New dense families of triple loop networks

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Abstract

Multi-loop digraphs are widely studied mainly because of their symmetric properties and their applications to loop networks. A multi-loop digraph \( G = G(N; s_1, \ldots, s_A) \) has set of vertices \( V = \mathbb{Z}_N \) and adjacencies given by \( v \rightarrow v + s_i \mod N, \ i = 1, \ldots, A \). For every fixed \( N \), an usual extremal problem is to find the minimum value

\[
D(N) = \min_{s_1, \ldots, s_A \in \mathbb{Z}_N} D(N; s_1, \ldots, s_A),
\]

where \( D(N; s_1, \ldots, s_A) \) is the diameter of \( G \). A closely related problem is to find the maximum number of vertices for a fixed value of the diameter. For \( A = 2 \), all optimal families have been found by using a geometrical approach. For \( A = 3 \), only some dense (but possibly not optimal) families are known. In this work some new dense families are given for \( A = 3 \) using a geometrical approach. This technique was already adopted in several papers for \( A = 2 \) (see for instance [7, 10]). These families improve all previous known results. © 1999 Elsevier Science B.V. All rights reserved

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1. Introduction

Multi-loop networks have been widely studied in the past years because of their relevance to the design of some interconnection or communication computer networks. Multi-loop digraphs model such networks, and are usually called circulant because their adjacency matrices are circulant. These digraphs can be defined as Cayley digraphs of cyclic groups; thus a circulant digraph on \( N \) vertices, denoted by \( G = G(N; s_1, \ldots, s_A) \), has set of vertices \( V = \mathbb{Z}_N \) (the set of integers modulo \( N \)) and set of adjacencies

\[
A = \{(v, v + s_i \mod N) \mid v \in V, \ i = 1, \ldots, A\}.
\]
The integers $s_1, \ldots, s_d$ are usually called the \textit{steps} of $G$. These digraphs are regular of in-degree and out-degree $\Delta$ and are strongly connected iff $\gcd(N, s_1, \ldots, s_d) = 1$. For a survey about multi-loop networks and multi-loop digraphs, see the papers of Bermond, et al. [5] or Hwang [12] for the case $\Delta = 2$.

Double-loop (case $\Delta = 2$) and triple-loop digraphs (case $\Delta = 3$) are the most studied cases of multi-loop digraphs. The minimization of the diameter in the digraph, which is the most studied problem, corresponds to the problem of minimizing the message transmission delay of the related network. Many authors have studied this problem for $\Delta = 2$ since the paper of Wong and Coppersmith [15]: for instance Cheng and Hwang [4], Erdőš and Hsu [6], Esquè et al. [7], Fiol et al. [10], Hwang and Xu [13], Aguiló and Fiol [1], etc. Double-loop networks have generated a rich set of results using different ideas ranging from Number Theory to geometrical approaches.

We will use a geometrical approach and use the terminology the 2D case when $\Delta = 2$ and the 3D case when $\Delta = 3$. As we will detail in Section 1.1, much more is known in the former than in the latter case. While some ideas of the 2D case are still useful in the 3D case, this is no longer valid for more than three dimensions.

Let $D(N; s_1, \ldots, s_d)$ be the diameter of $G = G(N; s_1, \ldots, s_d)$, i.e.

$$D(N; s_1, \ldots, s_d) = \max_{u,v \in V(G)} d(u,v),$$

where $d(u,v)$ is the distance from vertex $u$ to vertex $v$ in $G$. The property of $G$ being vertex symmetric reduces the complexity of (1) to

$$D(N; s_1, \ldots, s_d) = \max_{v \in V(G)} d(0,v).$$

Let $D_\Delta(N)$ be the optimal diameter which can be attained with $\Delta$ steps, that is

$$D_\Delta(N) = \min_{s_1, \ldots, s_\Delta \in \mathbb{Z}_N} D(N; s_1, \ldots, s_\Delta).$$

It is obvious that, for a fixed $N$, $D_\Delta(N)$ decreases while $\Delta$ increases. It could be expected that, for fixed $\Delta$, the value of $D_\Delta(N)$ should be an increasing function of the integral values of $N$. However this is not true. For instance,

$$D_2(20) = D(20; 1, 4) = 7 \geq D_2(21) = D(21; 1, 9) = 6,$$

In general, the behaviour of $D_2(N)$ as $N$ increases seems to be a hard problem. No closed expression of $D_2(N)$ is known, but there is a sharp lower bound given by Wong and Coppersmith in [15]:

$$D_2(N) \geq \text{lb}(N) = \lfloor \sqrt{3N} \rfloor - 2.$$  

The characterization of optimal double loop networks in [1] is based on this lower bound.

In Section 2.2 we will see some analogous properties for the behaviour of $D_3(N)$. However, the fact that no sharp lower bound for $D_3(N)$ is known makes the study of this function more difficult than in the 2D case.
1.1. Known results and facts

The usual treatment of the original problem is the maximization of the number of vertices for a fixed diameter. There are some known families with a good $D-N$ ratio, and this is the sense of ‘density’ which we want to attain. More precisely, the following families are known to be dense:

- Hsu and Jia [11] gave a family of order asymptotically equivalent to
  \[ N(D) = \frac{1}{16}D^3 + O(D^2) \approx 0.062D^3 + O(D^2). \]  

- Aguiró et al. [2] gave three families of order
  \[ N(D) = \frac{3}{27}D^3 + O(D^2) \approx 0.074D^3 + O(D^2). \]  

- Chen and Gu [3] gave a family of order
  \[ N(D) = \frac{5}{64}D^3 + O(D^2) \approx 0.078D^3 + O(D^2). \]  

Note that all these dense families have their number of vertices being \( N = \Theta(D^3) \) or equivalently \( D = \Theta(\sqrt[3]{N}) \).

2. Triple commutative step digraphs

Notwithstanding the fact that no analogue to (3) is known in the 3D case, we can still use a generalization of triple-loop networks: the triple commutative step digraphs. These digraphs allow us to work with a kind of 3D geometrical forms which can help us on the study of metrical properties of triple-loop digraphs. In particular, these geometrical shapes provide an intuitive study of the diameter. See [2] for more details.

Let \( M \) be a \( 3 \times 3 \) integral matrix, with \( |\det(M)| = N > 0 \). Let \( \mathbb{Z}^3 \) denote the additive group of (column) 3-vectors with integral coordinates. The set \( M\mathbb{Z}^3 \), whose elements are linear combinations (with integral coefficients) of the column vectors of \( M \) is said to be the lattice generated by \( M \). The concept of congruence in \( \mathbb{Z} \) has the following natural generalization to \( \mathbb{Z}^3 \) (see [8] for a generical study): let \( u, v \in \mathbb{Z}^3 \), we say that \( u \) is congruent to \( v \) modulo \( M \), and we write \( u \equiv v \pmod{M} \) iff \( u - v \in M\mathbb{Z}^3 \), that is

\[ u \equiv v \pmod{M} \iff \exists \lambda \in \mathbb{Z}^3: u = v + M\lambda. \]

The quotient group \( \mathbb{Z}^3 / M\mathbb{Z}^3 \) will be denoted by \( \mathbb{Z}^3 / M\mathbb{Z}^3 \).

**Definition 1.** Let \( M \) be a matrix as above. A triple commutative step digraph, \( G(M) \), is the Cayley digraph of the group \( V = \mathbb{Z}^3 / M\mathbb{Z}^3 \) with the generator set \( \{e_1, e_2, e_3\} \), where \( e_1 = (1, 0, 0)^T \), \( e_2 = (0, 1, 0)^T \) and \( e_3 = (0, 0, 1)^T \). So the set of adjacencies is given by

\[ A = \{(u, u + e_i \pmod{M}) \mid u \in V, \ i = 1, 2, 3\}. \]
Fiol showed in [9] that all finite Abelian Cayley digraphs of degree 3 can be written in the form of the above definition. In particular, the following proposition is obtained in [9].

**Proposition 1.** A triple commutative step digraph \( G(M) \) is isomorphic to a triple-loop digraph if \( V = \mathbb{Z}^3/M\mathbb{Z}^3 \) is a cyclic group.

We will make use of this isomorphism, defined through the Smith normal form (SNF) of the matrix \( M \). To this end, let us make some comments about the SNF of a matrix and the explicit construction of this isomorphism.

The SNF, \( S \), of \( M \) is the matrix \( S = LMR \), where \( L, R \) are unimodular integral matrices generated by the necessary elemental transformations to obtain \( S \) from \( M \). The entries of \( S \) are given by \( S = \text{diag}(i_1, i_2, i_3) \) with \( i_1 | i_2 \) and \( i_2 | i_3 \). These invariant factors can be obtained by \( i_1 = \gcd(\text{integral entries of } M) \), \( i_1i_2 = \gcd(2 \times 2 \text{ minors of } M) \) and \( i_1i_2i_3 = |\det(M)| \).

From the definition of \( G(M) \) we have

\[
u \equiv v \pmod{M} \iff u - v \in M\mathbb{Z}^3 \iff u - v \in L^{-1}SR^{-1} \mathbb{Z}^3
\]

\[
\iff Lu - Lv \in S\mathbb{Z}^3 \iff Lu \equiv Lv \pmod{S}
\]

and we can consider the isomorphism

\[
\mathbb{Z}^3/M\mathbb{Z}^3 \to \mathbb{Z}^3/S\mathbb{Z}^3,
\]

\[
u \to Lu = u',
\]

which is also a digraph isomorphism. So we have \( G(M) \cong G(S) \).

Moreover, considering \( G(S) \), we also have

\[
u' \equiv v' \pmod{S} \iff \exists \lambda = (\lambda_1, \lambda_2, \lambda_3)^T \in \mathbb{Z}^3: u' = v' + (i_1\lambda_1, i_2\lambda_2, i_3\lambda_3)^T
\]

\[
\iff \exists \lambda_k \in \mathbb{Z}: u'_k = v'_k + i_k\lambda_k, \quad k = 1, 2, 3
\]

\[
u'_k \equiv v'_k \pmod{i_k}, \quad k = 1, 2, 3,
\]

so it follows that \( \mathbb{Z}^3/S\mathbb{Z}^3 \cong \mathbb{Z}_{i_1} \times \mathbb{Z}_{i_2} \times \mathbb{Z}_{i_3} \).

From \( G(M) \cong G(S) \cong \mathbb{Z}_{i_1} \times \mathbb{Z}_{i_2} \times \mathbb{Z}_{i_3} \), we can conclude that \( V \) is cyclic iff

\[
i_1 = i_2 = 1.
\]

Then \( i_3 = |\det(M)| = N > 0 \) and \( G(M) \cong \mathbb{Z}_N \). If we denote by \( L_3 \) the third row of the matrix \( L \), we have

\[
u \equiv v \pmod{M} \iff Lu \equiv Lv \pmod{S} \iff L_3u \equiv L_3v \pmod{N},
\]

and then the isomorphism \( \phi: \mathbb{Z}^3/M\mathbb{Z}^3 \to \mathbb{Z}_N \) given by \( \phi(u) = L_3u \) is a digraph isomorphism. So \( G(M) \cong G(N; s_1, s_2, s_3) \), where the steps \( s_1, s_2, s_3 \) are given by \( L_3 = (s_1, s_2, s_3) \).
and

\[ \varphi : G(M) \rightarrow G(N; s_1, s_2, s_3), \]
\[ u = (u_1, u_2, u_3)^T \text{ (mod } M) \rightarrow \varphi(u) = u_1 s_1 + u_2 s_2 + u_3 s_3 \text{ (mod } N). \] (8)

Note that \( s_k = \varphi(e_k) \) for \( k = 1, 2, 3 \).

The isomorphism \( \varphi \) has been used in [7, 1] to classify optimal double loop networks (in the 2D case there is an analogous isomorphism using \( 2 \times 2 \) integral matrices). Also it has been used in [2] to find dense triple-loop families of digraphs.

2.1. 3D tessellations

The isomorphism \( \varphi \) of Proposition 1 can be generalized to multi-loop networks. In the 3D case a three-dimensional tessellation can be related to \( G(M) \). This tessellation contains the metric information of the digraph.

The same technique used in the plane case can be applied in the space for constructing a three-dimensional tile. However, contrarily to the 2D case, a generic basic 3D dense tile is not known for any number of vertices \( N \). Therefore, we neither know how the dimensions of a generic 3D-shaped dense tile should be.

Given a triple commutative step digraph \( G(M) \), each of its vertices is assigned to a unit cube of the space. The vertex \( a e_1 + b e_2 + c e_3 \text{ (mod } M) \) in \( G(M) \) is assigned to the unit cube centered at the point \( (a, b, c) \). In the cyclic case, that is in a triple-loop network \( G(N; s_1, s_2, s_3) \), such a cube represents the vertex \( a s_1 + b s_2 + c s_3 \text{ (mod } N) \). This representation allows us to compute the diameter in the basic 3D tile through its dimensions.

One basic tile can be obtained as a result of the following steps:

- Consider the 3D space divided into unit cubes. Assign one vertex of \( V = \mathbb{Z}^3 / M \mathbb{Z}^3 \) to every unit cube following the rule: from a cube labelled with \( u \in V \), we label by \( u + e_k \text{ (mod } M) \), \( k = 1, 2, 3 \), the next cube given by the direction of the vector \( e_k \). This assignment introduces the same equivalence between the labelled unit cubes as we had between vertices of \( G(M) \).

- Let \( S \) be the set of labels in \( V \). Choose a unit cube with label \((0,0,0)\) (this cube will be called the zero cube), mark it and set \( S = S - \{(0,0,0)\} \). Mark the other cubes, one per label in \( S \), close to the zero cube. Extract the corresponding label from \( S \) per each marked cube. This marking process can be done by considering those labelled cubes that minimize the norm \( \|(x, y, z)\| = |x| + |y| + |z| \) (only those placed in the direction and positive sense of the vectors \( e_1, e_2, \text{ and } e_3 \)).

- We stop when \( S = \emptyset \) and all the \( N \) labels of \( V \) have been assigned. Then all marked cubes form a basic 3D-shaped tile which tessellates the space.

For instance, take

\[ M = \begin{pmatrix} 6 & -3 & -1 \\ -1 & 6 & -3 \\ -3 & -1 & 6 \end{pmatrix}, \quad \det(M) = 134. \]
with Smith normal form, \( S \), and related unimodular matrices, \( L \) and \( R \), given by

\[
S = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 134
\end{pmatrix}, \quad
L = \begin{pmatrix}
0 & -1 & 0 \\
-4 & -9 & -5 \\
15 & 33 & 19
\end{pmatrix}, \quad
R = \begin{pmatrix}
1 & -3 & -105 \\
0 & 0 & 1 \\
0 & 1 & 37
\end{pmatrix}.
\]

By Proposition 1 the diameter of \( G(M) \) is the same as \( D(134; 15, 33, 19) = 10 \). The basic tile related to \( G(M) \) and \( G(134; 15, 33, 19) \) can be found by applying the above steps. This is the tile on the left-hand side of Fig. 1.

This tile is a particular case of a family given in [2] with generic basic tile depicted on the right-hand side of Fig. 1 and it is related to the matrix

\[
A = \begin{pmatrix}
l & -m & -n \\
-n & l & -m \\
-m & -n & l
\end{pmatrix}
\]

for \( m \geq n \geq 0 \) and \( l > m + n \).

It has order \( N(A) = l^3 - m^3 - n^3 - 3lmn \) (volume of the tile) and diameter \( D(A) = 3l - 2n - m - 3 \) (distance from the unit cube labelled with zero to the farthest unit cube in the tile), which are directly derived from the dimensions of the generic tile (see [2] for more details). This basic tile was called hyper-L by analogy with the 2D case. The related tessellation is showed in Fig. 2.

However, the above digraph is not optimal. The selected steps from the above basic tile, i.e. \( s_1 = 15, s_2 = 33 \) and \( s_3 = 19 \), are not the best possible ones. Consider the matrix

\[
B = \begin{pmatrix}
1 & 3 & 2 \\
3 & -3 & 5 \\
4 & 2 & -4
\end{pmatrix}
\]

with \( \det(B) = 134 \).
Fig. 2. Generical 3D tessellation of the family (5).

\[ N=134, \ s_1=76, \ s_2=72, \ s_3=61, \ D=9 \]

Fig. 3. Basic 3D tile related to \( G(134; 76, 72, 61) \).

Its associated Smith normal form and unimodular matrices are given by

\[
S = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 134
\end{pmatrix}, \quad L = \begin{pmatrix}
1 & 0 & 0 \\
-1 & -1 & 1 \\
-58 & -62 & 61
\end{pmatrix}, \quad R = \begin{pmatrix}
1 & 17 & -37 \\
0 & -5 & 11 \\
0 & -1 & 2
\end{pmatrix}.
\]

The steps \( s_1 = 76 \equiv -58 \pmod{134} \), \( s_2 = 72 \equiv -62 \pmod{134} \) and \( s_3 = 61 \) give the optimal diameter \( D_3(134) = 9 \). The related basic tile is given in Fig. 3.

2.2. Heuristic behaviour of \( D_3(N) \)

As in the 2D case, the plot of \( D_3(N) \) versus \( N \) has a kind of unit jumps with some strange local behaviour. Data obtained by exhaustive search show this fact. For
instance, it does not increase when increasing $N$. On the left-hand side of Fig. 4 we can see its values for $N = 5, \ldots, 848$. On the right-hand side we can observe with detail the values $D_3(600), \ldots, D_3(660)$, where some of these unit jumps appear.

These unit jumps can be up and down in the sense that there are integral values, $N_0$, having a neighbourhood $\mathcal{N}$ in $\mathbb{N}$ where $D_3(N)$ is a unit more (or a unit less) than $D_3(N_0)$ for all $N \neq N_0$ in $\mathcal{N}$. For instance, $D_3(623) = 18$, $D_3(622) = D_3(624) = 17$ and $D_3(638) = 17$, $D_3(637) = D_3(639) = 18$.

The global behaviour of $D_3(N)$ on the left-hand side of Fig. 4 seems to be like an integral function of type $\Theta(\sqrt[3]{N})$ (at least for $N \leq 848$). In fact, as a referee pointed out, according to [15] this assumption is true:

- To prove that $D_3(N) = O(\sqrt[3]{N})$ one can take $s_1 = 1$, $s_2 = \lceil \sqrt[3]{N} \rceil$ and $s_3 = s_2$. The distance from 0 to the vertices $\{1, \ldots, s_2\}$ is at most $\lceil \sqrt[3]{N} \rceil$ using the step $s_1$. The vertices $\{1, \ldots, s_2^2\}$ are covered with at most $2 \lceil \sqrt[3]{N} \rceil - 1$ jumps using the steps $s_1$ and $s_2$. Finally, all the vertices are reached with a distance at most $3 \lceil \sqrt[3]{N} \rceil - 2$ using $s_1$, $s_2$ and $s_3$.
- The generic bound $N > \left(\frac{D_3(N) + A}{A}\right)$ (see [15,10]) for $A = 3$ leads to $D_3(N) > \sqrt[3]{6N} - 2$.

The value $v(N) = \lceil \sqrt[3]{13N} \rceil - 3$ (note some analogy with (3)) is close to $D_3(N)$ for $N = 5, \ldots, 848$. In fact, we have

$$v(N) - 1 \leq D_3(N) \leq v(N)$$

for those values of $N$. However, this is not true for large values of $N$. See the comments after the proof of Theorem 2.

3. New dense families

The idea of finding new families of dense triple loop networks is based on the works [7,1,2]. We can use without any confusion the word diameter with respect a basic 3D tile, which can be computed from its metrical dimensions. From a particular basic 3D
tile with a good ratio between its diameter and its volume, we derive a family of tiles which preserves this ratio as the volume increases. From now on we will call a basic 3D tile a \textit{hyper-L}.

From a tessellation of the space, we consider its related hyper-L, \( H \), and integral matrix \( M \). This matrix defines a triple commutative step digraph with order \( |\det(M)| = \text{vol}(H) \) and diameter value at most the diameter of \( H \). We can obtain a family of hyper-Ls by increasing the value of the dimensions of \( H \) and preserving the ratios between them. Finally we must select a subfamily with set of vertices being a cyclic group. This method is used to obtain the following two dense families.

\textbf{Theorem 1.} Let

\[ N(t) = 1485t^3 + 648t^2 + 90t + 4, \]
\[ s_1(t) = 9t^3 + 3t, \]
\[ s_2(t) = -189t^2 - 57t - 4 \pmod{N(t)}, \]
\[ s_3(t) = 3t(15t + 2), \]

for \( t \equiv 1 \pmod{2} \), then the family \( G_1(t) = G(N(t); s_1(t), s_2(t), s_3(t)) \) has a diameter of \( D(t) \leq 27t + 1 \) and a density of

\[ N(t) \geq \frac{1485}{273}D(t)^3 + O(D(t)^2) \approx 0.075D(t)^3 + O(D(t)^2). \]

\textbf{Proof.} Consider the generic dimensions \( h, m, n \) of the hyper-L \( H_i \) shown in Fig. 5. This 3D tile was suggested by the particular tile linked to \( G(161; 2, 117, 7) \), where the ratios between its dimensions are

\[ h = m = 2n. \] (10)

Consider the integral matrix \( M_1 \) given by

\[ M_1(h, m, n) = \begin{pmatrix} n & -m & -m \\ n + m & -m \\ 2h & h & 2h - n \end{pmatrix}. \] (11)

The column vectors of \( M_1 \) give the distribution of the basic 3D tiles in the space. Moreover the value of \( \det(M_1) \) corresponds to the volume of this hyper-L, no holes between shapes appear and there are no intersection between any pair of these tiles. So these hyper-Ls form a tessellation of the space. The technical proof of this fact is left out. See [2] for an explicit analogous 3D tessellation proof. This tessellation is depicted in Fig. 6.

The diameter of this hyper-L is given by

\[ D(h, m, n) = \max\{3m + h + n, 2m + 2h + n, 3h + 3n\} - 3. \] (12)

This value bounds the diameter of the related triple commutative step digraph \( G(M_1) \).
As mentioned above, we want the ratios (10) to be preserved as \( n \to \infty \). However we also want the conditions (7) on the Smith normal form of \( M_1 \) in Section 2 to be fulfilled too. Set \( h = 2x + a, m = 2x + b \) and \( n = x + c \) where the parameters \( a, b, c \) will be found in such a way that (7) are satisfied on the matrix (11). These conditions are fulfilled by \( c = -1, a = 2c + 1 = -1, b = 2c + 1 = -1 \) and \( x \equiv 0 \mod 3 \). These equalities give the following two possibilities:

\[
x = 3t + 1 \quad \text{for} \quad t \equiv 0 \pmod{2}, \quad \text{and} \quad x = 3t + 2 \quad \text{for} \quad t \equiv 0 \pmod{2}.
\]

The family stated in this theorem corresponds to \( x = 3t + 1 \). The related steps, now in the ring \( \mathbb{Z}[t]/N(t) \), are derived from the Smith normal form (see the isomorphism (8) in the proof of Proposition 1) of \( M_1(6t + 1, 6t + 1, 3t) \), where \( N(t) = \det(M_1(6t + 1, 6t + 1, 3t)) \). The diameter is bounded by the corresponding value from (12), so \( D(t) \leq 27t + 1 \). The case \( x = 3t + 2 \) for \( t \equiv 0 \pmod{2} \) produce another different family of triple-loop networks, however its density is the same and we omit the details. Finally, the density is the one stated before by considering \( t \geq (D(t) - 1)/27 \). □
Note that this family improves those in (4) and (5), but not in (6) which has the better density known up to now.

**Theorem 2.** Let

\[ N(t) = 860t^3 + 1762t^2 + 1202t + 273, \]
\[ s_1(t) = -19t^2 - 29t - 11 \pmod{N(t)}, \]
\[ s_2(t) = -39t^2 - 53t - 18 \pmod{N(t)}, \]
\[ s_3(t) = 92t^2 + 123t + 41, \]

for \( t \not\equiv 2, 7 \pmod{10} \), then the family \( G_2(t) = G(N(t); s_1(t), s_2(t), s_3(t)) \) has a diameter of \( D(t) \leq 22t + 12 \) and has a density of

\[ N(t) \geq \frac{860}{22^3} D(t)^3 + O(D(t)^3) \approx 0.08 D(t)^3 + O(D(t)^2). \]

**Proof.** In this case we consider the hyper-L \( H_2 \) shown in Fig. 7 with associated matrix

\[ M_2(l, m, n) = \begin{pmatrix} l + m & -2l & l \\ 2l + n & 3l + n & -2l - n \\ l + n & l & l + m + 2n \end{pmatrix}, \]  

and volume \( V = \det(M_2) \), which also tessellates the space.

Now we want the ratio of dimensions to be \( l \approx 3x, m \approx 2x \) and \( n \approx x \). This generical tile was inspired on the hyper-L linked to \( G(860; 91, 51, 12) \) with \( l = 3, m = 2 \) and \( n = 1 \) (which is in fact the one in Fig. 7 with unit cubes). As in the above proof of Theorem 1, we modify the dimensions by using three parameters in such a way that the conditions (7) are satisfied.
From these considerations we obtain the dimensions \( l = 3x + 2, \ m = 2x + 1 \) and \( n = x + 1 \). These values of \( l, m, n \) must be entered into (13) with \( x = t \not\equiv 2, 7 \pmod{10} \) and the family stated in this theorem is found. The density is derived from \( t \geq (D(t) - 12)/22 \).

Note that the density of the family in Theorem 2 improves (6). A generical classification of dense, or even optimal, hyper-L (as it has been done in the 2D case in [1]), seems to be a hard task. However, a good strategy may reduce the search to some well selected subset of tiles.

As a referee noted in one of his remarks, the value \( v(N) = \lceil \sqrt[3]{13N} \rceil - 3 \) is not a sharp upper bound of \( D_3(N) \) for large values of \( N \). By the above theorem, there is a family of digraphs with \( N(t) \approx (860/22^3)D(t)^3 \) vertices. Note that, in this case, we have

\[
v(N(t)) \approx \frac{\sqrt[3]{860 \cdot 13}}{22} D(t) > D(t) \geq D_3(N(t)),
\]

and so, the inequalities (9) do not hold for large values of \( N \).

The associated matrices \( M_i \) to the tiles \( H_i, \ i = 1,2 \), generate dense families of triple commutative step digraphs, i.e. dense Cayley digraphs associated to non-cyclic groups, when conditions (7) are not satisfied.

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