Decay results in a doubly diffusive problem

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ABSTRACT

In this work we investigate temporal decay bounds of a system of doubly diffusive problems arising from Luikov’s system of equations under Dirichlet or Neumann boundary conditions. Under various assumptions on the coefficients we derive decay bounds for $L^2$ integrals.

1. Introduction

One of the theories that has been developed for heat and mass diffusion is well known as Luikov’s system of equations. A derivation of this system is found in [2, p. 401]. Specifically, the system defined on a three-dimensional bounded region $\Omega$ and for time $t$ relates the temperature $u$ and the mass diffusion $v$ as follows:

$$
\frac{\partial u}{\partial t} = a\Delta u + b\frac{\partial v}{\partial t} \quad \text{in } \Omega \times \{t > 0\},
$$

$$
\frac{\partial v}{\partial t} = c\Delta v + d\Delta u \quad \text{in } \Omega \times \{t > 0\},
$$

where $a$, $b$, $c$, and $d$ are positive constants and the symbol $\Delta$ denotes the Laplace operator in $\mathbb{R}^3$. Various initial and boundary value problems associated with this system have been investigated in the literature (see [2]). In this work, by relaxing the positivity assumptions on some of the constants $a$, $b$, $c$, and $d$, we are primarily concerned with the question of decay in time for solutions of the system (1.1) and (1.2) under appropriate initial and boundary conditions. Similar systems of equations dealing with the temperature and heat flux have received considerable attention in the recent literature [1,3–5].

For convenience, letting

$$
\alpha = a + bd, \quad \beta = bc, \quad \gamma = d, \quad \delta = c,
$$

we rewrite (1.1) and (1.2) as

$$
\frac{\partial u}{\partial t} = \alpha\Delta u + \beta\Delta v \quad \text{in } \Omega \times \{t > 0\},
$$

$$
\frac{\partial v}{\partial t} = \gamma\Delta u + \delta\Delta v \quad \text{in } \Omega \times \{t > 0\},
$$

where $\alpha$, $\beta$, $\gamma$, and $\delta$ are constants.
Under assumptions for the coefficients $\alpha$, $\beta$, $\gamma$, and $\delta$ we investigate the question of decay in time for solutions of the system (1.3) and (1.4) subject to the initial conditions
\begin{equation}
{u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x),}
\end{equation}
and either the Dirichlet boundary conditions
\begin{equation}
{u(x, t) = 0, \quad v(x, t) = 0 \quad \text{on } \partial \Omega \times \{t > 0\},}
\end{equation}
or the Neumann boundary conditions
\begin{equation}
{\frac{\partial u}{\partial n}(x, t) = 0, \quad \frac{\partial v}{\partial n}(x, t) = 0 \quad \text{on } \partial \Omega \times \{t > 0\},}
\end{equation}
where $\partial / \partial n$ is the outward directed normal derivative on $\partial \Omega$.

In our investigation we make use of the well-known results which hold for solutions of the problem
\begin{equation}
{\frac{\partial w}{\partial t} - k \Delta w = 0 \quad \text{in } \Omega \times \{t > 0\},}
\end{equation}
\begin{equation}
{w(x, 0) = w_0(x),}
\end{equation}
with either
\begin{equation}
{w = 0 \quad \text{on } \partial \Omega \times \{t > 0\},}
\end{equation}
or
\begin{equation}
{\frac{\partial w}{\partial n} = 0 \quad \text{on } \partial \Omega \times \{t > 0\}.}
\end{equation}

First, for the initial and Dirichlet boundary value problem (1.8)–(1.10) we have
\begin{equation}
{\int_{\Omega} w^2 \, dx \leq \int_{\Omega} w_0^2 \, dx \exp(-2k\lambda_1 t),}
\end{equation}
where $\lambda_1$ is the first eigenvalue of the problem
\begin{equation}
{\Delta \phi + \lambda \phi = 0 \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \partial \Omega.}
\end{equation}

Second, for the initial and Neumann boundary value problem (1.8), (1.9) and (1.11) we have
\begin{equation}
{\int_{\Omega} (w - \bar{w})^2 \, dx \leq \int_{\Omega} (w_0 - \bar{w}_0)^2 \, dx \exp(-2k\mu_2 t),}
\end{equation}
where
\begin{equation}
{\bar{w} = \frac{1}{|\Omega|} \int_{\Omega} w \, dx,}
\end{equation}
with $|\Omega|$ the measure of $\Omega$ and $\mu_2$ the first positive eigenvalue of the problem
\begin{equation}
{\Delta \phi + \mu \phi = 0 \quad \text{in } \Omega, \quad \frac{\partial \phi}{\partial n} = 0 \quad \text{on } \partial \Omega.}
\end{equation}

The proof of (1.12) and (1.14) may be found in [6, p. 11]. In the Neumann boundary case, since
\begin{equation}
{\frac{\partial}{\partial t} \int_{\Omega} w \, dx = 0,}
\end{equation}
we know that the mean value of $w$ is preserved for all time, i.e.
\begin{equation}
{\bar{w} = \bar{w}_0.}
\end{equation}
By substituting (1.16) into (1.14) we obtain
\begin{equation}
{\int_{\Omega} w^2 \, dx \leq \int_{\Omega} w_0^2 \, dx \exp(-2k\mu_2 t) + |\Omega| \bar{w}_0^2 [1 - \exp(-2k\mu_2 t)].}
\end{equation}

We investigate the question of $L_2$ decay bounds in time for solutions under conditions of $\beta$ and $\gamma$ having the same sign in the next section and under conditions of $\beta$ and $\gamma$ having different signs in the final section.
2. Case where $\beta$ and $\gamma$ are of the same sign

In this section, we investigate temporal decay bounds of (1.3)–(1.5) under either Dirichlet boundary conditions (1.6) or Neumann boundary conditions (1.7). To this end, we shall make the assumptions for the coefficients

(i) $\alpha > 0$, $\delta > 0$, $\beta$ and $\gamma$ are of same signs,  
(ii) $\alpha \delta > \beta \gamma$.  

We first note that if $\beta$ or $\gamma$ is zero the problem decouples and bounds are easily obtained and so we consider the problem for $\gamma \neq 0$.

To transform (1.3) and (1.4) into heat equations, we form

$$ \frac{\partial}{\partial t}(u + \sigma v) = (\alpha + \sigma \gamma) \Delta u + (\beta + \sigma \delta) \Delta v $$  

for some constant $\sigma$ to be determined. We then choose $\sigma$ so that

$$ \beta + \sigma \delta = \sigma(\alpha + \sigma \gamma), $$

i.e., $\sigma = \sigma_1$ or $\sigma_2$ where

$$ \sigma_{1,2} = \frac{(\delta - \alpha) \pm [(\delta - \alpha)^2 + 4\beta \gamma]^{1/2}}{2\gamma} $$

with the plus sign being for $\sigma_1$ and the minus sign for $\sigma_2$. Then we write (2.3) as

$$ \frac{\partial}{\partial t}(u + \sigma_1 v) = \rho_1 \Delta (u + \sigma_1 v), $$

$$ \frac{\partial}{\partial t}(u + \sigma_2 v) = \rho_2 \Delta (u + \sigma_2 v), $$

where

$$ \rho_{1,2} = \frac{(\delta + \alpha) \pm [(\delta - \alpha)^2 + 4\beta \gamma]^{1/2}}{2} $$

with the plus sign yielding $\rho_1$ and the minus sign $\rho_2$. Condition (2.2) ensures that $\rho_2 > 0$, so that both $\rho_1$ and $\rho_2$ are positive. Then, in the Dirichlet case it follows from (1.12) that

$$ \int_{\Omega} (u + \sigma_1 v)^2 \, dx \leq \int_{\Omega} (u_0 + \sigma_1 v_0)^2 \, dx \exp(-2\rho_1 \lambda_1 t) := Q_1, $$

$$ \int_{\Omega} (u + \sigma_2 v)^2 \, dx \leq \int_{\Omega} (u_0 + \sigma_2 v_0)^2 \, dx \exp(-2\rho_2 \lambda_1 t) := Q_2. $$

Thus

$$ \int_{\Omega} \left[2u^2 + 2(\sigma_1 + \sigma_2)uv + (\sigma_1^2 + \sigma_2^2)v^2 \right] \, dx \leq Q_1 + Q_2. $$  

(2.4)

We note that

$$ \sigma_1 + \sigma_2 = \frac{\delta - \alpha}{\gamma}, \quad \sigma_1^2 + \sigma_2^2 = \frac{(\delta - \alpha)^2 + 2\beta \gamma}{\gamma^2}. $$

Using the arithmetic–geometric mean inequality we may write

$$ 2(\sigma_1 + \sigma_2)uv \geq -u^2 - \frac{(\delta - \alpha)^2}{\gamma^2} v^2. $$  

(2.5)

Assuming (2.1) and (2.2), it follows then upon inserting (2.5) into (2.4) that

$$ \int_{\Omega} u^2 \, dx + \frac{2\beta}{\gamma} \int_{\Omega} v^2 \, dx \leq Q_1 + Q_2, $$

(2.6)

which yields decay bounds for both $u$ and $v$.

Turning to the Neumann boundary conditions (1.7), making use of (1.17) we obtain

$$ \int_{\Omega} (u + \sigma_1 v)^2 \, dx \leq \int_{\Omega} (u_0 + \sigma_1 v_0)^2 \, dx \exp(-2\rho_1 \mu_2 t) + |\Omega| (\overline{u_0} + \sigma_1 \overline{v_0})^2 \{1 - \exp(-2\rho_1 \mu_2 t)\}, $$

$$ \int_{\Omega} (u + \sigma_2 v)^2 \, dx \leq \int_{\Omega} (u_0 + \sigma_2 v_0)^2 \, dx \exp(-2\rho_2 \mu_2 t) + |\Omega| (\overline{u_0} + \sigma_2 \overline{v_0})^2 \{1 - \exp(-2\rho_2 \mu_2 t)\}. $$

A bound for $\int_{\Omega} u^2 \, dx + \frac{2\beta}{\gamma} \int_{\Omega} v^2 \, dx$ then follows as in the Dirichlet case.
When the mean values of \( u_0 \) and \( v_0 \) are zero, proceeding in a manner analogous to that for the homogeneous Dirichlet conditions, we may obtain the same type of inequality (2.6) for the homogeneous Neumann boundary conditions.

We conclude this section by showing that if \( \beta \gamma > \alpha \delta \) then the problem is ill-posed. To show this, in the Dirichlet case we consider solutions of the form
\[
\begin{align*}
  u &= A \exp(-\alpha \delta t) \phi(x), \\
  v &= B \exp(-\beta \gamma t) \phi(x),
\end{align*}
\]
where \( A \) and \( B \) are nonzero constants and \( \phi(x) \) is an eigenfunction of (1.13). These functions are solutions of our system (1.3)–(1.6) whenever
\[
\omega = \frac{\lambda}{2} \left[ (\alpha + \delta) \pm \sqrt{(\alpha + \delta)^2 - 4(\alpha \delta - \beta \gamma)} \right],
\]
with \( \lambda \) given by (1.13). We note that
\[
(\alpha + \delta)^2 - 4(\alpha \delta - \beta \gamma) = (\alpha - \delta)^2 + 4\beta \gamma
\]
which is always positive (\( \beta \) and \( \gamma \) have the same sign) and then \( \omega \) is always real. The sign of \( \omega \) is determined by
\[
(\alpha + \delta) \pm \sqrt{(\alpha + \delta)^2 - 4(\alpha \delta - \beta \gamma)}.
\]
If we consider the minus sign and \( \beta \gamma > \alpha \delta \), this quantity is negative and we have instability. Now suppose that \( \lambda \) is the \( n \)th eigenvalue \( \lambda_n \). Then since for \( \Omega \subset \mathbb{R}^d \), \( \lim_{n \to \infty} \lambda_n^{d/2}/n = C_d/|\Omega| \), where \( C_d = (4\pi)^{d/2} \Gamma(d/2 + 1) \) (in our case, \( \lim_{n \to \infty} \lambda_n^{3/2}/n = 6\pi^2/|\Omega| \)), we conclude by the usual arguments that if \( \beta \gamma > \alpha \delta \) the problem is ill-posed. In the Neumann case with the \( \phi(x) \) now an eigenfunction of (1.15), then replacing \( \lambda \) by \( \mu \) in (2.7), we arrive at the same conclusion.

3. Case where \( \beta \) and \( \gamma \) have different signs

In this case the solutions are always stable. We only need the positivity of \( \alpha \) and \( \delta \). To this end, we define
\[
E(t) = \int_\Omega u^2 dx + \mu \int_\Omega v^2 dx
\]
for a positive constant \( \mu \) to be determined. We compute
\[
\frac{dE}{dt} = 2 \left\{ \int_\Omega u(\alpha \Delta u + \beta \Delta v) dx + \mu \int_\Omega v(\gamma \Delta u + \delta \Delta v) dx \right\}
\]
\[
\leq -2 \left\{ \alpha \int_\Omega |\nabla u|^2 dx + \beta |\nabla v|^2 dx + \mu \right\} \int_\Omega \Delta v dx + \mu \delta \int_\Omega |\nabla v|^2 dx \right\}.
\]
We now choose \( \mu \) such that the right hand side of (3.1) is negative. Since \( \gamma \neq 0 \), if \( \alpha > 0 \) and \( \delta > 0 \), we choose
\[
\mu = -\beta / \gamma > 0.
\]
Then from (3.1) the Poincaré inequality allows us to conclude that
\[
\frac{dE}{dt} \leq -2 \left\{ \alpha \int_\Omega |\nabla u|^2 dx + \beta |\nabla v|^2 dx \right\} \leq -2C\lambda_1 E,
\]
where \( C = \max(\alpha, -\beta \gamma) \) and \( \lambda_1 \) is the first eigenvalue of (1.13). An integration leads to
\[
\int_\Omega u^2 dx + \mu \int_\Omega v^2 dx \leq \left( \int_\Omega u_0^2 dx + \mu \int_\Omega v_0^2 dx \right) \exp(-2C\lambda_1 t).
\]
For the Neumann boundary conditions, from (1.17) we have
\[
\int_\Omega u^2 dx + \mu \int_\Omega v^2 dx \leq \left( \int_\Omega u_\Omega^2 dx + \mu \int_\Omega v_\Omega^2 dx \right) \exp(-2C\mu_2 t) + |\Omega| \left( u_\Omega^2 + v_\Omega^2 \right) \{ 1 - \exp(-2C\mu_2 t) \}.
\]

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References