifold. In fact, any principal linear slice $S \subset B$ determines a C^∞ parameterization of the open subset $\phi(S)$ of X via the quotient projection $\phi : B \to X$; hence, by our assumptions, X is one-dimensional. Moreover, since $\mathcal{H}$ is proper, X must be Hausdorff.

Select a sequence $\{S_i\}_{i=1,2,\ldots}$ of principal linear slices ($S_i \approx \mathbb{R}$ and $\mathcal{H}|_{S_i} \approx H \times \mathbb{R}$ for some compact Lie group H) with $B = \bigcup_{i=1}^\infty \mathcal{H} \cdot S_i$. For each k, put $B_k := \bigcup_{i=1}^k \mathcal{H} \cdot S_i$. By rearranging the sequence if necessary, we may assume $S_{k+1} \cap B_k \neq \emptyset$ for all k (here we use the connectedness of $\mathcal{H}$).

Next, select an invading sequence of open subsets

$$\cdots \subset V_p \subset \overline{V}_p \subset V_{p+1} \subset \cdots \subset B$$

with $\phi(\overline{V}_p)$ compact. It is no loss of generality to assume V_p to be invariant. For each k, let $p(k)$ be the greatest $p \leqq k$ such that $\overline{V}_p \subset B_k$.

(Inductive step.) Suppose a representation $\eta_k : \mathcal{H}_k \to GL(E_k)$ as in the statement of the lemma has been constructed for $\mathcal{H}_k := \mathcal{H}|_{B_k}$. Put $S := S_{k+1}$ and $V := V_{p(k)}$. By the initial remarks, the intersection $S \cap B_k$ is of the form $(-\infty, a) \cup (b, \infty)$ or (a, b) in any parameterization $S \approx \mathbb{R}$. It is then clear that we can find open subsets $\Sigma \subset S$ and $U \subset B_k$ with

$$\overline{V} \subset U, \qquad \overline{V} \cap \Sigma = \emptyset, \qquad U \cup (\mathcal{H} \cdot \Sigma) = B_k \cup (\mathcal{H} \cdot S)$$

so that there is an isomorphism of representations on $U \cap \Sigma$ between E_k and R. By a standard Morita equivalence argument, we obtain a representation $\eta_{k+1} : \mathcal{H}_{k+1} \to GL(E_{k+1})$ which still satisfies the inductive hypothesis and moreover is globally isomorphic to η_k over V.

Finally, define the representation η on E to be the inductive limit of the partial representations $\eta_k|V_{p(k)}$ on $E_k|V_{p(k)}$. $\square$

Remark 7. The above proof makes essential use of the 'codimension = 1' hypothesis and of the fact that the component representation is defined intrinsically in terms of the groupoid structure.

8. Let $\mathcal{G} \rightrightarrows M$ be a connected, proper Lie groupoid. Assume that the orbits of $\mathcal{G}$ have codimension at most two – in other words, that

$$\sup_{m \in M} \dim(\boldsymbol{T}_m M/\boldsymbol{T}_m O_m) \leqq 2. \tag{4}$$

By the connectedness of $\mathcal{G}$, the dimension of the normal space $N_m := \boldsymbol{T}_m M/\boldsymbol{T}_m O_m$ is the same for all $m \in M^{\mathrm{pr}}$. We shall call the common dimension of these spaces the *principal codimension* of $\mathcal{G}$. The latter equals the dimension of the manifold $X^{\mathrm{pr}} := M^{\mathrm{pr}}/\mathcal{G}^{\mathrm{pr}}$. Observe that X^{pr} is a dense open subspace of the orbit space $X = M/\mathcal{G}$.

We will now indicate how to obtain an effective representation of $\mathcal{G}$ in different ways, according to whether the principal codimension of the groupoid equals zero, one, or two.

9. *Principal codimension zero.* In this case the singular set $M^! := M \setminus M^{\mathrm{pr}}$ is empty and $\mathcal{G}$ is transitive. Hence there actually is a faithful representation, by Morita equivalence to a compact Lie group.

10. *Principal codimension one.* Let $m \in M^!$ be a singular base point, and consider any linear slice at m

$$\mathbb{R}^d \approx V \subset M, \qquad \mathcal{G}|_V \approx G \ltimes \mathbb{R}^d$$

for some let us say orthogonal action $G \to GL(\mathbb{R}^d)$ of the isotropy group $G := \mathcal{G}_m$. One has the following possibilities:

(i) $d = 1$, and $G \to O(\mathbb{R}) = \{\pm Id\}$ is a nontrivial action with orbits $\{\pm t\}$ where $t \in \mathbb{R}$;
(ii) $d = 2$, and $G \to O(2) := O(\mathbb{R}^2)$ is a *spherical* action, i.e. one whose orbits are the circles $x^2 + y^2 = r^2$, $r \geqq 0$.

From this remark it follows in particular that $X^! := X \setminus X^{\mathrm{pr}}$ is a discrete subset of the orbit space $X = M/\mathcal{G}$ (that is, all its points are isolated in X). Hence we can assign each $x \in X^!$ an open neighbourhood Ω_x so that $\Omega_x \cap \Omega_{x'} = \emptyset$ for $x \neq x' \in X^!$. Let us fix for each $x \in X^!$ some $m := m_x \in M^!$ with $x = \phi(m)$ where $\phi : M \to X$ denotes the projection onto the quotient, and some (orthogonal) linear slice $V := V_x$ at m with $\phi(V) \subset \Omega_x$.

We contend that for each orthogonal compact Lie group action $G \to O(\mathbb{R}^d)$ as in (i) or (ii) one can find an effective representation $\Phi : G \ltimes \mathbb{R}^d \to GL(\underline{\mathbb{C}}^{2N})$ (where $\underline{\mathbb{C}}^{2N} = \mathbb{R}^d \times \mathbb{C}^{2N}$) so that its restriction to the isotropy bundle over $(\mathbb{R}^d)^{\mathrm{pr}} = \mathbb{R}^d \setminus 0$ is locally isomorphic to twice the component representation. This is clear in the case (i) (compare Remark 12 below) and shall be proved in the next section for actions of type (ii). Taking this for granted, fix one such representation for each linear slice V_x and call it Φ_x. Choose also a representation $\eta : \mathcal{G}^{\mathrm{pr}} \to GL(E)$ as in the statement of Lemma 6. Then there are invariant open subsets $U \subset M^{\mathrm{pr}}$ and $B_x \subset V_x$ such that

$$U \cup \bigcup \{U_x \colon x \in X^!\} = M$$

where $U_x := \mathcal{G} \cdot B_x$ and moreover $U \cap U_x = \mathcal{G} \cdot \Sigma$ for some $\Sigma := \Sigma_x \subset V_x$ of the form $(a, b) \times \mathbb{R}^{d-1} \subset \mathbb{R}^d$ ($a < b$ positive) with

$$\Phi_x|_\Sigma \approx R^{\mathcal{G}|_\Sigma} \oplus R^{\mathcal{G}|_\Sigma} \approx \eta|_\Sigma \oplus \eta|_\Sigma$$

as representations of $\mathcal{G}|_\Sigma = I(\mathcal{G}|_\Sigma)$. The usual Morita equivalence argument allows one to glue together these representations into a global representation which has to be isomorphic to $\eta \oplus \eta$ on U and to Φ_x on B_x and is therefore effective.

11. *Principal codimension two.* In this case the groupoid is regular. Hence the result follows immediately from [11, Corollary 4.12].

12. *Remark on spherical orthogonal actions of rank one.* Let $\mu : G \to O(1) = \{\pm I\}$ be a rank one orthogonal linear action of a compact Lie group G with spherical orbits $\{\pm t\}$. Put $K := \ker \mu$. Clearly $K = G_t$ for all $t \neq 0$ where G_t denotes the stabilizer at t. Also, the connected components of the identity in G_t and in G are the same; in symbols, $(G_t)^{(e)} = G^{(e)}$. Hence, it will be no loss of generality to assume that G is discrete.

Now let $\varrho^G : G \hookrightarrow GL(R^G)$ be the (right) regular representation of the (finite) group G. One has

$$R^G = C^0(G, \mathbb{C}) = C^0(K, \mathbb{C}) \oplus C^0(G \setminus K, \mathbb{C}) \approx R^K \oplus R^K$$

equivariantly as K-modules. The obvious extension of ϱ^G

$$\Phi : G \ltimes \mathbb{R} \to GL\big(\mathbb{R} \times R^G\big)$$

is then an effective (faithful) representation whose restriction to the isotropy subbundle over $\mathbb{R} \setminus 0$ is isomorphic to twice the component representation.

4. Study of rank two spherical actions

Let G be a compact Lie group acting orthogonally and spherically on $\mathbb{R}^2$. In other words, let a continuous homomorphism $\mu : G \to O(2) = O(\mathbb{R}^2)$ be given such that the orbits of the corresponding action $G \circlearrowright \mathbb{R}^2$ are the circles

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2\}, \quad r \geqq 0.$$

Our goal in this section is to construct an effective representation $\Phi : G \ltimes \mathbb{R}^2 \to GL(\underline{\mathbb{C}}^{2N})$ such that its restriction to the isotropy bundle over $\mathbb{R}^2 \setminus 0$ is locally isomorphic to the direct sum of two copies of the component representation. Clearly, it will be enough to find a continuous homomorphism of groups $\varphi : G \to GL(\mathbb{C}^{2N})$ such that $\ker \varphi \subset K$ and $\varphi|_{G_1} \approx \varrho^{G_1} \oplus \varrho^{G_1}$ where G_1 denotes the stabilizer subgroup at $(1, 0) \in \mathbb{R}^2$. For then Φ, defined by

$$\Phi(g, z) \cdot (z, v) := \big(\mu(g)(z), \varphi(g)v\big),$$

will have the desired properties.

We start with some remarks. Let $G^{(e)}$ denote the identity component of G and $\mu^{(e)} : G^{(e)} \to SO(2)$ the restriction of μ to $G^{(e)}$. Our first remark is that $G^{(e)} \cap K = G^{(e)} \cap G_1$ where $K = \ker \mu$ and G_1 denotes the stabilizer at $(1, 0) \in \mathbb{R}^2$. Indeed, for every $x \in G^{(e)}$ the matrix $\mu^{(e)}(x) \in SO(2)$ has no nonzero fixed vectors unless $x \in K$. Next, we observe that the composite $G_1 \subset G \to G/G^{(e)}$ is a surjective homomorphism of groups. Indeed, if U is a connected component of G and $g_0 \in U$, then ${g_0}^{-1} \cdot U = G^{(e)}$ and hence

$$U \cdot (1, 0) = g_0 \cdot G^{(e)} \cdot (1, 0) = g_0 \cdot S^1 = S^1 \quad \text{by sphericity,}$$

so there exists some $g \in U$ with $g \cdot (1, 0) = (1, 0)$; this says that $U \cap G_1 \neq \emptyset$. From the first remark it follows that $(G_1)^{(e)} \subset G^{(e)} \cap K$ and therefore that $(G_1)^{(e)} = (G^{(e)} \cap K)^{(e)}$ is a normal subgroup of G contained in K. Hence by factoring out $(G_1)^{(e)}$ we are reduced to the special case where the stabilizer G_1 is finite. This will be our assumption for the remainder of the section.

By sphericity, the homomorphism $\mu^{(e)} : G^{(e)} \to SO(2) = S^1$ must be surjective. In particular, $\mu^{(e)}$ must be nontrivial, hence submersive. The kernel $\ker \mu^{(e)} = G^{(e)} \cap K = G^{(e)} \cap G_1$ is finite. It follows that $\mu^{(e)}$ is a finite-sheeted covering of S^1 and that $G^{(e)}$ is one-dimensional. Let $\alpha : \mathbb{R} \to G^{(e)}$ be the unique Lie group homomorphism such that $\mu^{(e)} \circ \alpha = \exp : \mathbb{R} \to S^1$. Being surjective, α induces an isomorphism of Lie groups $S^1 \approx G^{(e)}$. Thus the subgroup $C := G^{(e)} \cap K \subset G^{(e)}$ must be cyclic. Let us pick a generator:

$$C = \langle c_0 \rangle = \{c_0, {c_0}^2, \dots, {c_0}^q = e\} \tag{5}$$

where $q = |C|$ is the order. Let H denote the kernel of the restriction $\mu|_{G_1} : G_1 \to O(2)$, and put $H' := G_1 \setminus H$. We will now show that for each $g \in G_1$

$$\begin{cases} gxg^{-1} = x \quad \forall x \in G^{(e)} & \text{when } g \in H, \\ gxg^{-1} = x^{-1} \quad \forall x \in G^{(e)} & \text{when } g \in H'. \end{cases} \tag{6}$$

To begin with, $G^{(e)} \approx S^1$ implies $\mathrm{Aut}(G^{(e)}) = \{id, \chi\}$ where χ stands for the inverse $x \mapsto x^{-1}$. Since $G^{(e)}$ is normal in G, each $g \in G_1$ acts on $G^{(e)}$ by conjugation and thus induces an automorphism $c_g \in \mathrm{Aut}(G^{(e)})$. Hence, either $gxg^{-1} = x$ for all $x \in G^{(e)}$, or $gxg^{-1} = x^{-1}$ for all $x \in G^{(e)}$. Suppose first $g \in H$. By definition of H, $\mu(g) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Therefore, if $gxg^{-1} = x^{-1}$ $\forall x \in G^{(e)}$ then $\mu(x)^{-1} = \mu(x)$ $\forall x \in G^{(e)}$ and hence, by connectedness, $\mu(x) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ $\forall x \in G^{(e)}$, which contradicts the finiteness of $G^{(e)} \cap K$. Suppose on the other hand that $g \in H'$. Then $\mu(g) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. If one assumes that $gxg^{-1} = x$ $\forall x \in G^{(e)}$ then one gets a contradiction as before.

The induced representation $\mu|_{G_1} : G_1 \to O(2)$ embeds into the direct sum of two copies of the (real) regular representation $R \oplus R := R^{G_1} \oplus R^{G_1}$ as the two-dimensional submodule

$$\mathrm{T} := \mathrm{T}_1 \oplus \mathrm{T}_2 := \left\langle \sum_{g \in G_1} \boldsymbol{e}_g \right\rangle \oplus \left\langle \sum_{h \in H} \boldsymbol{e}_h - \sum_{h' \in H'} \boldsymbol{e}_{h'} \right\rangle \subset R \oplus R \tag{7}$$

where $\boldsymbol{e}_g$ denotes the standard basis vector given by the function with value one at $g \in G_1$ and zero everywhere else. Identify $R := C^0(G_1, \mathbb{R}) = L^2(G_1, \mathbb{R})$ where G_1 is given the probability Haar measure. The orthogonal complement of the submodule (7) is then

$$\Lambda := \Lambda_1 \oplus \Lambda_2 := \left\{ f \in C^0(G_1) \,\middle|\, \sum_{g \in G_1} f(g) = 0 \right\} \oplus \left\{ f \in C^0(G_1) \,\middle|\, \sum_{h \in H} f(h) = \sum_{h' \in H'} f(h') \right\} \subset R \oplus R \tag{8}$$

where $R = \mathrm{T}_i \oplus \Lambda_i$ $(i = 1, 2)$.

We now proceed to show how to extend the action of G_1 on Λ to an action of the whole group. (In the end we shall define φ as the complexified direct sum of μ and this extended representation.) We distinguish two cases.

13. *The odd order case.* Suppose $q = |C| = 2p + 1$ for some $p \in \mathbb{N}$. Let us put $N = |G_1|$ (recall that G_1 is a finite group) and $n = N - 1$.

Choose an arbitrary orthonormal basis for the G_1-submodule (8) $\Lambda \subset R \oplus R$, and let $\Pi_1 : G_1 \to O(2N-2) = O(2n)$ be the corresponding orthogonal representation. Note that N coincides with the rank of the G_1-module R. We contend that Π_1 can be extended to a continuous representation $\Pi : G \to O(2n)$. Of course, we may assume $n \geqq 1$.

To begin with, we remark that $\Pi_1(C) \subset SO(2n)$. Indeed, on the one hand, from the inclusion $C = G^{(e)} \cap K \subset K$ we deduce that $\det \mu(c) = \det(id) = +1$ for all $c \in C$. On the other hand, from the identity (of G_1-modules) $\mathrm{T} \oplus \Lambda = R \oplus R$ we obtain

$$\det \mu(c) \det \Pi_1(c) = \det \begin{bmatrix} \mu(c) & 0 \\ 0 & \Pi_1(c) \end{bmatrix} = \det \begin{bmatrix} \varrho(c) & 0 \\ 0 & \varrho(c) \end{bmatrix} = \left[\det \varrho(c)\right]^2 = +1$$

(because $\varrho : G_1 \to O(N)$ is orthogonal) so that, in fact, $\det \Pi_1(c) = +1$ for all $c \in C$.

Let c_0 be the chosen generator for the cyclic group C as in (5). Put $P_0 = \Pi_1(c_0) \in SO(2n)$. Then ${P_0}^q = I_{2n}$ (identity $2n \times 2n$ matrix). Every element of $SO(2n)$ is conjugated in $SO(2n)$ to an element of the standard maximal torus $T(n)$ consisting of all block-diagonal matrices of the form

$$R(\theta_1, \ldots, \theta_n) := \mathrm{diag}\big(R(\theta_1), \ldots, R(\theta_n)\big), \quad \theta_1, \ldots, \theta_n \in \mathbb{R},$$

where $R(\theta)$ denotes the 2×2 matrix $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right) \in SO(2)$. Compare [15, Theorem (3.4)]. Thus, at the expense of replacing the representation Π_1 by an orthogonally equivalent one, we may assume that $P_0 = R(\theta_1, \ldots, \theta_n)$ for some real numbers $-\pi \leqq \theta_1, \ldots, \theta_n < \pi$.

We shall presently prove the existence of an extension $\Pi : G \to O(2n)$ under the assumption that $-\pi < \theta_1, \ldots, \theta_n < \pi$. This assumption is certainly true when q is odd, because of the identity ${P_0}^q = I_{2n}$. By the previously noticed surjectivity of the map $G_1 \subset G \twoheadrightarrow G/G^{(e)}$, it will be enough to show that there is a continuous group homomorphism $\Upsilon : G^{(e)} \to SO(2n)$ which restricts to Π_1 on $C = G^{(e)} \cap G_1 \subset G^{(e)}$ and satisfies the equation

$$\Upsilon\big(g_1 x {g_1}^{-1}\big) = \Pi_1(g_1) \Upsilon(x) \Pi_1(g_1)^{-1} \quad \forall x \in G^{(e)}\ \forall g_1 \in G_1, \tag{9}$$

for then we can define $\Pi(g) := \Pi_1(g_1)\Upsilon(x)$ for all $g = g_1 x \in G$. In order to construct Υ, we observe that, by our assumption on $\theta_1, \ldots, \theta_n$ and by Proposition 15 in combination with Remark 16 (see Appendix A below), there exists a unique one-parameter subgroup

$$\alpha : \mathbb{R} \to SO(2n) \quad \text{with} \quad \alpha(1) = R(\theta_1, \ldots, \theta_n) = P_0, \quad \big\|\dot\alpha(0)\big\| < \pi, \tag{10}$$

namely $t \mapsto \exp[tX(\theta_1, \ldots, \theta_n)]$ (the notations of Appendix A are in use). As observed above, for each $g_1 \in G_1$ we must have $g_1 x {g_1}^{-1} = x^{\pm 1}$ for all $x \in G^{(e)}$, the sign $\pm$ depending only on g_1. Accordingly,

$$\alpha(1)^{\pm 1} = \Pi_1(g_1) \alpha(1) \Pi_1(g_1)^{-1}.$$

Now, for each orthogonal matrix $R \in O(2n)$ the curves $t \mapsto R\alpha(t)R^{-1}$ and $t \mapsto \alpha(t)^{-1}$ define one-parameter subgroups $R\alpha R^{-1}$ and α^{-1} in $SO(2n)$ for which the estimates

$$\big\|\big(R\alpha R^{-1}\big)'(0)\big\| = \big\|\mathrm{Ad}(R) \cdot \alpha'(0)\big\| \leqq \big\|\alpha'(0)\big\| < \pi$$

and $\|(\alpha^{-1})'(0)\| = \|-\alpha'(0)\| < \pi$ still hold, by Remark 17. By the uniqueness argument mentioned before, we see that

$$\alpha(t)^{\pm 1} = \Pi_1(g_1) \alpha(t) \Pi_1(g_1)^{-1} \quad \forall t \in \mathbb{R}.$$

14. *The even order case.* Suppose $q = |C| = 2p$, $1 \leqq p \in \mathbb{N}$. Define N and n as before. For each $g_1 \in G_1$, introduce the following subspace of $R = C^0(G_1)$

$$\Theta_{g_1} := \left\{ f \in C^0(G_1) \;\middle|\; \sum_{i=1}^{q} f\big(g_1 {c_0}^i\big) = 0,\ \mathrm{supp}\, f \subset g_1 C \right\}, \tag{11}$$

c_0 being the chosen generator for the cyclic group C. Since $g_1 C \subset H$ when $g_1 \in H$, and $g_1 C \subset H'$ when $g_1 \in H'$, Θ_{g_1} is always a subspace of both Λ_1 and Λ_2 and hence so is

$$\Theta := \bigoplus_{\tilde{g}_1 \in G_1/C} \Theta_{g_1} \subset R. \tag{12}$$

The dimension of Θ is $[G_1 : C](q-1) = (N/q)(q-1)$. The orthogonal complement of Θ_i in Λ_i (where the notation Θ_i simply indicates that we regard Θ as a subspace of Λ_i) is the subspace ${\Theta_i}' = \{f \in \Lambda_i \mid f(g_1) = f(g_1 c_0)\ \forall g_1 \in G_1\}$ (of dimension $N/q - 1$). Now,

(i) the generator $c_0 \in C$ fixes each Θ_{g_1}, hence in particular the whole of Θ, and acts as the identity on the complement ${\Theta_i}'$;
(ii) for each $g, g_1 \in G_1$ and every $f \in C^0(G_1)$, one has $f \in \Theta_g \Leftrightarrow g_1 \cdot f \in \Theta_{g{g_1}^{-1}}$.

The second property follows, by (6), from the fact that $g_1{c_0}^i{g_1}^{-1} = {c_0}^{\pm i}$ (the sign being a plus or a minus according to whether $g_1 \in H$ or $g_1 \in H'$).

Next, let us concentrate on the action of c_0 on a single subspace Θ_g, for $g \in G_1$ fixed. The (-1)-eigenspace for the action of c_0 on Θ_g

$$\Lambda_g := \langle \lambda_g \rangle := \langle \boldsymbol{e}_g - \boldsymbol{e}_{gc_0} + \cdots + \boldsymbol{e}_{g{c_0}^{q-2}} - \boldsymbol{e}_{g{c_0}^{q-1}} \rangle = \left\langle \sum_{i=1}^{q} (-1)^i \boldsymbol{e}_{g{c_0}^i} \right\rangle \subset \Theta_g \tag{13}$$

is one-dimensional. Note that since $q = 2p$ is even, the (-1)-eigenvector $\lambda_g \in \Theta_g$ possesses the axial symmetry

$$\lambda_g\big(g{c_0}^i\big) = \lambda_g\big(g{c_0}^{-i}\big)$$

so that $g_1 \cdot \lambda_g = \lambda_{g{g_1}^{-1}}$ for all $g, g_1 \in G_1$. Consider the orthogonal decomposition $\Theta_g = \Lambda_g \oplus \Lambda'_g$. Any $g_1 \in G_1$ will map each complement Λ'_g bijectively onto $(\Lambda_{g{g_1}^{-1}})'$. Introduce the vectors

$$\begin{cases} \lambda_1^g := \lambda_g & (g \in G_1), \\ \lambda_2^h := \lambda_h & (h \in H), \\ \lambda_2^{h'} := -\lambda_{h'} & \big(h' \in H'\big). \end{cases} \tag{14}$$

From the equation $g_1 \cdot \lambda_g = \lambda_{g{g_1}^{-1}}$ and the fact that $g{g_1}^{-1} \in H \Leftrightarrow$ both g and g_1 belong to H or both belong to H', we obtain the following transformation rules: (a) $g_1 \cdot \lambda_1^g = \lambda_1^{g{g_1}^{-1}}$ for all $g, g_1 \in G_1$; (b) $h \cdot \lambda_2^g = \lambda_2^{gh^{-1}}$ for all $h \in H$, $g \in G_1$; (b$'$) $h' \cdot \lambda_2^g = -\lambda_2^{g{h'}^{-1}}$ for all $h' \in H'$, $g \in G_1$. Put $\Lambda_i^g := \langle \lambda_i^g \rangle$ $(i = 1, 2)$ and

$$\Lambda^g := \Lambda_1^g \oplus \Lambda_2^g \subset \Lambda. \tag{15}$$

On this two-dimensional subspace of Λ, we fix once and for all the distinguished basis $\{(\lambda_1^g, 0), (0, \lambda_2^g)\}$.

We have a decomposition of Λ into G_1-invariant direct summands

$$\Lambda = \Bigg[\bigoplus_{\widetilde{g}_1 \in G_1/C} \Lambda^{g_1}\Bigg] \oplus \Bigg[\bigoplus_{\widetilde{g}_1 \in G_1/C} \big({\Lambda_{g_1}}' \oplus \Lambda'_{g_1}\big)\Bigg] \oplus \big[\Theta'_1 \oplus \Theta'_2\big]. \tag{16}$$

As in the preceding subsection, we are given a representation $\Pi_1 : G_1 \to GL(\Lambda)$, namely the G_1-module Λ, and we would like to extend it to a continuous representation $\Pi : G \to GL(\Lambda)$ of the whole of G on the same vector space. Of course, we may deal with each direct summand separately.

The rightmost summand in (16) is a trivial G_1-module; so, on that summand, we may extend Π_1 by the trivial representation.

The middle summand falls into the case already studied in Subsection 13, because it corresponds to an orthogonal G_1-action in which c_0 acts with no -1 eigenvalue. By reasoning as before, one can obtain an extension on that summand by using the results of Appendix A.

On the first summand—call it W for brevity—one can construct an extension directly. Define a one-parameter subgroup $\alpha : \mathbb{R} \to GL(W)$ by

$$\alpha(t) := \bigoplus_{\widetilde{g}_1 \in G_1/C} \alpha^{g_1}(t) \tag{17}$$

where $\alpha^{g_1}(t) \in GL(\Lambda^{g_1})$ is the linear map represented by the 2×2 matrix $\left(\begin{smallmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{smallmatrix}\right)$ with respect to the distinguished basis $\{(\lambda_1^{g_1}, 0), (0, \lambda_2^{g_1})\}$ for Λ^{g_1}. Note that the linear map $\alpha^{g_1}(t)$ is well defined, i.e., independent of the choice of a representative g_1 for the given coset $\widetilde{g}_1 \in G_1/C$; indeed $\lambda_i^{g_1 c_0} = -\lambda_i^{g_1}$ for $i = 1, 2$, so the distinguished basis associated to $g_1 c_0$ is minus that associated to g_1 and therefore the same matrix represents the same linear map in either basis. Observe that $\alpha(\pi) = -id = \Pi_1(c_0)|_W$. As in Subsection 13, we need to check that

$$\begin{cases} \alpha(t) = \Pi_1(h)\alpha(t)\Pi_1(h)^{-1} & \text{for } h \in H, \\ \alpha(t)^{-1} = \Pi_1(h')\alpha(t)\Pi_1(h')^{-1} & \text{for } h' \in H'. \end{cases} \tag{18}$$

For each $g_1, g \in G_1$, the linear automorphism $\Pi_1(g)|_W \in GL(W)$ maps the subspace Λ^{g_1} onto the subspace $\Lambda^{g_1 g^{-1}}$. On the other hand, $\alpha(t)$ maps Λ^{g_1} and $\Lambda^{g_1 g^{-1}}$ into themselves by construction. Thus we can check the identities (18) for $g = h, h'$ in the distinguished bases of Λ^{g_1} and $\Lambda^{g_1 g^{-1}}$. This is straightforward, in view of the transformation rules (a), (b) and (b$'$) stated immediately after (14).

Appendix A

The Lie group $G = SO(n)$ is semisimple for all $n \geqq 3$. In fact, $Z(SO(2p-1)) = \{I\}$ and $Z(SO(2p)) = \{\pm I\}$ for all $p \geqq 2$. Compare [15, Remark (3.14) p. 201], and [16, p. 102]. Thus Lazard–Tits' criterion [17, Théorème (2.1)], which requires the connected component of the centre $Z_0(G)$ to be simply connected, can be applied when $G = SO(n)$.

The Lie algebra $\mathfrak{g} = \mathfrak{so}(n)$ is given by the $n \times n$, skew-symmetric real matrices $X \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, ${}^tX = -X$ (the Lie bracket being the usual anti-commutator $[X,Y] := XY - YX$). Make the identification $\mathrm{Mat}_{n\times n}(\mathbb{R}) = \mathrm{End}(\mathbb{R}^n)$, put the euclidean norm on $\mathbb{R}^n$, and let $\|X\|$ denote the usual operator norm on $\mathrm{End}(\mathbb{R}^n)$. Since for this norm one has the inequality $\|XY\| \leqq \|X\|\|Y\|$, one obtains the following estimate for the Lie bracket

$$\big\|[X,Y]\big\| \leqq 2\|X\|\|Y\|. \tag{19}$$

We will see that the factor two here is actually the norm of the bilinear form $(X,Y) \mapsto [X,Y]$; in other words, (19) is the best estimate possible. If we put $|X| := 2\|X\|$ then the inequality (19) implies that the latter norm (on the Lie algebra $\mathfrak{g}$) is *admissible*, i.e. satisfies

$$\big|[X,Y]\big| \leqq |X||Y|. \tag{20}$$

Then it follows from [17, Théorème (2.1)] that the exponential mapping

$$X \mapsto \exp(X) = I + X + \frac{X^2}{2!} + \cdots$$

of the Lie algebra $\mathfrak{g} = \mathfrak{so}(n)$ into the respective Lie group $G = SO(n)$ is injective on the open ball $\{X \in \mathfrak{g}\colon |X| < \pi\} = \{X \in \mathfrak{g}\colon \|X\| < \pi/2\}$. It is our goal in the present appendix to show that the injectivity radius of the exponential mapping $\exp : \mathfrak{so}(n) \to SO(n)$ is actually twice as much.

Proposition 15. *The injectivity radius of the exponential mapping with respect to the admissible norm $|X| = 2\|X\|$ on the Lie algebra $\mathfrak{g} = \mathfrak{so}(n)$ of the special orthogonal group $G = SO(n)$ is exactly 2π.*

Equivalently, the injectivity radius of the exponential mapping with respect to the operator norm $\|X\|$ on $\mathfrak{g}$ is exactly π. This is actually the best we can hope for, in view of the first of the following two remarks (which we will need in the proof of the proposition).

Remark 16. The skew-symmetric, 2×2 matrix $X(\theta) := \left(\begin{smallmatrix} 0 & -\theta \\ \theta & 0 \end{smallmatrix}\right)$ exponentiates to $R(\theta) := \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$. More generally,

$$\exp\begin{pmatrix} X(\theta_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X(\theta_n) \end{pmatrix} = \begin{pmatrix} R(\theta_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R(\theta_n) \end{pmatrix}.$$

We shall let $X(\theta_1,\dots,\theta_n)$ denote the skew-symmetric, $2n \times 2n$ matrix on the right-hand side. Note that $\|X(\theta_1,\dots,\theta_n)\| = \max_{i=1,\dots,n} |\theta_i|$.

Remark 17. The orthogonal matrices $R \in O(n)$, $R \cdot {}^tR = I$ act on the Lie algebra $\mathfrak{so}(n)$ by $X \mapsto RXR^{-1}$. In fact, $RXR^{-1} = \mathrm{Ad}(R)\cdot X$ is precisely the adjoint representation $\mathrm{Ad} : O(n) \to GL(\mathfrak{so}(n))$. Notice that each orthogonal matrix $R \in O(n)$ preserves the euclidean distance as an operator in $\mathrm{End}(\mathbb{R}^n)$ and therefore $\|R\| = \|{}^tR\| = 1$. It follows that

$$\big\|\mathrm{Ad}(R)\cdot X\big\| = \big\|R \cdot X \cdot {}^tR\big\| \leqq \|R\|^2\|X\| = \|X\|, \tag{21}$$

hence for each $R \in O(n)$ the adjoint automorphism $\mathrm{Ad}(R)$ of $\mathfrak{g} = \mathfrak{so}(n)$ maps the open ball $B(0,r) \subset \mathfrak{g}$ of radius $r > 0$ into itself. The same remark applies of course to the other norm: $|\mathrm{Ad}(R)(X)| \leqq |X|$ for all $X \in \mathfrak{g}$, $R \in O(n)$.

Proof of Proposition 15. For simplicity we shall assume that is n even, as this is the only case of practical interest to us; there are obvious modifications for n odd. So, let $X, Y \in \mathfrak{g} = \mathfrak{so}(2n)$ be given with $|X|, |Y| < 2\pi$, and suppose $\exp(X) = \exp(Y) \in SO(2n)$.

Since the norm $|X|$ on $\mathfrak{g}$ is admissible, it follows from [17, Lemme (6.1)] that $[X,Y] = 0$. Therefore the linear subspace $\mathrm{Span}_{\mathbb{R}}\{X,Y\}$ is an abelian subalgebra of $\mathfrak{g}$ and hence there is a maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{g}$ such that $X, Y \in \mathfrak{t}$. Let $T = \exp(\mathfrak{t})$ be the maximal torus in $SO(2n)$ corresponding to this maximal abelian subalgebra. The one-parameter subgroups $\alpha_X := \{t \mapsto \exp(tX)\}$ and $\alpha_Y := \{t \mapsto \exp(tY)\}$ are contained in the maximal torus T. Since all maximal tori in a connected compact Lie group are conjugated to each other [15, Theorem (1.6) p. 159, and p. 5], there will be an element $g_0 \in SO(2n)$ with $g_0Tg_0{}^{-1} = T(n)$ where $T(n)$ denotes the standard torus in $SO(2n)$ consisting of all block-diagonal matrices of the form $R(\theta_1,\dots,\theta_n)$. Since by Remark 17 the linear automorphism $\mathrm{Ad}(g_0) \in GL(\mathfrak{g})$ maps the open ball $B(0,2\pi)$ into itself, if we put $X_0 = \mathrm{Ad}(g_0)(X)$ and $Y_0 = \mathrm{Ad}(g_0)(Y)$ we still have $|X_0|, |Y_0| < 2\pi$. Clearly, $X = Y$ if and only if $X_0 = Y_0$.

The one-parameter subgroups $\alpha_{X_0} = g_0 \alpha_X {g_0}^{-1}$ and $\alpha_{Y_0} = g_0 \alpha_Y {g_0}^{-1}$ (recall that by the naturality of the exponential mapping one has

$$\exp\bigl(\mathrm{Ad}(g_0) \cdot X\bigr) = g_0 \exp X {g_0}^{-1};$$

compare [15, (3.2) p. 23] for instance) are contained in the torus $T(n)$. Now, $\|X_0\|, \|Y_0\| < \pi$. From Remark 16 it follows that $X_0 = Y_0$ and hence that $X = Y$. □

References

[1] I. Moerdijk, Orbifolds as groupoids: An introduction, in: Contemp. Math., vol. 310, 2002, pp. 205–222.
[2] A. Henriques, D.S. Metzler, Presentations of noneffective orbifolds, Trans. Amer. Math. Soc. 356 (6) (2004) 2481–2499.
[3] A. Connes, Noncommutative Geometry, Academic Press, New York, 1994.
[4] I. Moerdijk, J. Mrčun, Introduction to Foliations and Lie Groupoids, Cambridge University Press, 2003.
[5] H.-H. Tseng, C. Zhu, Integrating Lie algebroids via stacks, Compos. Math. 142 (2006) 251–270.
[6] K. Behrend, P. Xu, Differentiable stacks and gerbes, arXiv:math/0605694 [math.DG], 2006.
[7] A. Cannas da Silva, A. Weinstein, Geometric Models for Noncommutative Algebras, Berkeley Math. Lecture Notes, vol. 10, Amer. Math. Soc., 1999.
[8] J.-P. Dufour, N.T. Zung, Poisson Structures and Their Normal Forms, Progr. Math., vol. 242, Birkhäuser, Basel, 2005.
[9] N.P. Landsman, Mathematical Topics Between Classical and Quantum Mechanics, Springer-Verlag, New York, 1998.
[10] J. Kališnik, Representations of orbifold groupoids, Topology Appl. 155 (11) (2008) 1175–1188.
[11] G. Trentinaglia, On the role of effective representations of Lie groupoids, Adv. Math. 225 (2) (2010) 826–858, doi:10.1016/j.aim.2010.03.014.
[12] G. Trentinaglia, Tannaka duality for proper Lie groupoids, PhD thesis, Utrecht University, available for download (September 2008).
[13] G.E. Bredon, Introduction to Compact Transformation Groups, Academic Press, New York, 1972.
[14] I. Moerdijk, Lie groupoids, gerbes, and non-Abelian cohomology, K-Theory 28 (2003) 207–258.
[15] T. Bröcker, T. tom Dieck, Representations of Compact Lie Groups, second ed., Springer-Verlag, 1995.
[16] W. Fulton, J. Harris, Representation Theory: A First Course, Grad. Texts in Math., vol. 129, Springer-Verlag, New York, 1991.
[17] M. Lazard, J. Tits, Domaines d'injectivité de l'application exponentielle, Topology 4 (1966) 315–322.