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Automatic semigroups

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Abstract

The area of automatic groups has been one in which significant advances have been made in recent years. While it is clear that the definition of an automatic group can easily be extended to that of an automatic semigroup, there does not seem to have been a systematic investigation of such structures. It is the purpose of this paper to make such a study.

We show that certain results from the group-theoretic situation hold in this wider context, such as the solvability of the word problem in quadratic time, although others do not, such as finite presentability. There are also situations which arise in the general theory of semigroups which do not occur when considering groups; for example, we show that a semigroup S is automatic if and only if S with a zero adjoined is automatic, and also that S is automatic if and only if S with an identity adjoined is automatic. We use this last result to show that any finitely generated subsemigroup of a free semigroup is automatic.

Keywords: Automatic; Group; Regular; Semigroup

1. Introduction

The area of automatic groups has been one in which significant advances have been made in recent years. Starting with the work of Epstein et al. [5], there have been many beautiful results, and a coherent theory has been built up (see [1, 5, 7–10, 20, 28, 29] for example).

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A natural question is if there is an analogous theory for semigroups. There are several strong reasons for developing such a theory. Semigroups already play a significant role in the theory of automatic groups. The definition of an automatic group treats the group as a semigroup, and therefore readily applies to semigroups (as noted in [14]). Furthermore, the theory is framed in terms of automata and regular languages, both of which can be naturally interpreted in terms of semigroups. From a computer science perspective, semigroups are more natural objects than groups, since they are more amenable to combinatorial and algorithmic analysis. Also, from the perspective of semigroup theory, one could expect to obtain a class in which computation is easy, as is the case with automatic groups.

The purpose of this paper is to initiate the development of such a theory.

We find that certain results from the group-theoretic situation generalize when we consider semigroups. For example, if S is an automatic semigroup, then we can solve the word problem for S in quadratic time (see Corollary 3.7 below), and the free product of two semigroups is automatic if and only if the factors are automatic (see Theorem 6.1, although it should be noted, as pointed out after that result, that the free product of two semigroups is not quite a generalization of the notion of the free product of two groups). On the other hand, we show (Theorem 6.2) that the monoid free product of two automatic monoids is automatic, and this is a genuine generalization of the group-theoretical situation.

However, certain properties do not generalize. For example, an automatic semigroup need not be finitely presented (see Examples 3.9 and 4.4) and a semigroup may be automatic with respect to one finite generating set but not another (see Example 4.5). The main reason for this appears to be that the obvious generalizations of the fellow traveller property for groups (see Definition 3.8) do not characterize automatic semigroups. This means that the strong geometric theory that can be applied when considering automatic groups does not seem to work here. One consequence is that some results (such as finite presentability) no longer hold; a second is that, even when results can be generalized, we often need to construct new proofs. The theory of automatic semigroups (at least, as constructed here) is therefore more deeply based in automata and formal languages when compared to that of automatic groups.

There are also situations which arise in the general theory of semigroups which do not occur when considering groups. For example, one can adjoin a zero 0 or an identity 1 to a semigroup S to get a new semigroup $S^{\square}$ or S^{I} respectively. We show that S is automatic if and only if $S^{\square}$ is automatic (see Proposition 3.13) and that S is automatic if and only if S^{I} is automatic (see Theorem 7.2). This last result enables us to show that any finitely generated subsemigroup of a free semigroup is necessarily automatic (see Theorem 8.1).

In many respects, this paper is the beginning of a theory of automatic semigroups, in that we set the scene, establish the basic theory and pose some questions. We indicate several further avenues for future research. In particular, it would be interesting to see to what extent one can extend the rich connections between automaticity and topology which hold in the group case into this new setting, or how one can use the theory to develop effective methods of computing in automatic semigroups akin to those used for automatic groups.

For background material pertinent to the topics discussed in this paper, we refer the reader to [1, 5] as basic references for automatic groups, [11, 12] for formal language theory and [13, 16] for semigroups.

2. Definitions and regular sets

For any finite set A, let A^+ denote the set of all non-empty words over A, and let A^* denote the set of all words over A (including the empty word ε). For any word α in A^*, we let $|\alpha|$ denote the length of α (where $|\varepsilon| = 0$).

If A is a set of generators of a semigroup S, then there is a natural homomorphism $\theta : A^+ \to S$ where each word α of A^+ is mapped to the corresponding element of S. We will normally be concerned with finite sets A, so that the semigroup S is finitely generated. Where there is no danger of confusion, we will sometimes suppress the reference to θ, writing α for the element $\alpha\theta$ of the semigroup. In this context, if α and β are elements of A^+, we will write $\alpha \equiv \beta$ if α and β are identical as words, and $\alpha = \beta$ if α and β represent the same element of S (i.e. if $\alpha\theta = \beta\theta$). We may also write $\alpha = s$, where $\alpha \in A^+$ and $s \in S$, instead of $\alpha\theta = s$ in S.

As in the case of automatic groups, we will want to consider automata which accept pairs (α, β) of words with $\alpha, \beta \in A^+$. If $\alpha \equiv a_1 a_2 \ldots a_n$ and $\beta \equiv b_1 b_2 \ldots b_m$, $a_i, b_i \in A$, this is accomplished by having an automaton with input alphabet $A \times A$ and reading pairs (a_1, b_1), (a_2, b_2), and so on. To deal with the case where $n \neq m$, we introduce a *padding symbol* \$. More formally, as with automatic groups, we define a mapping δ_A : $A^* \times A^* \to A(2, \$)^*$, where $\$ \notin A$ and $A(2, \$) = ((A \cup \{\$\}) \times (A \cup \{\$\})) - \{(\$, \$)\}$, by

$$(\alpha, \beta)\delta_A = \begin{cases} (a_1, b_1) \ldots (a_n, b_n) & \text{if } n = m, \\ (a_1, b_1) \ldots (a_n, b_n)(\$, b_{n+1}) \ldots (\$, b_m) & \text{if } n < m, \\ (a_1, b_1) \ldots (a_m, b_m)(a_{m+1}, \$) \ldots (a_n, \$) & \text{if } n > m. \end{cases}$$

Given this, we make the following definition:

Definition 2.1. If S is a semigroup, A is a finite set, L is a regular subset of A^+, and $\phi : A^+ \to S$ is a homomorphism with $L\phi = S$, we say that (A, L) is an *automatic structure* for S if
- $L_= = \{(\alpha, \beta) \colon \alpha, \beta \in L, \alpha = \beta\}\delta_A$ is regular in $A(2, \$)^*$, and
- $L_a = \{(\alpha, \beta) \colon \alpha, \beta \in L, \alpha a = \beta\}\delta_A$ is regular in $A(2, \$)^*$ for each $a \in A$.

If a semigroup S has an automatic structure (A, L) for some A and L, then we say that S is *automatic*.

Note that the above definition of automaticity is precisely that given in [5], where a group is considered as a semigroup. The definition of "automatic" in [1] uses a set of monoid, as opposed to semigroup, generators, but this distinction does not make any difference as to whether or not a group is automatic. Another equivalent definition for groups is the one in terms of the fellow traveller property of the Cayley graph. This definition does not generalize to semigroups in a straightforward way; see Definition 3.8 and the subsequent discussion.

Note that the concept of an automatic semigroup does not coincide with that of a rational semigroup as discussed in [27, 22, 21]; the latter is a very interesting notion and is similar in flavour in its definition to that of an automatic semigroup, but the classes of structures are quite different; for example, there are many interesting examples of infinite automatic groups whereas any rational group is finite.

In Definition 2.1, we should really say that (A, L, ϕ) is an automatic structure for S, as opposed to (A, L); in practice, as above, we will usually identify A with $A\phi$, so that A is a subset of S, and we will then suppress the reference to ϕ.

Note that δ_A is an injection. It is also convenient to have a notation for its inverse $\mu_A : (A^* \times A^*)\delta_A \to A^* \times A^*$, so that $\gamma\mu_A = (\alpha, \beta) \Leftrightarrow (\alpha, \beta)\delta_A = \gamma$.

We now go on to list some properties of regular sets.

Proposition 2.2. *Suppose that A and B are finite sets. Then*:

(i) $\emptyset$, A^+ *and* A^* *are regular*;

(ii) *any finite subset of* A^* *is regular*;

(iii) *if* $K \subseteq A^*$ *and* $L \subseteq A^*$ *are regular, then* $K \cup L$, $K \cap L$, $K - L$, KL *and* K^* *are regular*;

(iv) *if* $K \subseteq A^*$ *is regular and* $\phi : A^* \to B^*$ *is a monoid homomorphism, then* $K\phi$ *is regular*;

(v) *if* $K \subseteq B^*$ *is regular and* $\phi : A^* \to B^*$ *is a monoid homomorphism, then* $K\phi^{-1}$ *is regular*;

(vi) *if* $K, L \subseteq A^*$ *are regular, then* $(K \times L)\delta_A$ *is regular*;

(vii) *if* $U \subseteq (A^* \times A^*)\delta_A$ *is regular, then*

$$\{\alpha \in A^* : (\alpha, \beta)\delta_A \in U \text{ for some } \beta \in A^*\}$$

is regular.

Proof. Parts (i)–(v) are well-known properties of regular languages; see [12] for example. For parts (vi) and (vii), see Lemmas II.5.1 and II.5.2 of [1]. □

Note that, in parts (iv) and (v) of Proposition 2.2, we could equally have had $K \subseteq A^+$ (respectively, $K \subseteq B^+$) regular and $\phi : A^+ \to B^+$ a semigroup homomorphism (since we can trivially extend ϕ to a monoid homomorphism from A^* to B^* by defining $\varepsilon\phi = \varepsilon$).

The following two results are also useful:

Proposition 2.3. *Suppose that A is a finite set and that U and V are subsets of $A^* \times A^*$ such that $U\delta_A$ and $V\delta_A$ are regular. Let*

$$W = \{(\alpha, \gamma) \in A^* \times A^* : \text{there exists } \beta \in A^* \text{ such that } (\alpha, \beta) \in U \text{ and } (\beta, \gamma) \in V\}.$$

Then $W\delta_A$ is regular.

Proof. This follows from Theorem 1.4.6 of [5]. □

Proposition 2.4. *If L is a regular subset of A^*, then $\{(\alpha,\alpha)\colon \alpha\in L\}\delta_A$ is regular.*

Proof. See Lemma II.5.6 of [1]. □

3. Basic properties

In this section, we prove some basic properties of automatic semigroups. First we have:

Proposition 3.1. *If S is a semigroup with an automatic structure (A,L) and $s\in S$, then the set $\{\alpha\in L\colon \overline{\alpha}=s\}$ is regular.*

Proof. Pick $\beta\in L$ with $\overline{\beta}=s$. If $\alpha\in A^+$, then $(\alpha,\beta)\delta_A\in L_=$ if and only if $\alpha\in L$ and $\overline{\alpha}=s$. Now the language

$$K=\{(\alpha,\beta)\colon \alpha\in L, \overline{\alpha}=s\}\delta_A = L_= \cap \{(\gamma,\beta)\colon \gamma\in A^+\}\delta_A$$

is regular by Proposition 2.2. So

$$\begin{aligned}\{\alpha\in A^+\colon (\alpha,\gamma)\delta_A\in K \text{ for some } \gamma\in A^+\} &= \{\alpha\in A^+\colon (\alpha,\beta)\delta_A\in K\}\\ &= \{\alpha\in L\colon \overline{\alpha}=s\}\end{aligned}$$

is regular by Proposition 2.2 as required. □

Next we have:

Proposition 3.2. *If S is a semigroup with an automatic structure (A,L) and $\gamma\in A^+$, then the set $L_\gamma=\{(\alpha,\beta)\in L\times L\colon \overline{\alpha\gamma}=\overline{\beta}\}\delta_A$ is regular.*

Proof. Let $\gamma\equiv a_1a_2\dots a_n$. Since (A,L) is an automatic structure, the sets

$$L_{a_1}=\{(\alpha,\alpha_1)\in L\times L\colon \overline{\alpha a_1}=\overline{\alpha_1}\}\delta_A,$$

$$L_{a_2}=\{(\alpha_1,\alpha_2)\in L\times L\colon \overline{\alpha_1 a_2}=\overline{\alpha_2}\}\delta_A,$$

$$\vdots$$

$$L_{a_{n-1}}=\{(\alpha_{n-2},\alpha_{n-1})\in L\times L\colon \overline{\alpha_{n-2}a_{n-1}}=\overline{\alpha_{n-1}}\}\delta_A,$$

$$L_{a_n}=\{(\alpha_{n-1},\beta)\in L\times L\colon \overline{\alpha_{n-1}a_n}=\overline{\beta}\}\delta_A$$

are regular. So the sets

$$L_{a_1a_2} = \{(\alpha,\alpha_2) \in L\times L\text{: there exists } \alpha_1 \in L \text{ such that } (\alpha,\alpha_1)\in L_{a_1}, (\alpha_1,\alpha_2)\in L_{a_2}\}\delta_A,$$

$$L_{a_1a_2a_3} = \{(\alpha,\alpha_3) \in L\times L\text{: there exists } \alpha_2 \in L \text{ such that } (\alpha,\alpha_2)\in L_{a_1a_2}, (\alpha_2,\alpha_3)\in L_{a_3}\}\delta_A,$$

$$\vdots$$

$$L_{a_1a_2\ldots a_n} = \{(\alpha,\beta) \in L\times L\text{: there exists } \alpha_{n-1} \in L \text{ such that } (\alpha,\alpha_{n-1})\in L_{a_1a_2\ldots a_{n-1}}, (\alpha_{n-1},\beta)\in L_{a_n}\}\delta_A$$

are regular by Proposition 2.3; in particular, L_γ is regular as required. □

An important concept when studying automatic groups is that of the Cayley graph of the group. We generalize this notion to semigroups:

Definition 3.3. Let S be a semigroup generated by a finite set A. The (right) *Cayley graph* Γ of S with respect to A is the directed graph with vertex set S and an edge labelled a from s to sa for every vertex $s\in S$ and every $a\in A$.

The following result essentially carries over from the group-theoretic case:

Proposition 3.4. *Let S be a semigroup with an automatic structure (A,L) and let Γ be the Cayley graph of S with respect to A. Then there exists a constant N such that, for any $\alpha\in L$ and any vertex s of Γ with $s=\overline{\alpha}$ or $s=\overline{\alpha a}$, $a\in A$, the following statements are true*:

(i) *there exists $\beta\in L$ such that $|\beta|\leqslant|\alpha|+N$ and $s=\overline{\beta}$, and*

(ii) *if $\gamma\in L$ with $|\gamma|>|\alpha|+N$ and $\overline{\gamma}=s$, then there exist infinitely many $\zeta\in L$ with $\overline{\zeta}=s$.*

Proof. We essentially follow the proof of Lemma 2.3.9 of [5].

Consider a collection $\mathscr{M}$ of finite state automata accepting the regular languages $L_=$ and L_a, $a\in A$, and let N be greater than the number of states in any of them. Choose $\beta_1\in L$ such that $\overline{\beta_1}=s$. Then $(\alpha,\beta_1)\delta_A$ is accepted by some automaton M in $\mathscr{M}$.

If $|\beta_1|>|\alpha|+N$, then M visits the same state, q say, twice after reading all of α. We can shorten β_1 to give β_2 by removing the subword between successive visits to q; the resulting pair $(\alpha,\beta_2)\delta_A$ is still accepted. Repeating this process as necessary yields (i).

If we have $\gamma\in L$ with $|\gamma|>|\alpha|+N$ and $\overline{\gamma}=s$, then, when reading $(\alpha,\gamma)\delta_A$, such a repetition of states must occur; so, as we repeat the subword read between successive visits to the state, we obtain an infinite sequence of words ζ in L with $\overline{\zeta}=s$ as required. □

Note that, unlike the group case, we cannot include the possibility that $\alpha = sa$ in the hypotheses of Proposition 3.4, since we may have $sa = s'a$ without having $s = s'$.

If S is a semigroup, we let S^I denote S with an identity element 1 adjoined (regardless as to whether or not S already has such an element). Before proving the next theorem, we need the following:

Proposition 3.5. *If S is automatic, then S^I is automatic.*

Proof. Suppose that (A, L) is an automatic structure for S. Let $B = A \cup \{e\}$, where $e \notin A$ and e is mapped to the identity element of S^I. Let $K = L \cup \{e\}$.

Since L is mapped onto S, K is mapped onto S^I; also, K is regular by Proposition 2.2. Using Proposition 2.2 again, we see that $K_= = L_= \cup \{(e, e)\}$ and $K_e = L_= \cup \{(e, e)\}$ are regular. It remains to check that each K_a is regular (for $a \in A$).

If $a \in A$, then $\{\alpha \in L\colon\ \alpha = a\}$ is regular by Proposition 3.1. So $H_a = \{(e, \alpha)\colon\ \alpha \in L, \alpha = a\}\delta_B$ is regular by Proposition 2.2. Hence $K_a = L_a \cup H_a$ is regular as required, and (B, K) is an automatic structure for S^I. □

We prove the converse of Proposition 3.5 later (see Theorem 7.2). Given Proposition 3.5, we can (as in the group-theoretic case) prove the following:

Theorem 3.6. *If S is a semigroup with an automatic structure (A, L), then, for any word $\alpha \in A^+$, we can find a word in L representing the same element of S as α in time proportional to $|\alpha|^2$.*

Proof. We essentially follow the proof of Theorem 2.3.10 of [5], modifying it to take account of the fact that S need not contain an identity element by using Proposition 3.5.

If S has an automatic structure (A, L), then S^I has an automatic structure $(A \cup \{e\}, L \cup \{e\})$ as in Proposition 3.5. Let $\alpha \equiv a_1 a_2 \ldots a_n$ with $a_i \in A$ for each i.

Let $M_1, M_2, \ldots, M_n$ be finite state automata accepting $L_{a_1}, L_{a_2}, \ldots, L_{a_n}$ respectively. We first follow a path in M_1 where the first components of the labels of the edges are $e\$\$\$\ldots$ until we reach a final state; the second components of the labels of these edges give a word α_1 in L with $\alpha_1 = ea_1 = a_1$. We now repeat this, inputting $\alpha_1\$\$\$\ldots$ into M_2 until we reach a final state, when the second components of the labels of the edges give a word α_2 in L with $\alpha_2 = \alpha_1 a_2 = a_1 a_2$, and so on. Eventually, we get $\alpha_n \in L$ with $\alpha_n = a_1 a_2 \ldots a_n \equiv \alpha$.

We can find each α_i in time proportional to $|\alpha_i|$, and $|\alpha_i| \leqslant |\alpha_{i-1}| + N$ for each i as in Proposition 3.4. So we can find α_n in time proportional to $|\alpha|^2$. □

This has the following consequence (as for groups):

Corollary 3.7. *If S is an automatic semigroup, we can solve the word problem for S in quadratic time.*

Proof. Let M be a finite state automaton accepting $L_=$. Given $\alpha, \beta \in A^+$, we can find $\gamma, \eta \in L$ with $\alpha = \gamma$ and $\beta = \eta$ in quadratic time as in Theorem 3.6, and then input $(\gamma, \eta)\delta_A$ into M. □

Suppose that Γ is the Cayley graph of a group G with respect to a generating set A; for simplicity, we assume that A is closed under inversion, so that $A = B \cup C$ where the generators in C are the inverses of the generators in B. One consequence is that, if there is an edge e labelled by an element of B from g to h in Γ, then there is also an edge f from h to g labelled by the corresponding element of C (and conversely). One often thinks of such a pair $\{e, f\}$ of edges as a single edge e labelled by an element of B, and then interpret traversing the edge e in the "wrong direction" as traversing the edge f labelled by the corresponding element of C. One can then define a *path* between two vertices g and h in Γ to be a sequence of edges (regardless of direction) connecting g to h and the *length* of such a path to be the number of edges in it. We then define the *distance* $d(g,h)$ between g and h to be the minimum length of a path joining g to h. The distance function d so defined is a metric on the group G.

A critical notion in the theory of automatic groups is that of the *fellow traveller property*. To explain this, we need some more notation. If A is a finite set, $\alpha \equiv a_1 a_2 \ldots a_n$ is an element of A^+, and $t \geqslant 1$, then we define

$$\alpha(t) = \begin{cases} a_1 a_2 \ldots a_t & \text{if } t \leqslant n, \\ a_1 a_2 \ldots a_n & \text{if } t > n. \end{cases}$$

We then have:

Definition 3.8. If G is a group, Γ is the Cayley graph of G with respect to a generating set A, and L is a regular subset of A^+ such that L maps onto S, then Γ is said to have the *fellow traveller property* with respect to L if there exists a constant k such that, whenever $\alpha, \beta \in L$ with $d(\alpha, \beta) \leqslant 1$, then $d(\alpha(t), \beta(t)) \leqslant k$ for all $t \geqslant 1$.

It is well known that the fellow traveller property is equivalent to automaticity for groups (see Theorem 2.3.5 of [5] for example). The situation for semigroups appears to be considerably more complicated. While we will not undertake an in-depth study of this here, we would, at least, like to indicate the nature of the problem.

In the group case, we can choose to ignore the directions of the edges by choosing a generating set which is closed under inversion. Ignoring the directions of the edges seems to be much less natural in the context of semigroups. We could define a path from s to t in the Cayley graph Γ of a semigroup S to be a directed sequence of edges from s to t. However, if we do this, then an automatic structure does not necessarily give rise to the fellow traveller property as the following example shows:

Example 3.9. Consider the semigroup S with presentation

$$\langle a, b, x, y \colon a^n x = b^n y \ (n = 1, 2, \ldots) \rangle$$

and let $A = \{a, b, x, y\}$. In S we clearly have $a^k b^l y = a^{k+l} x = b^{k+l} y$ for all $k, l > 0$, so that

$$\langle a, b, x, y \colon a^n x = b^n y, \ a^k b^l y = b^{k+l} y \ (n, k, l = 1, 2, \ldots) \rangle$$

is also a presentation for S. The standard confluence test (see [5]) shows that we have a complete rewriting system. Therefore, every element of S is represented by a unique element of the regular language

$$\begin{aligned} L &= A^+ - A^*\{a\}^+\{x\}A^* - A^*\{a\}^+\{b\}^+\{y\}A^* \\ &= A^+ - A^*\{ax\}A^* - A^*\{a\}\{b\}^+\{y\}A^*. \end{aligned}$$

It follows immediately that $L_= = \{(\alpha,\alpha)\colon \alpha \in L\}\delta_A$, which is regular by Proposition 2.4. Also, it is clear that for every $\alpha \in L$ we have $\alpha a, \alpha b \in L$, so that

$$L_a = \{(\alpha,\alpha)\colon \alpha \in L\}\delta_A\{(\$,a)\} \qquad \text{and} \qquad L_b = \{(\alpha,\alpha)\colon \alpha \in L\}\delta_A\{(\$,b)\},$$

which again are regular languages by Propositions 2.2 and 2.4.

Now let $\alpha, \beta \in L$ be such that $\alpha x = \beta$ in S. If α does not end with a then $\alpha x \equiv \beta$. Otherwise, we can write $\alpha \equiv \alpha_1 a^i$, where α_1 does not end with a, and then $\beta \equiv \alpha_1 b^i y$. We conclude that

$$\begin{aligned} L_x = &\{(\alpha,\alpha)\colon \alpha \in L - A^*\{a\}\}\delta_A\{(\$,x)\} \\ &\cup \{(\alpha_1,\alpha_1)\colon \alpha_1 \in (L \cup \{\varepsilon\}) - A^*\{a\}\}\delta_A\{(a,b)\}^+\{(\$,y)\}; \end{aligned}$$

as before this is a regular language. Finally, by using a similar argument, we can show that

$$\begin{aligned} L_y = &\{(\alpha,\alpha)\colon \alpha \in L - A^*\{a\}\}\delta_A\{(\$,y)\} \\ &\cup \{(\alpha_1,\alpha_1)\colon \alpha_1 \in (L \cup \{\varepsilon\}) - A^*\{a\}\}\delta_A\{(a,b)\}^+\{(b,b)\}^+\{(\$,y)\}. \end{aligned}$$

We conclude that (A,L) is an automatic structure for S.

Now we note that for any $n > 0$ we have $(a^n, b^n y)\delta_A \in L_x$. On the other hand there is no word β with $a^t\beta = b^t$ (or a word γ with $b^t\gamma = a^t$ for that matter) for any t $(1 \leqslant t \leqslant n)$.

So the fellow traveller property based on a definition in terms of directed paths does not necessarily hold for automatic structures. An alternative strategy might be to take the definition of distance to be the minimum length of a path ignoring the directions of the edges:

Definition 3.10. Let S be a semigroup with generating set A and let Γ be the Cayley graph of S with respect to A. A *path* between two vertices s and t is a sequence of edges (regardless of direction) connecting s to t and the *length* of such a path is the number of edges it contains. We define the *distance* $d(s,t)$ from s to t to be the minimum length of a path joining s to t (if such a path exists).

Given this notion of distance, we could define the fellow traveller property for semigroups as follows:

Definition 3.11. If S is a semigroup, Γ is the Cayley graph of S with respect to a generating set A, and L is a regular subset of A^+ such that L maps onto S, then Γ is said to have the *fellow traveller property* with respect to L if there exists a constant k such that, whenever $\alpha, \beta \in L$ with $d(\alpha, \beta) \leqslant 1$, then $d(\alpha(t), \beta(t)) \leqslant k$ for all $t \geqslant 1$.

At least, with this definition, automaticity does imply the fellow traveller property (as in the group case):

Proposition 3.12. *If S is a semigroup with an automatic structure (A, L) and if Γ is the Cayley graph of S with respect to A, then Γ has the fellow traveller property with respect to L.*

Proof. Let $\alpha, \beta \in L$ with $d(\alpha, \beta) \leqslant 1$ in Γ. Without loss of generality, we may assume that $\beta = \alpha x$ with $x \in A \cup \{\varepsilon\}$. Consider a collection $\mathscr{M}$ of finite state automata accepting the regular languages $L_=$ and L_a, $a \in A$, and let N be the maximum number of states in any machine in $\mathscr{M}$. Let M be a finite state automaton in $\mathscr{M}$ accepting L_x (or $L_=$ if $x = \varepsilon$), and let $t > 0$.

After reading $(\alpha(t), \beta(t))\delta_A$, M is in state q (say). Since reading the remainder of $(\alpha, \beta)\delta_A$ reaches a final state f, there is a pair of words γ, η such that reading $(\gamma, \eta)\delta_A$ from q reaches f without looping. We see that $|\gamma| \leqslant N - 1$ and that $|\eta| \leqslant N - 1$. Now $\alpha(t)\gamma x = \beta(t)\eta$, and we have that

$$d(\alpha(t), \beta(t)) \leqslant |\gamma| + |\eta| + 1 \leqslant 2N - 1.$$

Setting k to be $2N - 1$ yields the result. □

Before we discuss the converse of Proposition 3.12, we prove a result about adjoining a zero to a semigroup. To be more precise, if S is a semigroup, we let $S^{\square}$ denote S with a zero element 0 adjoined (regardless as to whether or not S already has such an element). We have the following result:

Proposition 3.13. *S is automatic if and only if $S^{\square}$ is automatic.*

Proof. Suppose that (A, L) is an automatic structure for S, and let $B = A \cup \{z\}$, $K = L \cup \{z\}$, where z represents 0. Then $K_= = L_= \cup \{(z, z)\}$, $K_a = L_a \cup \{(z, z)\}$ if $a \neq z$, and $K_z = (K \times \{z\})\delta_B$ are all regular. So (B, K) is an automatic structure for $S^{\square}$.

Conversely, suppose that (A, L) is an automatic structure for $S^{\square}$. Let

$$B = \{a \in A\text{: } a \text{ does not represent } 0\}.$$

Let $K = L \cap B^+$, the set of words in L representing elements of S, which is regular by Proposition 2.2. Now the languages

$$K_= = L_= \cap (B^+ \times B^+)\delta_B,$$
$$K_b = L_b \cap (B^+ \times B^+)\delta_B \quad (b \in B)$$

are regular by Proposition 2.2, so that (B, K) is an automatic structure for S as required. □

In the case of groups, the converse of Proposition 3.12 is also true, in that, if we have a homomorphism $\phi : A^+ \to G$ and a regular language L such that $L\phi = G$, and if the Cayley graph of G on A has the fellow traveller property with respect to L, then (A, L) is an automatic structure for G. (See Theorem 2.3.5 of [5] for example.) In the case of semigroups, with this definition of the fellow traveller property, this is no longer true. For example, consider any non-automatic semigroup S containing a zero element z (such as $G^\square$ for a non-automatic group G). Since $sz = z$ for any $s \in S$ and $d(s,t) \leqslant d(s,z) + d(z,t)$, we see that $d(s,t) \leqslant 2$ for any s and t, so that S trivially has the fellow traveller property with respect to L for any regular subset L of A^+.

It appears, therefore, that neither of these naive approaches is the correct one in this context. It may well be that one can say something sensible with a different definition of the fellow traveller property or that one can characterize automaticity in a geometric way if one restricts oneself to various classes of semigroups. With this in mind, we finish this section with the following (rather loose) question:

Question 3.14. Is there a geometric condition on the Cayley graph of a semigroup which is equivalent to the semigroup being automatic?

4. Examples

In this section, we give some examples of automatic semigroups and show that not all properties enjoyed by automatic groups generalize to the semigroup case. We start with the obvious case of finite semigroups:

Example 4.1. Any finite semigroup is automatic. If S is a finite semigroup, we can take A to be S and L to be A. Since $L_=$ and L_a, $a \in A$, are finite, they are all regular by Proposition 2.2.

Example 4.2. Let M be the bicyclic monoid. It has a monoid presentation $\langle b, c \colon bc = 1 \rangle$. As a semigroup it may be defined by

$$\langle b, c, e \colon bc = e,\ be = eb = b,\ ce = ec = c \rangle.$$

We let $A = \{b, c, e\}$, and then let $L = (\{c\}^*\{b\}^* - \{\varepsilon\}) \cup \{e\}$. L is regular by Proposition 2.2; we claim that (A, L) is an automatic structure for M.

Since every element of M is represented by a unique element of L, we have that $L_= = \{(\alpha, \alpha) \colon \alpha \in L\}\delta_A$, which is regular by Proposition 2.4. It is clear that $L_e = L_=$, so that L_e is regular. Lastly, note that

$$L_b = [\{(c,c)\}^*\{(b,b)\}^* - \{(\varepsilon,\varepsilon)\}]\{(\$,b)\} \cup \{(e,b)\},$$

$$L_c = [\{(c,c)\}^*\{(b,b)\}^* - \{(\varepsilon,\varepsilon)\}]\{(b,\$)\} \cup \{(c,c)\}^+\{(\$,c)\} \cup \{(e,c)\},$$

so that L_b and L_c are regular by Proposition 2.2 as required.

We now turn to free semigroups. It is well known that finitely generated free groups are automatic, and we find that finitely generated free semigroups are also automatic:

Example 4.3. Any finitely generated free semigroup is automatic. If S is the free semigroup on the finite set A and $L = A^+$, then (A, L) is an automatic structure for S, since

$$L_= = \{(\alpha, \alpha)\colon\ \alpha \in L\}\delta_A$$

is regular by Proposition 2.4, and

$$L_a = \{(\alpha, \alpha a)\delta_A\colon\ \alpha \in L\} = L_=\{(\$, a)\}$$

is regular by Proposition 2.2.

In the case of groups, it follows immediately that a finitely generated subgroup of a free group is automatic, since a subgroup of a free group is free. It is not true that finitely generated subsemigroups of free semigroups are free; however, we will see later (Theorem 8.1) that finitely generated subsemigroups of free semigroups are automatic.

Any automatic group is finitely presented; this result does not pass over to semigroups as the following example shows:

Example 4.4. Consider the semigroup S defined by the presentation

$$\langle a, b\colon\ ab^i a = aba\ (i \geqslant 2)\rangle.$$

No relation can be applied to a proper subword of $ab^i a$ for any i, and so none of the relations $ab^i a = aba$ can be deduced from the remainder. In particular, S is not finitely presented.

Let $A = \{a, b\}$, and let L be the regular language

$$\{b\}^*\{a, ab\}^*\{b\}^* - \{\varepsilon\}.$$

We see that L maps onto S; in fact, every element of S is represented by precisely one element of L. Now

$$L_= = \{(\alpha, \alpha)\colon\ \alpha \in L\}\delta_A$$

is regular by Proposition 2.4,

$$L_a = \{(b, b)\}^*\{(a, a), (ab, ab)\delta_A\}^*\{(ab^2, aba)\delta_A\}\{(b, \$)\}^*$$
$$\cup\, \{(b, b)\}^*\{(a, a), (ab, ab)\delta_A\}^+\{(\$, a)\} \cup \{(b, b)\}^+\{(\$, a)\}$$

is regular by Propositions 2.2 and 2.4, and

$$L_b = L_=\{(\$, b)\}$$

is regular by Proposition 2.2. So (A, L) is an automatic structure for S.

We note that the automatic semigroup from Example 3.9 also cannot be finitely presented.

Yet another way of seeing that automatic semigroups need not be finitely presented is to use Theorem 8.1 where we show that finitely generated subsemigroups of free semigroups are automatic, since it is well known that finitely generated subsemigroups of free semigroups need not be finitely presented; however, the above example is more straightforward and gives a direct illustration of this fact.

A nice property of groups is that, if a group is automatic with respect to one finite generating set, then it is automatic with respect to *any* finite generating set; see Theorem 2.4.1 of [5] for example. This result does not generalize to semigroups as the following example shows:

Example 4.5. Let F be the free semigroup on a, b and c, and let S be the subsemigroup generated by $u=c$, $v=ac$, $w=ca$, $x=ab$ and $y=baba$. Let $A=\{u,v,w,x,y\}$. It is clear that we have relations $ux^{2i}v=wy^iu$, $i\geqslant 0$, in S. Let $\mathfrak{R}$ denote this set of relations; we claim that $\mathfrak{R}$ is a set of defining relations for S.

Assume that $\mathfrak{R}$ is not a set of defining relations for S and let $\mathfrak{I}$ denote the set of relations in S that cannot be deduced from $\mathfrak{R}$. Suppose that $\alpha_1\alpha_2\dots\alpha_n=\beta_1\beta_2\dots\beta_m$ is a relation in $\mathfrak{I}$, where $\alpha_i,\beta_i\in\{u,v,w,x,y\}$. By identifying each α_i and β_i with the corresponding word in F, we get that $\alpha_1\alpha_2\dots\alpha_n\equiv\beta_1\beta_2\dots\beta_m$ in F. We will assume that we have chosen a relation in $\mathfrak{I}$ with $n+m$ minimal. In particular, we have $\alpha_1\neq\beta_1$ since, if $\alpha_1=\beta_1$, we can just cancel the first terms to get a shorter relation in $\mathfrak{I}$.

Now, without loss of generality, $\alpha_1=u$ and $\beta_1=w$, and we have that $\alpha_2\alpha_3\dots\alpha_n\equiv a\beta_2\beta_3\dots\beta_m$ in F. If $\alpha_2=v=ac$, then

$$c\alpha_3\alpha_4\dots\alpha_n\equiv\beta_2\beta_3\dots\beta_m,$$

and $\beta_2=u=c$ or $\beta_2=w=ca$. We cannot have $\beta_2=c$, since we would either have the relation $uv=wu$, which is in $\mathfrak{R}$, or else $\alpha_3\dots\alpha_n\equiv\beta_3\dots\beta_m$ is a shorter relation in $\mathfrak{I}$. If $\beta_2=ca$, then we have $uv\alpha_3\dots\alpha_n=w^2\beta_3\dots\beta_m$, and we may apply $uv=wu$ from $\mathfrak{R}$ to cancel w and get a shorter relation in $\mathfrak{I}$, a contradiction. So assume that $\alpha_2=x=ab$ and then that $\beta_2=y=baba$. Continuing in this way, we see that our relation must be of the form $ux^{2i}v=wy^iu$, a contradiction. So $\mathfrak{R}$ is a set of defining relations for S as claimed and S has presentation

$$\langle u,v,w,x,y\colon\ ux^{2i}v=wy^iu\ (i\geqslant 0)\rangle.$$

Note that none of the relations is redundant, as we cannot apply any relation to a proper subword of another.

Suppose we have an automatic structure (A,L) for S. Let Γ be the Cayley graph of S with respect to A and let k be the constant as in Definition 3.11 and Proposition 3.12. Since wy^i and ux^{2i} are the only words in A^+ representing the corresponding elements in S, we must have that all these words lie in L. In a similar fashion, for any i, we must have at least one of $ux^{2i}v$ and wy^iu in L, since these are the only two words representing the corresponding element of S.

For the moment, let us fix i, and suppose first that $wy^iu \in L$. If we let $\alpha = ux^{2i}$ and $\beta = wy^iu$, then $\alpha v = \beta$, and so $d(\alpha,\beta) = 1$ in Γ. So, by Proposition 3.12, we have $d(\alpha(t),\beta(t)) \leqslant k$ for all t. Since $\alpha(t) \equiv ux^{t-1}$ has length $2t-1$ in F, $\beta(t) \equiv wy^{t-1}$ has length $4t-2$ in F, and no generator of S has length greater than 4 in F, we have that $d(\alpha(t),\beta(t)) \geqslant (2t-1)/4$ in Γ. On the other hand, for the same fixed i, if $ux^{2i}v \in L$, then a similar argument (with $\alpha = wy^i$ and $\beta = ux^{2i}v$) yields that $d(\alpha(t),\beta(t)) \geqslant (2t-1)/4$ in Γ. However, by taking $t > 2k+1$, we have a contradiction, as $d(\alpha(t),\beta(t)) \leqslant k$ for all t.

The upshot of this is that S cannot have an automatic structure (A, L). We could look ahead and use Theorem 8.1 to show that S is automatic. However, it is perhaps instructive to exhibit an explicit generating set here with respect to which S is automatic, and we now proceed to do this.

Let $B = A \cup \{z\}$, where z represents the element $abab$ of F. With respect to this new generating set, S has presentation

$$\langle u, v, w, x, y, z \colon uz^iv = wy^iu \ (i \geqslant 0),\ z = x^2 \rangle.$$

We now have a complete rewriting system

$$zx \to xz, \quad x^2 \to z, \quad wy^iu \to uz^iv \quad (i \geqslant 0)$$

for S. So every element of S is represented by a unique element of the set

$$L = B^+ - (B^*\{zx\}B^* \cup B^*\{w\}\{y\}^*\{u\}B^* \cup B^*\{x^2\}B^*).$$

Note that L is regular by Proposition 2.2. Also note that, if $\alpha \in L$ and if β is a subword of α, then $\beta \in L$. Since $L_= = \{(\alpha,\alpha) \colon \alpha \in L\}$ is regular by Proposition 2.4, it remains to check that the languages L_g are regular for all $g \in B$.

The languages L_v, L_w, L_y and L_z are straightforward; in each of these cases, $(\alpha,\beta)\delta_B \in L_g$ if and only if $\alpha g \equiv \beta$, and so $L_g = L_=\{(\$, g)\}$ is regular by Proposition 2.2.

If $(\alpha,\beta)\delta_B \in L_u$, then either $\alpha \not\equiv \gamma wy^i$ (for any γ and i) and $\beta \equiv \alpha u$, or else $\alpha \equiv \gamma wy^{i_1}wy^{i_2}\dots wy^{i_k}$ (for some $\gamma \in B^*$, $k \geqslant 1$ and $i_j \geqslant 0$ for each j, where γ is not of the form σwy^j) and $\beta \equiv \eta uz^{i_1}uz^{i_2}\dots uz^{i_k}v$. In the first case,

$$(\alpha,\beta)\delta_B \in K = (L_= - (B^*\{w\}\{y\}^* \times B^*\{w\}\{y\}^*)\delta_B)\{(\$, u)\},$$

which is regular by Proposition 2.2. In the second case, since α and β are in L, we have that γ and η are in L, and, as $\gamma = \eta$, we have that $\gamma \equiv \eta$. So $\alpha \equiv \gamma wy^{i_1}wy^{i_2}\dots wy^{i_k}$ and $\beta \equiv \gamma uz^{i_1}uz^{i_2}\dots uz^{i_k}v$. Thus

$$\begin{aligned} L_u = K \cup (&L_= \cup \{(e,e)\} - B(2,\$)^*\{(w,w)\}\{(y,y)\}^*) \\ &\{(wy^{i_1}\dots wy^{i_k}, uz^{i_1}\dots uz^{i_k}v) \colon k \geqslant 1,\ i_1 \geqslant 0,\dots,i_k \geqslant 0\}\delta_B. \end{aligned}$$

Since K and $L_= \cup \{(e,e)\} - B(2,\$)^*\{(w,w)\}\{(y,y)\}^*$ are regular, and

$$\begin{aligned} &\{(wy^{i_1}\dots wy^{i_k}, uz^{i_1}\dots uz^{i_k}v) \colon k \geqslant 1,\ i_1 \geqslant 0,\dots,i_k \geqslant 0\}\delta_B \\ &\quad = \{(w,u)\}(\{(y,z)\}^*\{(w,u)\})^*\{(y,z)\}^*\{(\$, v)\} \end{aligned}$$

is regular by Proposition 2.2, L_u is regular as required.

We now turn to L_x. Suppose that $(\alpha,\beta)\delta_B \in L_x$. We have four cases to consider here. If $\alpha \not\equiv \gamma x$ and $\alpha \not\equiv \gamma z$ for any γ, then $\beta \equiv \alpha x$ and

$$(\alpha,\beta)\delta_B \in L_1 = (L_= - (B^*\{z\} \times B^*\{z\})\delta_B - (B^*\{x\} \times B^*\{x\})\delta_B)\{(\$,x)\},$$

which is regular by Proposition 2.2. If $\alpha \equiv \gamma x$ for some γ, then $\beta \equiv \gamma z$ and

$$(\alpha,\beta)\delta_B \in L_2 = L_=\{(x,z)\},$$

which is regular by Proposition 2.2. If $\alpha \equiv \gamma z^i$ for some γ and some $i > 0$, where γ is not of the form ηx, then $\beta \equiv \gamma x z^i$ and

$$(\alpha,\beta)\delta_B \in L_3 = (L_= - (B^*\{x\} \times B^*\{x\})\delta_B)\{(z,x)\}\{(z,z)\}^*\{(\$,z)\},$$

which is also regular. Finally, if $\alpha \equiv \gamma x z^i$ for some γ and some $i > 0$, then $\beta \equiv \gamma z^{i+1}$ and

$$(\alpha,\beta)\delta_B \in L_4 = L_=\{(x,z)\}\{(z,z)\}^+,$$

which is regular. So $L_x = L_1 \cup L_2 \cup L_3 \cup L_4$ is regular as required.

There are other natural examples of automatic semigroups such as free commutative monoids. It would be interesting to know which other classes of naturally occurring semigroups are automatic; for example, we ask

Question 4.6. Is every finitely generated commutative semigroup automatic?

While all finitely generated abelian groups are automatic (see [5] for example), the structure of finitely generated commutative semigroups is somewhat more complicated, although they are always finitely presented (see [23]).

5. Modifying automatic structures

In this section, we show how, given an automatic structure, we can sometimes create a new one. First we prove:

Proposition 5.1. *Suppose that S is a semigroup with an automatic structure (A,L) and that $B \subseteq A$. Let T be the subsemigroup of S generated by B, and suppose that $L \cap B^+$ maps onto T. Then $(B, L \cap B^+)$ is an automatic structure for T.*

Proof. The set $K = L \cap B^+$ is regular by Proposition 2.2, and, by assumption, maps onto T. In addition,

$$K_= = \{(\alpha,\beta)\colon \alpha,\beta \in K,\ \alpha = \beta\}\delta_B = L_= \cap (B^+ \times B^+)\delta_B,$$

$$K_b = \{(\alpha,\beta)\colon \alpha,\beta \in K,\ \alpha b = \beta\}\delta_B = L_b \cap (B^+ \times B^+)\delta_B \quad (b \in B),$$

are all regular by Proposition 2.2. □

This has the following immediate consequence:

Corollary 5.2. *If S is a semigroup with an automatic structure (A,L), if $B \subseteq A$, and if $L \cap B^+$ maps onto S, then $(B, L \cap B^+)$ is an automatic structure for S.*

We next have:

Proposition 5.3. *If S is a semigroup with an automatic structure (A,L), $K \subseteq L$, K is regular and K maps onto S, then (A,K) is an automatic structure for S.*

Proof. We replace the regular languages $L_=$ and L_a, $a \in A$, by

$$K_= = L_= \cap (K \times K)\delta_A \quad \text{and} \quad K_a = L_a \cap (K \times K)\delta_A$$

respectively. Then (A,K) is an automatic structure for S as required. □

Note that the proof of Proposition 5.3 for groups in [5] (see Corollary 2.3.8) is rather different in that it makes use of the fact that a group is automatic if and only if it has the fellow traveller property; as we have seen, this does not hold for semigroups.

We next turn to the useful concept of an automatic structure "with uniqueness". As with groups, let A be a finite set $\{a_1, a_2, \ldots, a_n\}$ and choose an ordering $a_1 < a_2 < \cdots < a_n$ on A. Then the corresponding *shortlex ordering* on A^* is defined by

$\alpha < \beta$ if and only if either (i) $|\alpha| < |\beta|$ or else (ii) $|\alpha| = |\beta|$ and α precedes β lexicographically (with respect to the ordering on A).

We now have:

Proposition 5.4. *If S is a semigroup with an automatic structure (A,L) and*

$$K = \{\alpha \in L : \text{ if } (\alpha, \beta) \in L_= \text{ for any } \beta, \text{ then } \alpha \leqslant \beta \text{ in the shortlex order}\},$$

then (A,K) is an automatic structure for S.

Proof. K is regular as in Theorem 2.5.1 of [5], and the result follows from Proposition 5.3. □

This has the following immediate consequence:

Corollary 5.5. *If S is a semigroup with an automatic structure (A,L), then there exists an automatic structure (A,K) for S such that $K \subseteq L$ and every element of S is represented by precisely one element of K.*

As for groups, if (A,L) is an automatic structure for S such that every element of S is represented by precisely one element of L, then we say that (A,L) is an automatic structure for S *with uniqueness*. So we have:

Corollary 5.6. *If S is an automatic semigroup, then S has an automatic structure with uniqueness.*

The following result is also useful:

Proposition 5.7. *Let S be a semigroup with an automatic structure (A, L) and let $\alpha \in A^+$. If $K = L \cup \{\alpha\}$, then (A, K) is an automatic structure for S.*

Proof. First note that K is regular by Proposition 2.2. Next, let $J_\alpha = \{\beta \in L\colon\ \overline{\beta} = \overline{\alpha}\}$, which is regular by Proposition 3.1. So

$$K_= = L_= \cup (J_\alpha \times \{\alpha\})\delta_A \cup (\{\alpha\} \times J_\alpha)\delta_A \cup \{(\alpha, \alpha)\}\delta_A$$

is regular. Now consider

$$\begin{aligned} K_a = {} & \{(\gamma, \beta)\colon\ \gamma, \beta \in L,\ \overline{\gamma a} = \overline{\beta}\}\delta_A \cup \{(\alpha, \beta)\colon\ \beta \in L \cup \{\alpha\},\ \overline{\beta} = \overline{\alpha a}\}\delta_A \\ & \cup \{(\gamma, \alpha)\colon\ \gamma \in L \cup \{\alpha\},\ \overline{\gamma a} = \overline{\alpha}\}\delta_A. \end{aligned}$$

We will show that K_a is regular by showing that each of the three sets in this union is regular and then using Proposition 2.2.

The set $\{(\gamma, \beta)\colon\ \gamma,\ \beta \in L,\ \overline{\gamma a} = \overline{\beta}\}\delta_A$ is just L_a, which is regular. Next, choose $\theta \in L$ with $\overline{\theta} = \overline{\alpha}$. Since L_a is regular, we have that

$$L_a \cap (A^+ \times \{\theta\})\delta_A = \{(\gamma, \theta)\colon\ \gamma \in L,\ \overline{\gamma a} = \overline{\theta}\}\delta_A$$

is regular by Proposition 2.2. So $\{\gamma \in L\colon\ \overline{\gamma a} = \overline{\theta}\}$ is regular, and hence $\{(\gamma, \alpha)\colon\ \gamma \in L,\ \overline{\gamma a} = \overline{\theta}\}\delta_A$ is regular by Proposition 2.2. Since $\overline{\theta} = \overline{\alpha}$, we have that $\{(\gamma, \alpha)\colon\ \gamma \in L,\ \overline{\gamma a} = \overline{\alpha}\}\delta_A$ is regular, and so

$$\{(\gamma, \alpha)\colon\ \gamma \in L \cup \{\alpha\},\ \overline{\gamma a} = \overline{\alpha}\}\delta_A$$

is regular by Proposition 2.2.

Lastly we consider $\{(\alpha, \beta)\colon\ \beta \in L \cup \{\alpha\},\ \overline{\beta} = \overline{\alpha a}\}\delta_A$. We choose $\theta \in L$ with $\overline{\theta} = \overline{\alpha}$ as above, and then note that, since L_a is regular, we have that

$$L_a \cap (\{\theta\} \times A^+)\delta_A = \{(\theta, \beta)\colon\ \beta \in L,\ \overline{\theta a} = \overline{\beta}\}\delta_A$$

is regular by Proposition 2.2. So $\{\beta \in L\colon\ \overline{\theta a} = \overline{\beta}\}$ is regular, and hence $\{(\alpha, \beta)\colon\ \beta \in L, \overline{\theta a} = \overline{\beta}\}\delta_A$ is regular by Proposition 2.2. So $\{(\alpha, \beta)\colon\ \beta \in L,\ \overline{\alpha a} = \overline{\beta}\}\delta_A$ is regular, and hence

$$\{(\alpha, \beta)\colon\ \beta \in L \cup \{\alpha\}, \overline{\alpha a} = \overline{\beta}\}\delta_A$$

is regular by Proposition 2.2 as required. □

In addition to adding a word, we can also add a symbol:

Proposition 5.8. *Suppose that S is a semigroup with an automatic structure (A, L) and let $B = A \cup \{b\}$ where $b \notin A$. For any word $\alpha \in A^+$, we have an automatic structure (B, K) for S, where $K = L$ and b is mapped to the element of S represented by α.*

Proof. Since no word in K contains an occurrence of b, we have $K_= = L_=$ and $K_a = L_a$ for any $a \in A$. Now

$$K_b = \{(\beta,\gamma):\ \beta \in L,\ \gamma \in L,\ \beta b = \gamma\}\delta_B = \{(\beta,\gamma):\ \beta \in L,\ \gamma \in L,\ \beta\alpha = \gamma\}\delta_B$$

is regular by Proposition 3.2. So (B,K) is an automatic structure for S as required. □

We will also need the following fact about automatic monoids:

Proposition 5.9. *Let M be an automatic monoid. Then M has an automatic structure (A,L) with uniqueness such that A contains an element e representing the identity element of M, $e \in L$, and $L - \{e\} \subseteq (A - \{e\})^+$.*

Proof. By Corollary 5.6, M has an automatic structure (B,K) with uniqueness. Let $A = B \cup \{e\}$, where $e \notin B$ and e represents the identity element of M; we now have an automatic structure (A,K) for M by Proposition 5.8, and hence an automatic structure $(A, K \cup \{e\})$ for M by Proposition 5.7. Let α be the unique word in K^+ representing the identity element of M. Then $L = K \cup \{e\} - \{\alpha\}$ is regular and every element of M is represented by a unique element of L, so that (A,L) is an automatic structure for M with uniqueness by Proposition 5.3 as required. □

6. Free and direct products

In this section, we consider the operations of taking free products and direct products of semigroups.

The free product $S_1 * S_2$ of two (disjoint) semigroups S_1 and S_2 can be defined in terms of the usual universal property. If S_1 and S_2 are defined by presentations $\langle A_1 : R_1\rangle$ and $\langle A_2 : R_2\rangle$, respectively (where $A_1 \cap A_2 = \emptyset$) then $S_1 * S_2$ is defined by $\langle A_1 \cup A_2 : R_1 \cup R_2\rangle$. Every element of $S_1 * S_2$ can be written uniquely as an alternating product $t_1 t_2 \ldots t_k$, where $k \geqslant 1$, $t_i \in S_1 \cup S_2$ $(i = 1,\ldots,k)$, and $t_i \in S_1$ if and only if $t_{i+1} \in S_2$ $(i = 1,\ldots,k-1)$. For further details see [13].

Theorem 6.1. *Let S_1 and S_2 be semigroups. Then $S_1 * S_2$ is automatic if and only if both S_1 and S_2 are automatic.*

Proof. Suppose that S_1 and S_2 are automatic. By Corollary 5.6, S_1 and S_2 have automatic structures with uniqueness, say (A_1,L_1) and (A_2,L_2), respectively. Without loss of generality, we may assume that A_1 and A_2 are disjoint. Let $A = A_1 \cup A_2$ and

$$L = (L_1 \cup \{\varepsilon\})(L_2 L_1)^*(L_2 \cup \{\varepsilon\}) - \{\varepsilon\},$$

so that $L \subseteq A^+$ is regular.

Every element of $S_1 * S_2$ is represented by a unique element of L, and so $L_= = \{(\alpha,\alpha)\colon \alpha \in L\}\delta_A$ is regular by Proposition 2.4. Now let

$$K_1 = (L_1 \cup \{\varepsilon\})(L_2L_1)^* - \{\varepsilon\}, \qquad K_2 = (L_1 \cup \{\varepsilon\})(L_2L_1)^* L_2,$$

so that $L = K_1 \cup K_2$ and $K_1 \cap K_2 = \emptyset$.

If $a \in A_1$ and θ is the unique word in L_1 such that $\theta = a$, then

$$L_a = (L_1)_a \cup (\{(\alpha,\alpha)\colon\ \alpha \in K_2\}\delta_A)((L_1)_a \cup \{(\varepsilon,\theta)\}\delta_A).$$

Similarly, if $a \in A_2$ and $a = \theta \in L_2$, then

$$L_a = (L_2)_a \cup (\{(\alpha,\alpha)\colon\ \alpha \in K_1\}\delta_A)((L_2)_a \cup \{(\varepsilon,\theta)\}\delta_A).$$

In either case, L_a is regular by Propositions 2.2 and 2.4 as required.

Conversely, suppose that $S_1 * S_2$ is automatic with an automatic structure (A,L). Let

$$B = \{a \in A\colon\ a \text{ represents an element of } S_1\}.$$

Since $xy \in S_1$ implies that both $x \in S_1$ and $y \in S_1$, we have that $L \cap B^+$ maps onto S_1. So $(B, L \cap B^+)$ is an automatic structure for S_1 by Proposition 5.1. So S_1 is automatic, and the argument for S_2 is similar. □

One can define free products for groups and monoids in an analogous way to that for semigroups. The proof for groups that, if G_1 and G_2 are automatic, then $G_1 * G_2$ is automatic, is very similar to that of Proposition 6.1; see [1] for example. However, the proof that, if $G_1 * G_2$ is automatic, then G_1 and G_2 are automatic, is harder than the corresponding proof here. This is due to the fact that the semigroup free product is not the same as the group free product (even when the factors are themselves groups). One may think of the group free product as being the semigroup free product with the identity subgroups amalgamated, and it is this amalgamation that is the source of the difficulties. We can, however, define a monoid free product analogous to the group free product, and, in a similar manner to the group case, we can prove:

Theorem 6.2. *Let M_1 and M_2 be monoids. If both M_1 and M_2 are automatic, then the* (*monoid*) *free product $M_1 * M_2$ is automatic.*

Proof. Suppose that (A_1, L_1) and (A_2, L_2) are automatic structures with uniqueness for M_1 and M_2, respectively, as in Proposition 5.9, with $A_1 \cap A_2 = \{e\}$, e representing the identity element of each M_i, $e \in L_i$ $(i = 1,2)$, and $\bar{L}_i = L_i - \{e\} \subseteq (A_i - \{e\})^+$ $(i = 1,2)$. Let $A = A_1 \cup A_2$ and

$$L = \{e\}(\bar{L}_1 \cup \{\varepsilon\})(\bar{L}_2\bar{L}_1)^*(\bar{L}_2 \cup \{\varepsilon\}).$$

We claim that (A,L) is an automatic structure for $M = M_1 * M_2$.

Since $\bar{L}_1$ and $\bar{L}_2$ are regular, L is clearly regular. Since every element of M is represented by a unique element of L, $L_= = \{(\alpha,\alpha)\colon \alpha \in L\}\delta_A$ is regular by

Proposition 2.4. In a similar way to the proof of Theorem 6.1, let

$$K_1=\{e\}(\bar{L}_1\cup\{\varepsilon\})(\bar{L}_2\bar{L}_1)^*, \qquad K_2=\{e\}(\bar{L}_1\cup\{\varepsilon\})(\bar{L}_2\bar{L}_1)^*\bar{L}_2,$$

so that $L=K_1\cup K_2$ and $K_1\cap K_2=\emptyset$.

First note that $L_e=L_=$ is regular. Next, if $a\in A_1-\{e\}$, let θ be the unique word in $\bar{L}_1$ such that $\theta=a$; then $(\bar{L}_1)_a=(L_1)_a-\{(e,\theta)\}\delta_A$ is regular, and so

$$L_a=\{(e,e)\}(\bar{L}_1)_a\cup(\{(\alpha,\alpha)\colon\ \alpha\in K_2\}\delta_A)(\{(\varepsilon,\theta)\}\delta_A\cup(\bar{L}_1)_a)$$

is regular. Similarly, if $a\in A_2-\{e\}$, let θ be the unique word in $\bar{L}_2$ such that $\theta=a$. Again, $(\bar{L}_2)_a=(L_2)_a-\{(e,\theta)\}\delta_A$ is regular, and so

$$L_a=\{(e,e)\}(\bar{L}_2)_a\cup(\{(\alpha,\alpha)\colon\ \alpha\in K_1\}\delta_A)(\{(\varepsilon,\theta)\}\delta_A\cup(\bar{L}_2)_a)$$

is regular as required. □

The converse of Theorem 6.2 is true for groups, and so it is natural to ask:

Question 6.3. If M_1 and M_2 are monoids such that the monoid free product $M_1 * M_2$ is automatic, does it follow that M_1 and M_2 are automatic?

We now turn to direct products. Unlike the case of free products, there is no ambiguity here, in that, if G_1 and G_2 are groups, then $G_1\times G_2$ is the same both regarded as a semigroup direct product and as a group direct product.

It is well known that, even if S_1 and S_2 are finitely generated semigroups, then the direct product $S_1\times S_2$ need not be finitely generated; for example, if S_1 and S_2 are both free semigroups on one generator, then $S_1\times S_2$ is not finitely generated. This example shows that we can have S_1 and S_2 automatic but $S_1\times S_2$ not finitely generated (and hence certainly not automatic). For further results on the finite generation (and finite presentability) of direct products of semigroups, see [24]. So we obviously cannot have a generalization of the fact that the direct product of automatic groups is automatic to the semigroup case. The situation is rather different when we consider monoids, however, and we prove:

Theorem 6.4. *If M_1 and M_2 are automatic monoids, then their direct product $M_1\times M_2$ is automatic.*

Proof. Suppose that (A_1,L_1) and (A_2,L_2) are automatic structures with uniqueness for M_1 and M_2, respectively, as in Proposition 5.9, with $A_1\cap A_2=\emptyset$, $e_i\in A_i$, e_i representing the identity element of M_i, $e_i\in L_i$, and $L_i-\{e_i\}\subseteq(A_i-\{e_i\})^+$ $(i=1,2)$. Let $\bar{L}_i$ denote $L_i-\{e_i\}$. We let $A=A_1\cup A_2$. If $\alpha\equiv a_1\dots a_n\in L_1$ and $\beta\equiv b_1\dots b_m\in L_2$, then we define $\alpha \sharp \beta \in A^+$ by

$$\alpha \sharp \beta = \begin{cases} a_1 b_1 \ldots a_n b_n & \text{if } n = m, \\ a_1 b_1 \ldots a_n b_n e_1 b_{n+1} \ldots e_1 b_m & \text{if } n < m, \\ a_1 b_1 \ldots a_m b_m a_{m+1} e_2 \ldots a_n e_2 & \text{if } n > m. \end{cases}$$

If $\sigma : A(2,\$)^* \to A^*$ is the homomorphism defined by

$$(a,b) \mapsto ab, \qquad (a,\$) \mapsto ae_2, \qquad (\$,b) \mapsto e_1 b,$$

then $\alpha \sharp \beta = (\alpha,\beta)\delta_A \sigma$. Let

$$L = \{\alpha \sharp \beta\colon\ \alpha \in L_1, \beta \in L_2\} = (L_1 \times L_2)\delta_A \sigma,$$

which is regular by Proposition 2.2. We claim that (A, L) is an automatic structure for $M = M_1 \times M_2$.

Since (A_1, L_1) and (A_2, L_2) are automatic structures with uniqueness, $\alpha \sharp \beta = \alpha' \sharp \beta'$ in M if and only if $\alpha \equiv \alpha'$ and $\beta \equiv \beta'$. So

$$L_= = \{(\alpha \sharp \beta, \alpha \sharp \beta)\colon\ \alpha \in L_1, \beta \in L_2\}\delta_A = \{(\gamma,\gamma)\colon\ \gamma \in L\}\delta_A,$$

which is regular by Proposition 2.4.

Let $a \in A_1$. Then $(\alpha \sharp \beta)a = \alpha' \sharp \beta'$ in M if and only if $\alpha a = \alpha'$ in M_1 and $\beta = \beta'$ in M_2, i.e. if and only if $(\alpha,\alpha') \in (L_1)_a$ and $\beta \equiv \beta' \in L_2$. So

$$L_a = \{(\alpha \sharp \beta, \alpha' \sharp \beta)\colon\ (\alpha,\alpha') \in (L_1)_a, \beta \in L_2\}\delta_A.$$

Let $C = A(2,\$)$, and define a homomorphism $\psi : C(2,\$)^* \to A(2,\$)^*$ by

$$\begin{array}{lll l}
((a_1,a_1'),(a_2,a_2)) & \mapsto & (a_1a_2, a_1'a_2)\delta_A & a_1, a_1' \in A_1, a_2 \in A_2, \\
((a_1,\$),(a_2,a_2)) & \mapsto & (a_1a_2, e_1a_2)\delta_A & a_1 \in A_1, a_2 \in A_2, \\
((\$,a_1'),(a_2,a_2)) & \mapsto & (e_1a_2, a_1'a_2)\delta_A & a_1' \in A_1, a_2 \in A_2, \\
(\$,(a_2,a_2)) & \mapsto & (e_1a_2, e_1a_2)\delta_A & a_2 \in A_2, \\
((a_1,a_1'),\$) & \mapsto & (a_1e_2, a_1'e_2)\delta_A & a_1, a_1' \in A_1, \\
((a_1,\$),\$) & \mapsto & (a_1e_2, \$)\delta_A & a_1 \in A_1, \\
((\$,a_1'),\$) & \mapsto & (\$, a_1'e_2)\delta_A & a_1' \in A_1, \\
(x,y) & \mapsto & \varepsilon & \text{otherwise.}
\end{array}$$

We see that $L_a = (((L_1)_a \times \{(\alpha,\alpha)\colon\ \alpha \in L_2\}\delta_A)\delta_C)\psi$, and so L_a is regular by Propositions 2.4, 2.2(vi) and 2.2(iv). The argument for L_a with $a \in A_2$ is similar. □

We have the following (rather general) question:

Question 6.5. Under what conditions is the direct product of two automatic semigroups automatic?

In the case of groups, it is still unknown as to whether or not $G_1 \times G_2$ automatic implies that both G_1 and G_2 are automatic, and we can ask the same question for monoids:

Question 6.6. If M_1 and M_2 are monoids such that $M_1 \times M_2$ is automatic, are M_1 and M_2 necessarily automatic?

Naturally enough, we have the same question for semigroups:

Question 6.7. If S_1 and S_2 are semigroups such that $S_1 \times S_2$ is automatic, are S_1 and S_2 necessarily automatic?

Obviously if the answer to Question 6.7, is "yes", then the answer to Question 6.6 is also "yes".

7. Adjoining elements to semigroups

We have seen (Proposition 3.13) that a semigroup S is automatic if and only if $S^{\square}$ is automatic, and (Proposition 3.5) that, if S is automatic, then S^I is automatic. In this section we prove the converse of this last result, so that S is automatic if and only if S^I is automatic. We will use this result in the next section to prove that any finitely generated subsemigroup of a free semigroup is automatic. We first prove the following technical result:

Proposition 7.1. *Suppose that S is a semigroup and that S^I has an automatic structure (A,L) with uniqueness with $1 \in A$. Then there exists $h>0$ such that no word in L contains 1^h as a subword.*

Proof. Let M be an automaton accepting L and suppose that no such h exists. Let n be the number of states in M and let α be a word in L containing a subword 1^m with $m > n$, say $\alpha \equiv \beta 1^m \gamma$. There are more words β, $\beta 1$, $\beta 1^2, \ldots, \beta 1^m$ than there are states in M, so that, after reading the two words $\beta 1^u$ and $\beta 1^v$ for some $0 \leqslant u < v \leqslant m$, M must be in the same state q. Since M accepts $\alpha \equiv \beta 1^u 1^{m-u} \gamma$, M must also accept $\alpha' \equiv \beta 1^v 1^{m-u} \gamma$, and hence $\alpha' \in L$. Since $\alpha = \alpha'$ in S, this contradicts the assumption that (A,L) is an automatic structure with uniqueness. $\square$

We are now in a position to prove:

Theorem 7.2. *Let S be a semigroup. Then S^I is automatic if and only if S is automatic.*

Proof. As noted above, we proved in Proposition 3.5 that, if S is automatic, then S^I is automatic; we prove the converse here.

If S^I is automatic, then we may suppose that S^I has an automatic structure (A, L) with uniqueness with $1 \in A$. By Proposition 7.1, there exists $h > 0$ such that no word in L contains 1^h as a subword. Let

$$W = \{\alpha \in A^+ \colon \ |\alpha| \leqslant h\}$$

and let C be a set of symbols in a one-to-one correspondence with W. So we have a bijection $\phi : W \to C$. Let

$$B = (W - \{1\}^+)\phi \subseteq C.$$

We may extend ϕ to an injective mapping from A^+ to C^+ by inductively defining $\alpha\phi$ to be $(\beta\phi)(\gamma\phi)$ if $|\alpha| > h$, where $\alpha \equiv \beta\gamma$ and $|\beta| = h$; note that α and $\alpha\phi$ represent the same element of S^I. We may also define $\theta : C^+ \to A^+$ by mapping each element c of C onto $c\phi^{-1}$ and then extending θ to a homomorphism. Note that $\alpha\phi\theta \equiv \alpha$ for all α in A^+.

Let $C_1 = \{c \in C\colon \ |c\theta| = h\}$ and $C_2 = \{c \in C\colon \ |c\theta| < h\}$, so that $C = C_1 \cup C_2$. Then $D = C_1^* C_2 \cup C_1^+$ is regular and so $L\phi = L\theta^{-1} \cap D$ is regular by Proposition 2.2. We now define a homomorphism $\psi : C^+ \to B^*$ by

$$c\psi = \begin{cases} c & \text{if } c \in B, \\ \varepsilon & \text{if } c \notin B. \end{cases}$$

If $\alpha \in C^* - \{1\}^*$, then $\alpha\psi \in B^+$ and α, $\alpha\psi$ represent the same element of S. Since ψ is a homomorphism and $L\phi$ is regular, $L\phi\psi$ is also regular, and so $K = L\phi\psi \cap B^+$ is a regular subset of B^+ by Proposition 2.2. We claim that (B, K) is an automatic structure for S.

In order to prove this, we will show that, if $J \subseteq (A^+ \times A^+)\delta_A$ is regular in $A(2,\$)^*$, then $J\mu_A\phi\delta_C$ is a regular set in $C(2,\$)^*$, where we define $(\alpha_1, \alpha_2)\phi$ for $\alpha_1, \alpha_2 \in A^+$ to be $(\alpha_1\phi, \alpha_2\phi)$. To do this, we define a homomorphism $\bar{\theta} : C(2,\$)^* \to A(2,\$)^*$ by

$$(c, c') \mapsto (c\theta, c'\theta)\delta_A; \quad (c, \$) \mapsto (c\theta, \varepsilon)\delta_A; \quad (\$, c') \mapsto (\varepsilon, c'\theta)\delta_A.$$

Then

$$J\mu_A\phi\delta_C = J\bar{\theta}^{-1} \cap (D \times D)\delta_C,$$

so that $J\mu_A\phi\delta_C$ is regular by Proposition 2.2 as required.

We may now establish the claim that (B, K) is an automatic structure for S. Every element s of S^I is represented by a unique element of L and hence by a unique element α of $L\phi$. If $s \neq 1$, then $\alpha \notin \{1\}^*$ and s is represented by the unique element $\alpha\psi$ of $L\phi\psi \cap B^+ = K$; so every element of S is represented by a unique element of K. We then have that $K_= = \{(\alpha, \alpha)\colon \ \alpha \in K\}\delta_B$, so that $K_=$ is regular by Proposition 2.4.

Let $b \in B$ and let $\gamma \in A^+$ be such that $\gamma\phi = b$. By Proposition 3.2, the set $L_\gamma = \{(\alpha, \beta) \in L \times L\colon \ \alpha\gamma = \beta\}\delta_A$ is regular. By the above,

$$U = L_\gamma\mu_A\phi\delta_C = \{(\zeta, \eta) \in L\phi \times L\phi\colon \ \zeta b = \eta\}\delta_C$$

is a regular subset of $C(2,\$)^*$. Let $M=(Q,C(2,\$),\tau,s,F)$ be a deterministic finite automaton accepting U.

We have that $K_b = U\mu_C\psi\delta_B$ where we define $(\alpha_1,\alpha_2)\psi$ to be $(\alpha_1\psi,\alpha_2\psi)$. We let N denote the finite automaton $(Q,B(2,\$),\tau',s,F')$ where τ' is defined as follows:

$$
\begin{array}{llll}
\text{if} & (q,(x,y))\tau = r, & x,y\in B & \text{then} \quad (q,(x,y))\tau' = r,\\
\text{if} & (q,(x,y))\tau = r, & x\in C-B,\ y\in B & \text{then} \quad (q,(\$,y))\tau' = r,\\
\text{if} & (q,(x,y))\tau = r, & x\in B,\ y\in C-B & \text{then} \quad (q,(x,\$))\tau' = r.
\end{array}
$$

If $(q,(x,y))\tau = r$ in M, $x\psi = \varepsilon = y\psi$, then there is no corresponding transition defined in N. However, in this case, if $r\in F$, then we add q to the set F' of final states of N, so that F' consists of F together with all states q obtained in this way. Note that, since no word in L contains 1^h as a subword, if $\zeta \equiv c_1c_2\ldots c_r$ is a word in $L\phi$, then $|c_i| = h$ and so $c_i\psi \neq \varepsilon$ for $1\leqslant i\leqslant r-1$. We see that N accepts K_b, so that K_b is regular as required. □

We have discussed the relationship between S and $S^{\Box}$ and between S and S^I. In each of the cases, S is a subsemigroup of $S^{\Box}$ (respectively, S^I), and the number of elements in $S^{\Box}$ (respectively, S^I) outside S is finite. In general, a subsemigroup S of a semigroup T is said to have *finite index* in T if $T - S$ is finite. There are many interesting results on subsemigroups of finite index in the sense that, if S is a subsemigroup of finite index in T, then, for many properties $\wp$, S has property $\wp$ if and only if T has property $\wp$. This is true, for example, if $\wp$ is the property of being finitely generated [15, 2, 25], being finitely presented [25], having soluble word problem [25], being locally finite [25], being locally finitely presented [25], being periodic [25], having finitely many right ideals [25] or being residually finite [26]. It is natural to ask whether being automatic is such a property:

Question 7.3. Suppose that T is a semigroup and that S is a subsemigroup of T of finite index in T. Is it always the case that S is automatic if and only if T is automatic?

8. Subsemigroups of free semigroups

In this section we add to our list of naturally occurring automatic semigroups by proving the following result:

Theorem 8.1. *If F is a free semigroup on a set X and S is a finitely generated subsemigroup of F, then S is automatic.*

Proof. We will show that S^I is automatic; the result then follows from Theorem 7.2.

Suppose that S is generated by $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$, where each α_i is an element of F. Let α_i have length m_i when considered as a word in X^+. Let $A=\{a_1,a_2,\ldots,a_n,1\}$, and let

$$L=\{a_1 1^{m_1-1}, a_2 1^{m_2-1},\ldots,a_n 1^{m_n-1}\}^+\cup\{1\},$$

which is clearly regular. We have a natural homomorphism $\rho:(A\cup\{\$\})^+\to X^*$ defined by $a_i\mapsto\alpha_i$ for each i, $1\mapsto\varepsilon$, $\$\mapsto\varepsilon$. We claim that (A,L) is an automatic structure for S^1.

Let $K=\{a_1 1^{m_1-1},a_2 1^{m_2-1},\dots,a_n 1^{m_n-1}\}^+$, so that $L_= = K_= \cup\{(1,1)\}$, $L_1=L_=$ and $L_{a_i}=K_{a_i}\cup\{(1,w)\delta_A\colon w\in L, w\rho\equiv\alpha_i\}$ for each i. Since $\{(1,1)\}$ and $\{(1,w)\delta_A\colon w\in L, w\rho\equiv\alpha_i\}$ are finite sets, it is enough to show that $K_=$ is regular and that K_{a_i} is regular for each i.

Let W be the subset of X^* consisting of all subwords of the α_i, i.e.

$$W=\{\gamma\colon \alpha_i\equiv\beta\gamma\eta \text{ for some } \beta,\eta\in X^* \text{ and for some } i\}.$$

Note that $\varepsilon\in W$. Let $Q=(W\times\{\varepsilon\})\cup(\{\varepsilon\}\times W)$ and consider a finite automaton with states Q, start state $(\varepsilon,\varepsilon)$, inputs $(A\cup\{\$\})\times(A\cup\{\$\})$, and transitions defined as follows:

$$
\begin{aligned}
&(\beta,\varepsilon)(x,y)=(\varepsilon,\gamma) && \text{if } \beta(x\rho) \text{ is a prefix of } y\rho \text{ and } y\rho\equiv\beta(x\rho)\gamma,\\
&(\beta,\varepsilon)(x,y)=(\gamma,\varepsilon) && \text{if } y\rho \text{ is a prefix of } \beta(x\rho) \text{ and } \beta(x\rho)\equiv(y\rho)\gamma,\\
&(\beta,\varepsilon)(\$,y)=(\varepsilon,\gamma) && \text{if } \beta\gamma\equiv y\rho,\\
&(\beta,\varepsilon)(\$,y)=(\gamma,\varepsilon) && \text{if } \beta\equiv(y\rho)\gamma,\\
&(\varepsilon,\beta)(x,y)=(\gamma,\varepsilon) && \text{if } \beta(y\rho) \text{ is a prefix of } x\rho \text{ and } x\rho\equiv\beta(y\rho)\gamma,\\
&(\varepsilon,\beta)(x,y)=(\varepsilon,\gamma) && \text{if } x\rho \text{ is a prefix of } \beta(y\rho) \text{ and } \beta(y\rho)\equiv(x\rho)\gamma,\\
&(\varepsilon,\beta)(x,\$)=(\gamma,\varepsilon) && \text{if } \beta\gamma\equiv x\rho,\\
&(\varepsilon,\beta)(x,\$)=(\varepsilon,\gamma) && \text{if } \gamma\equiv(x\rho)\beta.
\end{aligned}
$$

In all other cases, no transition is defined. An inductive argument shows that, for any $u,v\in(A\cup\{\$\})^+$, we have

$$
\begin{aligned}
&(\beta,\varepsilon)(\eta,\zeta)=(\varepsilon,\gamma) \;\Leftrightarrow\; \zeta\rho\equiv\beta(\eta\rho)\gamma,\\
&(\beta,\varepsilon)(\eta,\zeta)=(\gamma,\varepsilon) \;\Leftrightarrow\; \beta(\eta\rho)\equiv(\zeta\rho)\gamma,\\
&(\varepsilon,\beta)(\eta,\zeta)=(\gamma,\varepsilon) \;\Leftrightarrow\; \eta\rho\equiv\beta(\zeta\rho)\gamma,\\
&(\varepsilon,\beta)(\eta,\zeta)=(\varepsilon,\gamma) \;\Leftrightarrow\; \beta(\zeta\rho)\equiv(\eta\rho)\gamma.
\end{aligned}
$$

If we make $(\varepsilon,\varepsilon)$ into the unique accept state, we have an automaton accepting $K_=$; if we make (ε,α_i) into the unique accept state, we have an automaton accepting K_{a_i} for any i. So $K_=$ and each K_{a_i} are regular as required. □

It is interesting to contrast the above argument with the proof in [27] that a subsemigroup of a free semigroup is rational.

9. Postscript

Since the first draft of this paper, there has been further work in the area of automatic semigroups (see for example [3, 4, 6, 17–19]). Some of this work is relevant to the problems posed in this paper. In particular, the converse of Theorem 6.2 has been shown to hold; see [6]. This is proved by first showing that the invariance under the change of generators holds for automatic monoids. Also in [3] it is shown that the direct product of two automatic semigroups is automatic provided it is finitely generated. The relationship between the fellow traveller property and automaticity has been studied further in the context of completely simple semigroups; see [4]. Finally, connections between automatic structures and rewriting systems are considered in [18, 19], and the Dehn functions of automatic monoids are considered in [17].

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