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Multicyclic treelike reflexive graphs

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Abstract

A simple graph is reflexive if its second largest eigenvalue does not exceed 2. A graph is treelike (sometimes also called a cactus) if all its cycles (circuits) are mutually edge-disjoint. In a lot of cases one can establish whether a given graph is reflexive by identifying and removing a single cut-vertex (Theorem 1). In this paper we prove that, if this theorem cannot be applied to a connected treelike reflexive graph G and if all its cycles do not have a common vertex (do not form a bundle), such a graph has at most five cycles (Theorem 2). On the same conditions, in Theorem 3 we find all maximal treelike reflexive graphs with four and five cycles.

Keywords: Graph theory; Second largest eigenvalue; Treelike graphs

1. Introduction

For an undirected graph G without loops and/or multiple edges (also called a simple graph) let $P_G(\lambda) = \det(\lambda I - A)$ be the characteristic polynomial of its $(0, 1)$ adjacency matrix A, also called the *characteristic polynomial* of G and denoted by $P(\lambda)$ if it is clear which graph it is related to. The roots of $P_G(\lambda)$ are the *eigenvalues* of G and, since they are real (A is a real and symmetric matrix), they can be designated in non-increasing order: $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$. The family of eigenvalues is the *spectrum* of G and the largest eigenvalue $\lambda_1(G)$ is also called the *index* of G. Note, if G is connected, then $\lambda_1(G) > \lambda_2(G)$, while for a disconnected graph, since its spectrum is the union of the spectra of its components, $\lambda_1(G) = \lambda_2(G)$ if these are the indices of two distinct components of G. We assume that all graphs we are looking for are connected.

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Recall, the interrelation between the spectra of a graph G and its induced subgraph H is expressed by the so-called *interlacing theorem*:

Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ *be the eigenvalues of a simple graph G and* $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m$ *the eigenvalues of its induced subgraph H. Then the inequalities* $\lambda_{n-m+i} \leqslant \mu_i \leqslant \lambda_i$ $(i = 1, 2, \ldots, m)$ *hold.*

Thus, if $m = n - 1$, we have $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \cdots$ and also $\lambda_1 > \mu_1$ if G is connected.

Graphs for which $\lambda_2 \leqslant 2$ are usually called *reflexive graphs*. In fact, these graphs correspond to sets of vectors in the Lorentz space $R^{p,1}$ having Gram matrix $2I - A$ (and consequently norm 2 and mutual angles 90° and 120°) and they are Lorentzian counterparts of the spherical and Euclidean graphs, which occur in the theory of reflexion groups, having direct application to the construction and the classification of such groups [7]. If $\lambda_2 \leqslant 2 \leqslant \lambda_1$, they are also known as *hyperbolic graphs*. In particular, reflexive trees have been studied in [5,6], and a class of bicyclic reflexive graphs in [10] (see also [8,3]).

A *cactus* or a *treelike graph* is a graph in which any two cycles have at most one common vertex.

In Section 2 we give some known results, which will be useful tools for further investigation. Section 3 contains a general result on reflexive graphs with a cut-vertex [10] (Theorem 1) and further discussion on the number of cycles of a reflexive cactus if this theorem is not applicable, which results in Theorem 2. In Section 4 we find all treelike reflexive graphs with four and five cycles on the conditions that Theorem 1 cannot be applied and that all cycles of such a graph do not make a bundle (Theorem 3). Since the graphic property $\lambda_2 \leqslant 2$ is *hereditary* (every induced subgraph maintains the property), the result is expressed through the set of maximal graphs. In some stages of the research theoretical reasoning is combined with some aid of a computer.

The terminology of the theory of graph spectra in this paper is in accord with [1], while for general graph theoretic concepts one can see [4].

2. Preliminaries

The following list of lemmas contains basic facts to be repeatedly used in Sections 3 and 4.

Lemma 1 (*Smith [12], see also [1, p. 79]*). *Let* $\lambda_1(G)$ *be the index of a graph G. Then* $\lambda_1(G) \leqslant 2$ $(\lambda_1(G) < 2)$ *if and only if each component of G is a subgraph* (*resp. proper subgraph*) *of one of the graphs depicted in Fig.* 1, *all of which have index equal to* 2.

These graphs are known as Smith graphs.

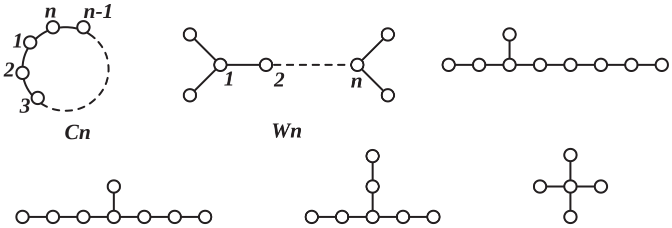

Fig. 1.

Very useful formulae, which express interrelations between the characteristic polynomial of a graph and the set of its subgraphs, are the two equalities of the following lemma.

Lemma 2 (*Schwenk [11]*). *Given a graph G, let $C(v)$ ($C(uv)$) denote the set of all its cycles containing a vertex v (resp. an edge uv). Then*

(1) $P_G(\lambda) = \lambda P_{G-v}(\lambda) - \sum_{u \in Adj(v)} P_{G-v-u}(\lambda) - 2\sum_{C \in C(v)} P_{G-V(C)}(\lambda)$,

(2) $P_G(\lambda) = P_{G-uv}(\lambda) - P_{G-v-u}(\lambda) - 2\sum_{C \in C(uv)} P_{G-V(C)}(\lambda)$,

where $Adj(v)$ denotes the set of neighbours of v, while $G - V(C)$ is the graph obtained from G by removing the vertices belonging to the cycle C.

Obvious consequences of these relations are the next formulae (due to E. Heilbronner—see, e.g. [1, p. 59]).

Corollary 1. *Let G be a graph obtained by joining a vertex v_1 of a graph G_1 to a vertex v_2 of a graph G_2 by an edge. Let G_1' (G_2') be the induced subgraph of G_1 (G_2) obtained by deleting the vertex v_1 (v_2) from G_1 (resp. G_2). Then*

$$P_G(\lambda) = P_{G_1}(\lambda)P_{G_2}(\lambda) - P_{G_1'}(\lambda)P_{G_2'}(\lambda).$$

Corollary 2. *Let G be a graph with a pendent edge v_1v_2 (v_1 being of degree 1). Then*

$$P_G(\lambda) = \lambda P_{G_1}(\lambda) - P_{G_2}(\lambda),$$

where G_1 (G_2) is the graph obtained from G (resp. G_1) by deleting vertex v_1 (v_2).

Now, we give a list of values of $P(2)$ for some small graphs, to be a very useful set of tools for solving many particular cases in the coming sections.

Lemma 3 (*The reduced list of Lemma 4.1 of [10]*). *Let $G_1, \ldots, G_7$ be the graphs displayed in Fig. 2 . Then*

(1) $P_{G_1}(2) = k + 2$,

(2) $P_{G_2}(2) = 4$,

(3) $P_{G_3}(2) = -klm + k + l + m + 2$,

(4) $P_{G_4}(2) = 4(1 - kl)$,

(5) $P_{G_5}(2) = -km$,

(6) $P_{G_6}(2) = -m(2kl+k+l)$,

(7) $P_{G_7}(2) = -4m$.

Also, we will meet the situation of adding a new vertex to a Smith graph.

Lemma 4 (*Radosavljević and Simić [10]*). *Let G be a (connected) graph obtained by extending any of the Smith graphs (see Fig. 1) by a vertex of arbitrary degree. Then* $P_G(2) < 0$.

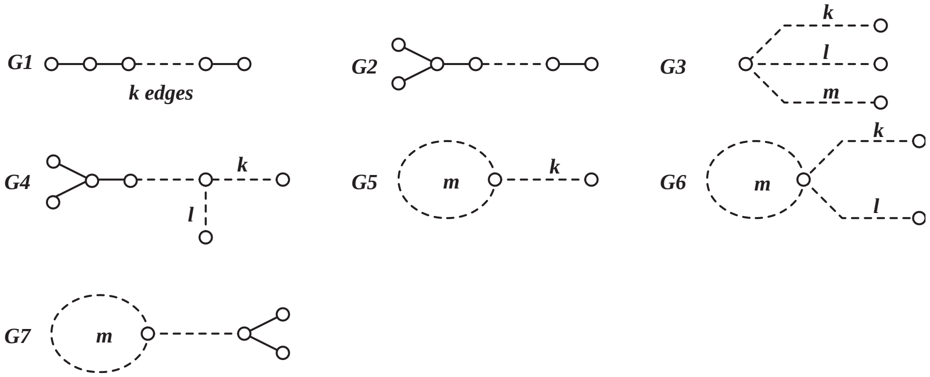

Fig. 2.

3. Some general results

If it comes about that the removal of a cut-vertex of a graph results in two components which are both Smith graphs (e.g. the graph of Fig. 3), it follows from the interlacing theorem that such a graph has $\lambda_2 = 2$.

A more general question—what happens if we have an arbitrary number of components among which there are Smith graphs, has its answer in the following theorem. Let us call a graph positive, null or negative depending on whether its index is greater than, equal to or less than 2, respectively.

Theorem 1 (*Radosavljević and Simić [10]*). *Let G be a graph as in Fig. 4 (u being a cut-vertex).*

1. *If at least two components of $G - u$ are positive, or if only one is positive and some of remaining null, then $\lambda_2(G) > 2$.*
2. *If at least two components of $G - u$ are null and any other non-positive, then $\lambda_2(G) = 2$.*
3. *If at most one component of $G - u$ is null and any other negative, then $\lambda_2(G) < 2$.*

This theorem solves a wide class of cases of treelike graphs, but cannot do if we have one positive and all other negative components. This means that, if we want to find all maximal reflexive graphs within a given class of treelike graphs, we actually have a problem of solving those cases when Theorem 1 cannot be applied. That is why the non-usability of Theorem 1 is a permanent assumption in the further investigation.

For one who wants to find all maximal reflexive treelike graphs, it may seem natural to start with the case when all cycles have a common vertex (forming a bundle), but it immediately turns out that this initial case is at the same time the most difficult one. The number of cycles is not limited and, though we can easily imagine trees hanging on some vertices of cycles (including their common vertex) and giving graphs for which we can establish, either by means of Theorem 1 or in some other way, whether they are maximal reflexive graphs, the multitude of cases shows that this problem will

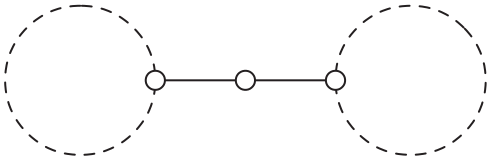

Fig. 3.

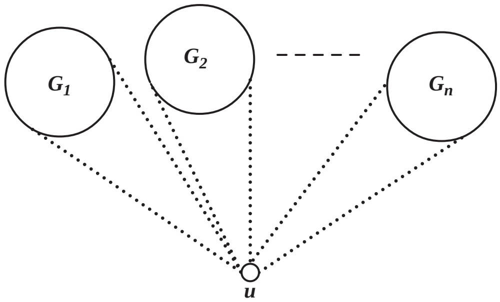

Fig. 4.

have to be solved in some particular classes (e.g. bicyclic graphs, etc.). Therefore, in this paper we assume that all the cycles of our cactus do not have a common vertex.

If a treelike graph has a bridge between two cycles, the case is still rather general, but at the same time much more tractable. All maximal reflexive bicyclic graphs with a bridge between the two cycles have been found in [10]. The result includes an exceptional case of a tricyclic cactus, which appeared naturally along with the result contained in the following lemma.

Let two cycles of arbitrary lengths be connected by a bridge whose vertices are c_1 and c_2 and let c_1c_3 be additional pendent edge.

Lemma 5 (*Radosavljević and Simić [10]*). *If in a bicyclic graph with a bridge between its cycles all vertices of the cycles except* c_i $(i = 1, 2)$ *are of degree two and if Theorem* 1 *is not applicable, it is reflexive if and only if it is an induced subgraph of a graph formed by identifying with* c_2 *and* c_3 *two vertices obtained by splitting any of the Smith trees S at any vertex into two trees* S_1 *and* S_2 (*Fig.* 5(a)), *or of the graph of Fig.* 5(b) *for* $l_1 = l_2 = 0$.

This lemma was proved in [10, Propositions 4.5 and 4.5′] by recognizing all particular cases of maximal reflexive graphs on the given assumptions. The fact that attaching S_1 and S_2 to the vertices c_3 and c_2 gives such a graph can be verified also in the following way.

Let S be a Smith tree and v any of its vertices dividing S into S_1 and S_2 (Fig. 6(a)).

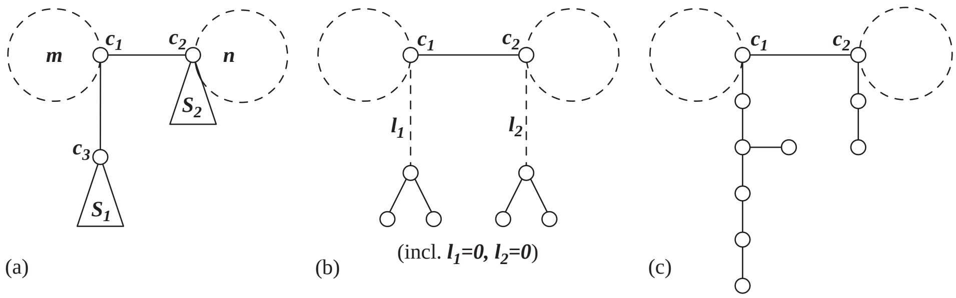

Fig. 5.

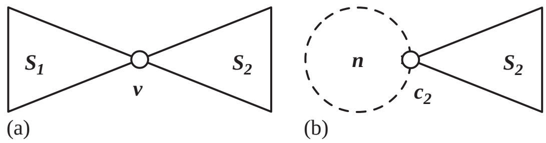

Fig. 6.

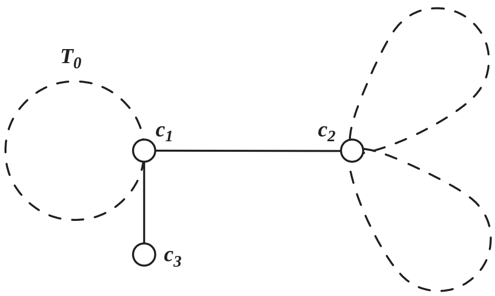

Fig. 7.

We have $P_S(2)=0$ and, according to Lemma 2(1)

$$P_S(2) = 2P_{S_1-v}(2)P_{S_2-v}(2) - \sum_{u'\in Adj(v)} P_{S_1-v-u'}(2)\cdot P_{S_2-v}(2) - \sum_{u'\in Adj(v)} P_{S_2-v-u'}(2)\cdot P_{S_1-v}(2).$$

Putting $P_{S_1-v}(2)=A$, $P_{S_2-v}(2)=B$, $\sum_{u'\in Adj(v)} P_{S_1-v-u'}(2)=\Sigma_1$, $\sum_{u'\in Adj(v)} P_{S_2-v-u'}(2) = \Sigma_2$, $P_{S_1}(2)=A_1$, $P_{S_2}(2)=B_1$, we get

$$2AB - \Sigma_2 A - \Sigma_1 B = 0. \tag{1}$$

Also, by applying Lemma 2(1) to S_1 and S_2 and the vertex v, we have

$$A_1 = 2A - \Sigma_1, \quad B_1 = 2B - \Sigma_2, \tag{2}$$

respectively. The application of Corollary 1 to the graph of Fig. 5(a) gives

$$\begin{aligned} P(2) &= -n\Sigma_2(-mA) - nBmA_1 = nm(A\Sigma_2 - B(2A - \Sigma_1)) \\ &= nm(A\Sigma_2 + B\Sigma_1 - 2AB) = 0, \end{aligned} \tag{3}$$

where we used the fact that for the graph of Fig. 6(b) $P(2) = -n\Sigma_2$ by applying Lemma 2(1) to c_2.

If we assume that S is a proper subgraph of a Smith tree, we obtain in (1) $P_S(2) > 0$ and consequently $P(2) < 0$ in (3), which proves that for the graphs of Fig. 5(a) $\lambda_2 = 2$. These graphs are maximal, for if we add a new vertex to S_1 or S_2, according to Lemma 4 S becomes a proper supergraph of a Smith graph for which $P_S(2) < 0$, and we get in (3) $P(2) > 0$.

Splitting of a given Smith tree S at any vertex and attaching of S_1 and S_2 to c_2 and c_3 (e.g. as in Fig. 5(c)) produces an interesting phenomenon of "pouring" between two vertices, which we will meet several times in this paper.

Of course, since this fact includes also attaching a whole Smith graph to the vertex c_2, while c_3 remains an end-vertex, a simple generalization gives rise to the tricyclic maximal reflexive graph T_0 displayed in Fig. 7, which will be repeatedly used in the further analysis.

Let us now consider the general case of two bundles of cycles with a bridge which joins their common vertices. For a single bundle of k cycles of lengths $n_1, n_2, \ldots, n_k$ according to Lemmas 2(1) and 3(1) it holds $P(2) = -2(k-1)\prod_{i=1}^{k} n_i$. Let $m_1, m_2, \ldots, m_k$ and $n_1, n_2, \ldots, n_l$ be the lengths of the cycles of the two bundles. By using Corollary 1 we get $P(2) = (4(k-1)(l-1) - 1)\prod_{i=1}^{k} m_i \prod_{i=1}^{l} n_i$. Thus, if one bundle contains only one cycle, $P(2) < 0$, i.e. $\lambda_2 < 2$ and the graph is reflexive (which is also clear by Theorem 1), while for $\min(k, l) \geqslant 2$ $P(2) > 0$. Also in the case $k = 1$ the graph T_0 of Fig. 7 shows that already by adding the single pendent edge at the vertex c_1 l becomes at most 2 (and if $l = 2$, adding any other pendent edge to the left cycle is not possible). On the other hand, if there are no pendent edges on the left cycle, we may apply Theorem 1. These facts, which supplement the results of [10], show that those results actually embrace all cases of reflexive treelike graphs with a bridge between cycles (of course, on the assumption of non-usability of Theorem 1).

Therefore, from now on we are interested only in treelike graphs without such a bridge. Assume now that every cycle has at most two vertices which belong also to some other cycles. Then the total number of such vertices is at most two (if there were three, the removal of the middle one gives $\lambda_2 > 2$).

However, if there is a cycle that has at least three common vertices with other cycles (in this case let us call it the central cycle), it can be at most quadrangle (otherwise, the removal of an appropriate vertex gives a subgraph to which Theorem 1 can be applied).

Now, in order to find all reflexive treelike graphs with more than three cycles, let us consider the graph in Fig. 8.

Of course, for $k \geqslant 3$ it is a supergraph (induced) of the graph T_0 of Fig. 7. Since λ_2 may remain unchanged by adding to a graph a new vertex of arbitrary degree, extension of T_0

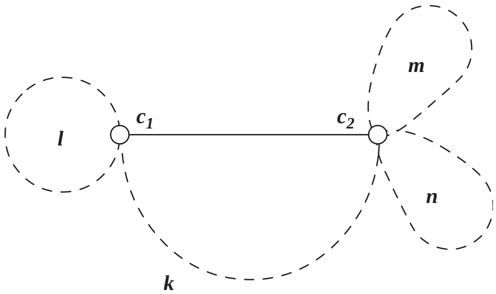

Fig. 8.

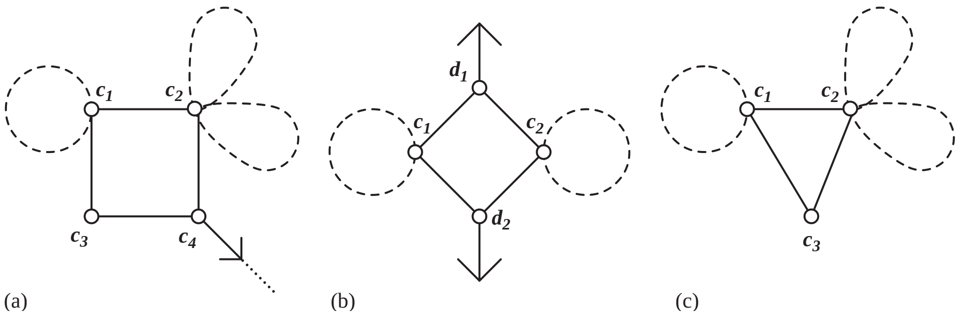

Fig. 9.

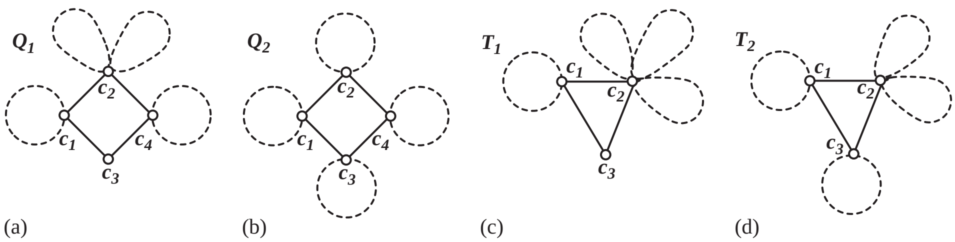

Fig. 10.

may be possible if we join the new vertex by two new edges with c-vertices, thus forming a graph without a bridge between the cycles. Indeed, by applying Lemma 2(1) to the vertex c_2 and making use of Lemma 3, we obtain $P(2) = 2lmn(k-3)$, which allows $k = 2$ and 3.

If $k = 3$ (Fig. 9(a)), we already have $\lambda_2 = 2$, but since also $\lambda_2(T_0) = 2$, it follows by induction that this graph can be extended infinitely at the vertex c_4 preserving $P(2) = 0$ (no extension at c_3 is possible because of T_0).

Of course, this simply means that after some steps of extension we only get $\lambda_3 = 2, \ldots,$ and we must find the limit before which we have $\lambda_2 = 2$. If we consider only maximum number of cycles, it is clear that T_0 allows a cycle at c_4, and indeed such a graph with five

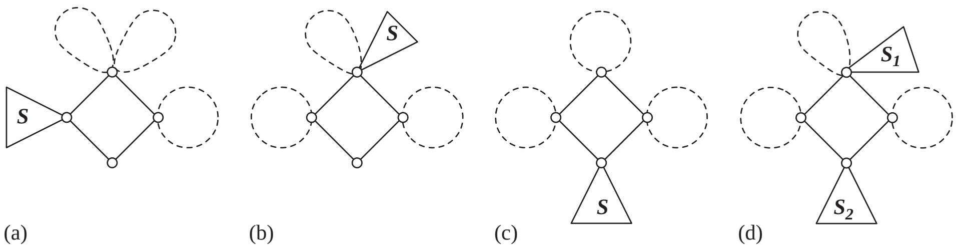

Fig. 11.

cycles has $\lambda_2 = 2$ (Q_1 of Fig. 10(a)). No additional cycle can be attached to c_4 because of T_0, but if we remove one cycle at c_2 and attach it to c_3, again $\lambda_2 = 2$ (the graph Q_2 in Fig. 10(b)). The same conclusions might be obtained by starting from the graph of Fig. 3 and adding the new vertex d_2 (Fig. 9(b)). This extension does not change $\lambda_2 = 2$ and further extension at d_1 and d_2 preserves $P(2) = 0$.

Both resulting graphs are maximal: for Q_1 we must not attach additional edges because of T_0, while for Q_2, if we add a pendent edge at any vertex, by removing a vertex of the quadrangle we get a proper supergraph of the graph of Fig. 3 to which Theorem 1 can be applied.

If $k = 2$ (Fig. 9(c)), we have $P(2) < 0$, i.e. $\lambda_2 < 2$. Possible new cycles may be added only at c_2 and c_3 and in both cases, by applying Lemma 2(1) to e.g. c_2, and using facts of Lemma 3, we get $P(2) = 0$ (the graphs T_1 and T_2 of Fig. 10(c) and (d)).

Both graphs are maximal. In the case of T_1 no pendent edge can be put at c_3, because if we apply Corollary 2 we get $P(2) > 0$, while any other additional edge is not possible because of T_0 or the results of [10]. In the case of T_2 the same conclusions hold for c_1 and c_3, and all other vertices, respectively.

Now, based upon the previous analysis and conclusions, we may formulate the following theorem.

Theorem 2. *A treelike reflexive graph to which Theorem* 1 *cannot be applied and whose cycles do not make a bundle has at most five cycles. The only such graphs with five cycles, which are all maximal, i.e. cannot be extended at any vertex, are the four families of graphs of Fig.* 10.

4. Treelike reflexive graphs with four cycles

In order to find all maximal reflexive cactuses with 4 cycles, let us have a look again at graphs of Fig. 10. Since a cycle is simply one of Smith graphs, it is natural to try to replace one cycle by other Smith graphs, the situation which we already had with the graphs of Lemma 5 and the graph T_0. Moreover, it will turn out that the effect of splitting Smith graphs and “pouring” them from one vertex to another (in Lemma 5 from c_1 to c_3 and vice versa) will play the crucial role.

Proposition 1. *If the cycle at c_1 (or c_4) of the graph Q_1 is replaced by any of Smith trees, attached to c_1 at any vertex, all obtained graphs are maximal reflexive cactuses* (*Fig.* 11(a)).

Proof. Direct checking shows that we always have $\lambda_2 = 2$. Also, no extension of these graphs is possible. The Smith tree cannot be extended because after removing c_2 and applying Theorem 1 to c_3 we get $\lambda_2 > 2$. The extension at other vertices is impossible because of T_0. □

Proposition 2. *If one of the cycles at c_2 of the graph Q_1 is replaced by any of Smith trees, attached to c_2 at any vertex, all obtained graphs are maximal reflexive cactuses* (*Fig*. 11(b)).

Proof. Checking of all cases gives $\lambda_2 = 2$. The extension at any vertex of the Smith tree or at c_3 is not possible because of Lemma 5, while adding a pendent edge at any vertex of the remaining cycle at c_2 is not allowed according to the results of [10, Theorem 4.6]. The same holds for the cycles at c_1 and c_4, but this is obvious also by removing c_2 and applying Theorem 1 to c_3. □

Proposition 3. *If any of the four non-central cycles of the graph Q_2 is replaced by any of Smith trees, attached at any vertex, all obtained graphs are maximal reflexive cactuses* (*Fig*. 11(c)).

The proof is analogous to previous cases.

Also, like in Lemma 5, we again come to the phenomenon of "pouring" of Smith trees.

Proposition 4. *If we remove one of the four non-central cycles of Q_2, say the one attached to c_3, and identify with c_3 and c_2 two vertices obtained by splitting any of Smith trees, at any vertex, into S_1 and S_2, all obtained graphs are maximal reflexive cactuses* (*Fig*. 11(d)).

Proof. Besides $P(2) = 0$, as in previous cases we have $\lambda_2 = 2$. All such graphs are maximal since no pendent edge can be added at S_1 and S_2 (after removing c_1 or c_4 we would have a proper supergraph of a member of the family described in Lemma 5) and the same holds for adding new edges at non-central cycles. □

Since the "pouring" of Smith trees from one vertex to another naturally includes attaching of a complete Smith tree to one of these vertices, we may also assume that cases (b) and (c) are embraced by (d).

Now, let us consider the graph T_1. We immediately guess that a cycle at c_2 can be replaced by an arbitrary Smith tree, but this time we will at once treat the general case.

Proposition 5. *If we remove one of the cycles at the vertex c_2 of the graph T_1, and identify with c_2 and c_3 two vertices obtained by splitting any of the Smith trees, at any vertex v, into S_1 and S_2, all obtained graphs are maximal reflexive cactuses, including cases when a whole Smith tree is attached to c_2 or c_3* (*Fig*. 12).

Proof. We will use the notation which follows Lemma 5.

Relations (1) and (2) imply

$$2AB - \Sigma_2 A - \Sigma_1 B = A(2B - \Sigma_2) - \Sigma_1 B = AB_1 - \Sigma_1 B = 0. \quad (4)$$

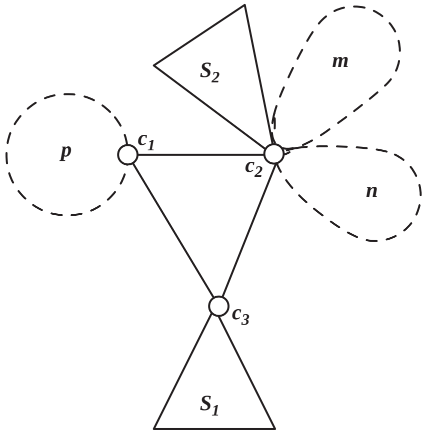

Fig. 12.

We will now apply Lemma 2(1) to the graph at Fig. 12 and its vertex c_2.

$$\begin{aligned} P(2) &= 2mn(-pB)A - pB_1mnA - 0 - 2(n-1)m(-pB)A \\ &\quad - 2(m-1)n(-pB)A - \Sigma_1(-pB)mn - 2(m+n)A(-pB) - 2mnpAB \\ &= -mnp(B_1A - \Sigma_1 B) = 0. \end{aligned} \tag{5}$$

If S were a proper subgraph of a Smith tree, it would be $P_S(2) > 0$, implying $AB_1 - \Sigma_1 B > 0$ in (4) and $P(2) < 0$ in (5), which proves that the eigenvalue of graphs of Fig. 12 which is equal to 2 is just λ_2. All these graphs are maximal, because if we assume that S is a proper supergraph of a Smith tree formed by adding one new vertex ($P_S(2) < 0$, see Lemma 4), we obtain in (5) $P(2) > 0$. □

Of course, even if we add nothing to c_2 and c_3 after removing one cycle attached to c_2, the vertex c_1 cannot be loaded by a new edge (i.e. no pendent edge can be attached to it) because of the graph T_0. The same follows for any vertex of the cycle at c_1 from the result of [10, Theorem 4.6]. Loading of vertices of the cycles at c_2 is not possible because of the results of [Proposition 8]. Thus, we can reformulate the last proposition.

Proposition 5′. *If we start from the graph with four cycles obtained by removing a cycle at c_2 from the graph T_1, and if we attach some trees only to its c-vertices, such a graph will be reflexive if and only if it is an induced subgraph of some of the graphs of the family displayed in Fig. 12.*

The cyclic structure of the graph T_2 suggests that a cycle at c_2 can be replaced by a Smith tree, which then can “pour” between c_2 and e.g. c_3.

Proposition 6. *Let G be a graph obtained by removing one of the cycles at the vertex c_2 of T_2 and identifying with c_2 and c_3 two vertices obtained by splitting any of Smith trees*

S at any vertex into S_1 *and* S_2*, including cases when a whole Smith tree is attached to* c_2 *or* c_3 (*Fig.* 13). *Then* $\lambda_2(G) = 2$ *and G is a maximal reflexive graph, with the exception of the case when, after removing* c_1 *from G, the remaining component with the bridge* c_2c_3 *is the graph of Fig.* 5(b).

Proof. Proceeding in the same manner and using the same designations as in the proof of Proposition 5, we have

$$P_G(2) = -mnp(B\Sigma_1 + A\Sigma_2 - 2AB),$$

implying $P_G(2) < 2$, $P_G(2) = 2$, $P_G(2) > 2$ if S is a proper subgraph of a Smith tree, a Smith tree, or a proper supergraph of a Smith tree, respectively. Therefore, the assumption that S is a Smith tree means $\lambda_2(G) = 2$. According to the results of [10], no additional edges can load vertices of the cycles at c_2 and c_3. As for the vertex c_1, let us load it by a pendent edge and apply Corollary 2 to the new graph G_1. We get $P_{G_1}(2) = 2P_G(2) - mP_{G'}(2)$, where G' is the bicyclic component obtained after removing c_1 from G. Thus, $P_{G_1}(2) = 0$ if and only if $P_{G'}(2) = 0$ and this happens in the case of the graph of Fig. 5(b). For all other possibilities of splitting a Smith tree into S_1 and S_2 we may verify by inspection that every such case is a proper subgraph of some case of Fig. 5(a) of Lemma 5, implying $P_{G_1}(2) > 2$. Also, it turns out that loading of other vertices of the cycle at c_1 is possible only in the described exceptional case, those graphs being covered by the results of [Proposition 8]. □

Of course, it follows by induction that in the described case when $P_{G_1}(2) = 0$ an infinite extension at c_1 preserves $P(2) = 0$. The maximal graph for the fact $\lambda_2 = 2$ is the graph of Fig. 13(b).

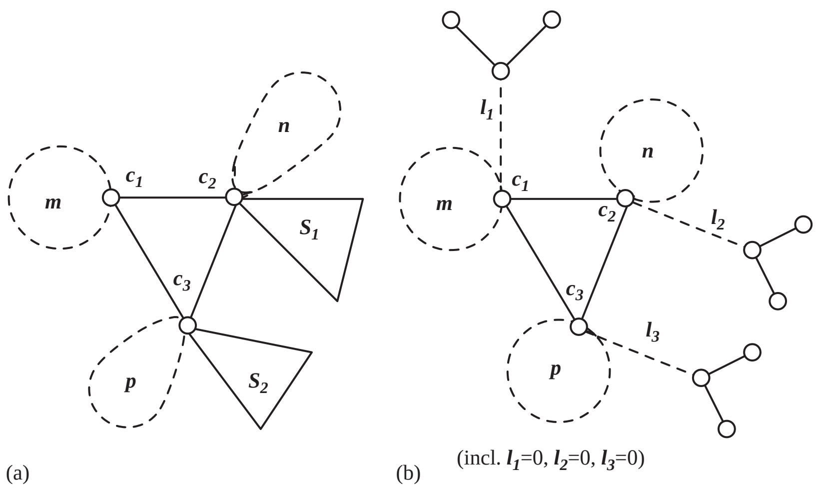

Fig. 13.

Since Proposition 6 covers all situations when one or two c-vertices are loaded, in order to find all maximal graphs with the cyclic structure as of the graphs of Fig. 13 we will have to suppose that now all c-vertices are of at least degree 5.

Proposition 7. *If we remove one of the cycles at the vertex c_2 of the graph T_2, and attach some trees to all its c-vertices, such a graph is a maximal reflexive cactus if and only if it belongs to one of the* 10 *families of graphs of Fig.* 14 *or the one of Fig.* 13(b).

Proof. If we want to load all c-vertices but not to get a graph of Fig. 13(b), the consequence of Proposition 6 is that at any two c-vertices, say c_2 and c_3, we must have such trees that, having glued them i.e. having identified c_2 with c_3, we get a proper subgraph of a Smith tree. This fact points to starting from all such subgraphs (Coxeter–Dynkin graphs) and splitting them into two parts to be attached to c_2 and c_3. At least one of the three trees at c-vertices is not a simple path (otherwise we would have a proper subgraph of a graph of Fig. 13(b)) and suppose that it is at c_2. If it were a proper subgraph of the Smith tree W_n displayed in Fig. 15 (let us denote it by Z_{m_1}), c_1 can be loaded by at most Z_{m_2} because of Proposition 6, and then at c_3 we also have Z_{m_3}. If we attach to c_2 and c_3 parts of a proper subgraph of some of the rest of Smith trees, after a simple discussion and a little aid of a computer we come to the resulting maximal graphs of Fig. 14, which all have $\lambda_2 = 2$. □

Attaching the result of Proposition 6 to Proposition 7 we can reformulate the latter one.

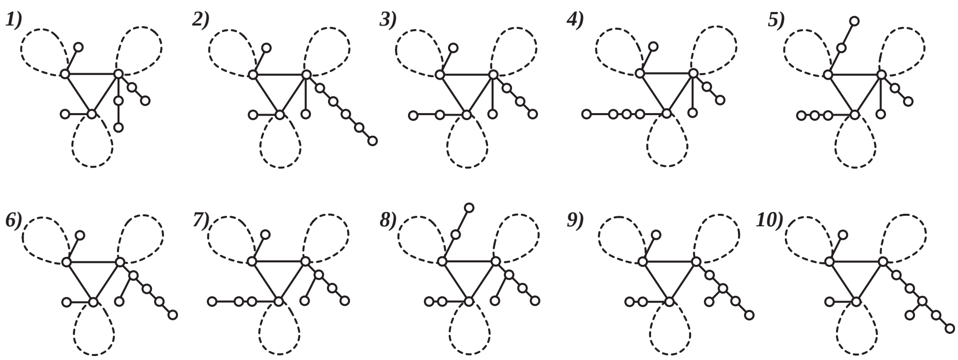

Fig. 14.

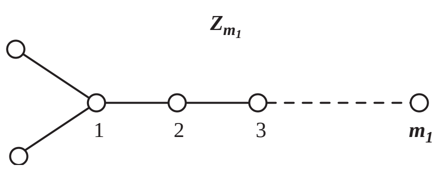

Fig. 15.

Proposition 7′. *If we start from the graph with four cycles with the cyclic structure as of the graphs of Fig.* 13, *and if we attach some trees only to its c-vertices, such a graph is reflexive if and only if it is an induced subgraph of some of the graphs displayed in Figs.* 13 *and* 14.

Besides the *c*-vertices, loading of other vertices of non-central cycles can also give rise to maximal reflexive cactuses. These possibilities have been discussed in [9] and completely solved by considering various particular cases.

Proposition 8 (*Theorem 3 of [9]*). *A treelike graph with four cycles to which Theorem* 1 *cannot be applied, whose cycles do not form a bundle and which, besides the c-vertices, has at least one vertex of non-central cycles loaded, is reflexive if and only if it is an induced subgraph of some of the* (*families of*) *graphs* $H_1 - H_{48}, I_1 - I_9, J_1 - J_{11}, K_1 - K_{36}, L_1 - L_{12}, M_1 - M_{12}$ *and* $N_1 - N_{42}$ *displayed in Figs*. 16–22[1] (*Figs*. 9, 13, 15, 17, 19, 21, 23 *and* 24 *of* [9]).

Those maximal graphs that have $\lambda_2 < 2$ are marked by asterisk.

All results contained in the previous propositions lead to the following conclusion.

Theorem 3. *A treelike graph with four cycles to which Theorem* 1 *cannot be applied and whose cycles do not make a bundle is reflexive if and only if it is an induced subgraph of some of the graphs of Figs*. 11–14 *and Figs.* 16–22[1] (*Figs*. 9, 13, 15, 17, 19, 21, 23, 24 *in* [9]).

A proper subgraph of a cycle is a path, and if we replace any of non-central cycles of any of the graphs of Fig. 10 by a path, attached to any of its vertices on the central cycle, we always have a graph that fits in the results of Theorem 3. Thus, according to Theorem 2, we can make the following formulation.

Theorem 3′. *A treelike graph with more than three cycles to which Theorem* 1 *cannot be applied and whose cycles do not make a bundle is reflexive if and only if it is an induced subgraph of some of the graphs of Figs*. 10–14 *and Figs*. 16–22[1] (*Figs*. 9, 13, 15, 17, 19, 21, 23, 24 *in* [9]).

Finally, let us mention that by following the ideas of replacing cycles by Smith trees and splitting and “pouring” Smith trees one can anticipate various new classes of maximal reflexive cactuses with less than four cycles.

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[1] Available in the electronic version of the paper on ScienceDirect.

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