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Robust and stable predictive control with bounded uncertainties

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Abstract

Min–Max optimization is often used for improving robustness in Model Predictive Control (MPC). An analogy to this optimization could be the BDU (Bounded Data Uncertainties) method, which is a regularization technique for least-squares problems that takes into account the uncertainty bounds. Stability of MPC can be achieved by using terminal constraints, such as in the CRHPC (Constrained Receding-Horizon Predictive Control) algorithm. By combining both BDU and CRHPC methods, a robust and stable MPC is obtained, which is the aim of this work. BDU also offers a guided method of tuning the empirically tuned penalization parameter for the control effort in MPC.

Keywords: Model predictive control; Min–Max optimization; Regularization; Robustness; Stability

1. Introduction

The *Least-Squares* (LS) method is used for solving problems such as $\mathbf{Ax} \approx \mathbf{b}$ for some known matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ (with $m \geqslant n$), being $\mathbf{x} \in \mathbb{R}^n$ an unknown vector and $\mathbf{b} \in \mathbb{R}^m$ the measurement vector. The approximation arises because $\mathbf{A}$ and $\mathbf{b}$ are never perfectly known, and so they present uncertainty. The LS criterion considers that matrix $\mathbf{A}$ is known exactly, and all the errors and uncertainties occur only in $\mathbf{b}$, meaning, $\mathbf{b} + \delta\mathbf{b}$, and $\delta\mathbf{b}$ being the uncertainty. So vector $\hat{\mathbf{x}}$ is estimated by solving

$$\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|_2 \tag{1}$$

being $\|.\|_2$ the *Euclidean* norm of its vector argument. The solution $\hat{\mathbf{x}}$, denoting $\mathbf{A}^+$ the pseudoinverse matrix of $\mathbf{A}$, is

$$\hat{\mathbf{x}} = \left[\mathbf{A}^T\mathbf{A}\right]^{-1}\mathbf{A}^T\mathbf{b} = \mathbf{A}^+\mathbf{b}. \tag{2}$$

A disadvantage of the LS method is its sensitivity to data error. More specifically, a design that is based on given data $(\mathbf{A}, \mathbf{b})$ can perform poorly if the true data happens to be the perturbed version $(\mathbf{A} + \delta\mathbf{A}, \mathbf{b} + \delta\mathbf{b})$ for some unknown $\delta\mathbf{A}$

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and $\boldsymbol{\delta}\mathbf{b}$. The *Regularized* LS is a variant method used for combating much of the ill-conditioning that arises in pure LS problems [11,12,30], and can provide a more robust solution (less sensitive to data errors). So the matrix inversion results easier and more accurate, and the system is more robust. *Regularization* consists of choosing in advance a positive parameter λ (regularization parameter) and selecting $\hat{\mathbf{x}}$ by solving

$$\min_{\mathbf{x}}\left[\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2+\lambda\|\mathbf{x}\|_2^2\right] \tag{3}$$

being the solution

$$\hat{\mathbf{x}}=\left[\mathbf{A}^T\mathbf{A}+\lambda\mathbf{I}\right]^{-1}\mathbf{A}^T\mathbf{b}. \tag{4}$$

The disadvantage is that λ is chosen in an intuitive way, so if it results too high (over-regularization), an overly conservative design is obtained, and if it results too low (under-regularization), the design becomes sensitive to errors. The *regularization* obtains *more robust*, but *less accurate* solutions, that is to say, it introduces a *bias*, providing lower values for $\|\hat{\mathbf{x}}\|_2$, which provide smoother and more robust control actions from the control point of view.

The desire for a more mathematically rigorous method for posing robust problems led to the development of the Min–Max problem, or BDU (*Bounded Data Uncertainties*) [5], also called *Robust* LS [10]. This is used in identification and estimation problems [21,23,24,28,29,31], but rarely applied to process control [17,18,21]. The BDU technique uses information about bounds on the sizes of the uncertainties to obtain the regularization parameter λ, and so its selection is neither intuitive nor arbitrary.

From the process control context, *Model Predictive Control* (MPC) can be found in a wide variety of industrial applications [15,16]. It integrates optimal control, that is to say, control actions are calculated by a cost function optimization along the prediction horizon (using the receding horizon concept). The cost function considers a penalization parameter for the control effort, which usually results in a diagonal and constant matrix $\rho\mathbf{I}$. The parameter both prevents the control actions from being too large and abrupt, and improves the system robustness. But the main drawback is that the parameter is generally tuned by empirical criteria, without the objective of improving the robustness, which is a fundamental matter in MPC when there is a mismatch between model and process due to model uncertainty and/or noise. A technique for improving robustness in MPC is the Min–Max optimization [3,13], which, in the presence of uncertainty, can be stated as a BDU problem. So, the main objective of this work is to use the BDU technique for tuning, in a guided way, the penalization parameter in MPC when uncertainty is present.

From a popular MPC algorithm such as GPC (*Generalized Predictive Control*) [1,6,7], a new controller GPC–BDU is stated [19], where the penalization parameter is tuned via BDU to improve system robustness.

Stability has also been considered by using a GPC variant, the *constrained receding-horizon predictive control* (CRHPC) [8], which imposes constraints on output, so that the reference and the output coincide on a horizon beyond the prediction horizon. So the cost function is forced to be monotonic and decreasing [26].

This work focuses on the CRHPC, because it covers a variety of differing stabilizing schemes [20]. The CRHPC ensures nominal closed-loop stability under certain conditions, but it cannot be ensured when a mismatch between process and model appears. To improve the system robustness, the CRHPC tuned via BDU is presented as the CRHPC–BDU, and this results in a robust and stable variant of GPC.

The structure of the work is the following: in Section 2 the fundamentals of BDU are presented, in Section 3 the CRHPC is shown, while the CRHPC–BDU is stated in Section 4. Section 5 shows the results, and finally in Section 6 the conclusions and future work are stated.

2. Fundamentals of BDU

The Bounded Data Uncertainties problem, BDU [4,5,14,21–24,29], the so-called *Min–Max* problem, or *Robust Least-Squares* (RLS) [10], was proposed and solved, via the secular equation in [5], and via *Linear Matrix Inequalities* (LMI) in [9]. With LMIs, the computational burden is smaller, although this is only noticeable when constraints are present in the problem, but otherwise the *secular equation* is simpler. The BDU problem seeks a solution $\hat{\mathbf{x}}$ that performs *best* in the worst-possible scenario inside a bounded region. There are several statements of the BDU problem [23,24], but one that is very useful in the context of identification and control is the following

$$\min_{\mathbf{x}}\ \max_{\substack{\|\boldsymbol{\delta}\mathbf{A}\|_2\leqslant\eta_A\\ \|\boldsymbol{\delta}\mathbf{b}\|_2\leqslant\eta_b}}\left[\left\|[\mathbf{A}+\boldsymbol{\delta}\mathbf{A}]\mathbf{x}-[\mathbf{b}+\boldsymbol{\delta}\mathbf{b}]\right\|_2^2+\rho\|\mathbf{x}\|_2^2\right], \tag{5}$$

where $(\mathbf{A}, \mathbf{b})$ represents the nominal model, and $(\mathbf{A} + \boldsymbol{\delta}\mathbf{A}, \mathbf{b} + \boldsymbol{\delta}\mathbf{b})$ the unknown perturbed model, because $\boldsymbol{\delta}\mathbf{A}$ and $\boldsymbol{\delta}\mathbf{b}$ are unknown, but a bound of them is known, η_A and η_b, such as $\|\boldsymbol{\delta}\mathbf{A}\|_2 \leqslant \eta_A$ and $\|\boldsymbol{\delta}\mathbf{b}\|_2 \leqslant \eta_b$ and ρ is the penalization parameter for the minimization variable $\mathbf{x}$. Eq. (5) can be reduced, defining the residual J, as

$$\min_{\mathbf{x}} \max_{\substack{\|\boldsymbol{\delta}\mathbf{A}\|_2 \leqslant \eta_A \\ \|\boldsymbol{\delta}\mathbf{b}\|_2 \leqslant \eta_b}} J(\mathbf{x}, \boldsymbol{\delta}\mathbf{A}, \boldsymbol{\delta}\mathbf{b}), \tag{6}$$

which can be regarded as a constrained two-player game problem, with the designer trying to pick an $\mathbf{x}$ that minimizes the residual J, while the opponents $\boldsymbol{\delta}\mathbf{A}$ and $\boldsymbol{\delta}\mathbf{b}$ try to maximize the residual. The goal consists of determining the solution $\hat{\mathbf{x}}$ whose maximum residual J, being $\|\boldsymbol{\delta}\mathbf{A}\|_2 \leqslant \eta_A$ and $\|\boldsymbol{\delta}\mathbf{b}\|_2 \leqslant \eta_b$, is the smallest possible. A nonzero solution $\hat{\mathbf{x}}$ is obtained if the following condition holds [5]

$$\eta_A < \frac{\|\mathbf{A}^T\mathbf{b}\|_2}{\|\mathbf{b}\|_2}. \tag{7}$$

Also in [5] it is shown that the Min–Max problem with constraints in (5), is equivalent to the following minimization problem without constraints

$$\min_{\mathbf{x}} \left[\left[\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 + \eta_A \|\mathbf{x}\|_2 + \eta_b \right]^2 + \rho \|\mathbf{x}\|_2^2 \right]. \tag{8}$$

It is noticeable that if $\boldsymbol{\delta}\mathbf{A} = \mathbf{0}$ and $\boldsymbol{\delta}\mathbf{b} = \mathbf{0}$ in (5), the original solution to the Regularized Least-Squares problem (3)

$$\hat{\mathbf{x}} = \left[\mathbf{A}^T\mathbf{A} + \rho\mathbf{I}\right]^{-1}\mathbf{A}^T\mathbf{b} \tag{9}$$

is obtained, being ρ the *empirical* regularization parameter in the absence of uncertainty. Nevertheless, when uncertainty is present, a new regularization parameter λ, which takes into account more information, is obtained

$$\hat{\mathbf{x}} = \left[\mathbf{A}^T\mathbf{A} + \lambda\mathbf{I}\right]^{-1}\mathbf{A}^T\mathbf{b}, \tag{10}$$

$$\lambda = \frac{\eta_A \|\mathbf{A}\hat{\mathbf{x}} - \mathbf{b}\|_2}{\|\hat{\mathbf{x}}\|_2} + \frac{\rho \|\mathbf{A}\hat{\mathbf{x}} - \mathbf{b}\|_2}{\|\mathbf{A}\hat{\mathbf{x}} - \mathbf{b}\|_2 + \eta_A \|\hat{\mathbf{x}}\|_2 + \eta_b}. \tag{11}$$

The solution is obtained by solving the non-linear equations system formed by (10) and (11), being (11) the non-linear secular equation which depends on λ, $\mathbf{A}$, $\mathbf{b}$, ρ, η_A and η_b. Defining $F(\lambda)$ as

$$F(\lambda) = \frac{\eta_A \|\mathbf{A}\hat{\mathbf{x}} - \mathbf{b}\|_2}{\|\hat{\mathbf{x}}\|_2} + \frac{\rho \|\mathbf{A}\hat{\mathbf{x}} - \mathbf{b}\|_2}{\|\mathbf{A}\hat{\mathbf{x}} - \mathbf{b}\|_2 + \eta_A \|\hat{\mathbf{x}}\|_2 + \eta_b} - \lambda \tag{12}$$

a unique solution $\hat{\lambda} > 0$ exists, so that $F(\hat{\lambda}) = 0$, which can be determined, for example, by employing a bisection-type algorithm [5].

3. CRHPC

The CRHPC [8] is a variant of the standard GPC which guarantees the closed-loop stability under certain conditions by imposing constraints on the output, so that the reference and the output coincide on a horizon beyond the prediction horizon (see Fig. 1, where $y(k+i|k)$ predicts the output at instant $k+i$ being at k). In fact, some degrees of freedom of the controller are used in the constraints, while the rest are used in the cost function minimization along the prediction horizon. In this statement, the *Controlled Auto-Regressive and Integrated Moving Average* (CARIMA) model from GPC is used

$$y(k) = \frac{\mathbf{B}(z^{-1})z^{-1}}{\mathbf{A}(z^{-1})}u(k) + \frac{\mathbf{T}(z^{-1})}{\boldsymbol{\Delta}\mathbf{A}(z^{-1})}\xi(k), \tag{13}$$

where $y(k)$ and $u(k)$ are the system output and control actions, respectively, $\xi(k)$ represents the disturbance, $\boldsymbol{\Delta} = (1 - z^{-1})$, $\mathbf{T}(z^{-1})$ is a noise stochastic characteristic, and $\mathbf{B}(z^{-1})z^{-1}$ and $\mathbf{A}(z^{-1})$ are the numerator and denominator of the discrete model.

The prediction model is stated as [2]

$$y(k+i|k) = \mathbf{G}_i\left(z^{-1}\right)\Delta u(k+i-1) + f(k+i|k), \tag{14}$$

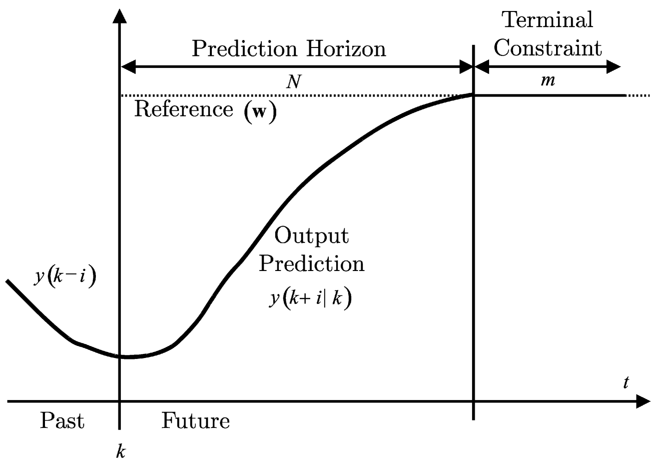

Fig. 1. Terminal constraint.

which predicts the output at instant $k+i$ being at k, being $f(k+i|k)$ the *free response* prediction, and $\mathbf{G}_i(z^{-1})\Delta u(k+i-1)$ the *forced response* prediction, based on the future control actions. Applying (14) for instant $i=1,\ldots,N$, the prediction in a matrix form is obtained assuming $\Delta u(k+N_u)=\Delta u(k+N_u+1)=\cdots=\Delta u(k+N-1)=0$ (being N and N_u the prediction and control horizon, respectively)

$$\mathbf{y}_{1_{N\times 1}}=\mathbf{G}_{1_{N\times N_u}}\mathbf{\Delta u}_{N_u\times 1}+\mathbf{f}_{1_{N\times 1}}, \tag{15}$$

where matrix $\mathbf{G}_1$ is formed by the g_i coefficients from the unitary step response [2]. If the prediction is extended for m instants beyond the prediction horizon ($i=N+1,\ldots,N+m$) results in

$$\mathbf{y}_{2_{m\times 1}}=\mathbf{G}_{2_{m\times N_u}}\mathbf{\Delta u}_{N_u\times 1}+\mathbf{f}_{2_{m\times 1}}, \tag{16}$$

where

$$\mathbf{G}_2=\begin{bmatrix} g_N & g_{N-1} & \cdots & g_{N+1-N_u} \\ g_{N+1} & g_N & \cdots & g_{N+2-N_u} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N+m-1} & g_{N+m-2} & \cdots & g_{N+m-N_u} \end{bmatrix}. \tag{17}$$

The CRHPC statement is the same as in GPC,

$$\min_{\mathbf{\Delta u}}\left[\sum_{i=N_1}^{N_2}\alpha_i\big[y(k+i|k)-w(k+i)\big]^2+\sum_{j=1}^{N_u}\rho_j\big[\Delta u(k+j-1)\big]^2\right] \tag{18}$$

but in this case, subject to $y(k+N+i)=w(k+N)$, $i\in[1,\ldots,m]$, where N_1, N_2 are the minimum and maximum cost horizons, $N=N_2-N_1+1$, N_u is the control horizon (degrees of freedom), $\sum_{i=N_1}^{N_2}\alpha_i[y(k+i|k)-w(k+i)]^2$ weights the error between the output y and the reference w, $\sum_{j=1}^{N_u}\rho_j[\Delta u(k+j-1)]^2$ weights the control effort. The minimization and the constraint can be expressed as

$$\begin{aligned}&\min_{\mathbf{\Delta u}}\big[[\mathbf{y}_1-\mathbf{w}_1]^T\mathbf{A}[\mathbf{y}_1-\mathbf{w}_1]+\mathbf{\Delta u}^T\mathbf{P}\mathbf{\Delta u}\big]\\ &\text{s.t.}\quad \mathbf{y}_2=\mathbf{w}_2,\end{aligned} \tag{19}$$

where $\mathbf{A}_{N\times N}=\operatorname{diag}(\alpha_1,\ldots,\alpha_i,\ldots,\alpha_N)$, $\mathbf{P}_{N_u\times N_u}=\operatorname{diag}(\rho_1,\ldots,\rho_j,\ldots,\rho_{N_u})$ are usually diagonal constant matrices $\mathbf{P}=\rho\mathbf{I}$ and $\mathbf{A}=\alpha\mathbf{I}$, being

$$\mathbf{w}_1=\big[w(k+1),\ldots,w(k+N)\big]^T,\qquad \mathbf{w}_2=\big[w(k+N),\ldots,w(k+N)\big]^T. \tag{20}$$

Assuming $\mathbf{A}=\mathbf{I}$, for the sake of simplicity, and denoting error as the difference between the reference and the free response prediction $\mathbf{e}_1=\mathbf{w}_1-\mathbf{f}_1$, $\mathbf{e}_2=\mathbf{w}_2-\mathbf{f}_2$, expression (19) can be stated as

$$\min_{\mathbf{\Delta u}}\left[\|\mathbf{G}_1\mathbf{\Delta u}-\mathbf{e}_1\|_2^2+\rho\|\mathbf{\Delta u}\|_2^2\right]$$
$$\text{s.t.}\quad \mathbf{G}_2\mathbf{\Delta u}=\mathbf{e}_2. \tag{21}$$

An analytical solution to this minimization problem can be calculated by solving two linear equations systems, the terminal constraint

$$\mathbf{G}_{2_{m\times N_u}}\mathbf{\Delta u}=\mathbf{e}_{2_{m\times 1}} \tag{22}$$

and the following regularized least-squares problem

$$\min_{\mathbf{\Delta u}}\left[\|\mathbf{G}_1\mathbf{\Delta u}-\mathbf{e}_1\|_2^2+\rho\|\mathbf{\Delta u}\|_2^2\right] \tag{23}$$

for the remaining degrees of freedom, obtaining

$$\mathbf{\Delta\hat{u}}=\tilde{\mathbf{M}}\mathbf{G}_1^T\mathbf{e}_1+\left[\mathbf{I}-\tilde{\mathbf{M}}\left[\mathbf{G}_1^T\mathbf{G}_1+\rho\mathbf{I}\right]\right]\mathbf{G}_2^+\mathbf{e}_2, \tag{24}$$
$$\tilde{\mathbf{M}}=\mathbf{H}\left[\mathbf{H}^T\left[\mathbf{G}_1^T\mathbf{G}_1+\rho\mathbf{I}\right]\mathbf{H}\right]^{-1}\mathbf{H}^T. \tag{25}$$

The CRHPC guarantees the closed-loop stability under the following conditions [25,32]: (i) $\alpha_i \geqslant 0$, $\rho_j \geqslant 0$, (ii) $m=n_a$, (iii) $N_u \geqslant m$, (iv) $N \geqslant N_u+\max(n_b,n_a)-n_a$, where $n_a=\deg(\mathbf{A}(z^{-1})\mathbf{\Delta})$ and $n_b=\deg(\mathbf{B}(z^{-1}))$, with the assumption that $\mathbf{A}(z^{-1})\mathbf{\Delta}$ and $\mathbf{B}(z^{-1})$ have no unstable common factor. So n_a is the minimum number of terminal constraints that must be taken into account. It is not useful to consider a higher number of constraints because the higher the number is, the larger the computational burden results. If condition $N_u \geqslant m$ holds as $N_u=m$, then $\mathbf{G}_2$ results in a square matrix, and therefore, solution (24) can be stated as $\mathbf{\Delta\hat{u}}=\mathbf{G}_2^{-1}\mathbf{e}_2$, because the N_u degrees of freedom are used for satisfying the constraint (22), and there are no additional degrees for the minimization. Due to the receding horizon concept in MPC, the previous problem must be solved at each sample time instant.

4. CRHPC–BDU

Most of the results obtained in MPC in general, and in CRHPC in particular, are based on the idea that model and process are the same, and disturbances are not present, but, in reality, there is always a mismatch between model and process [27]. The CRHPC ensures the nominal stability, but this is not the case when mismatch appears. Since the BDU can be used for tuning the GPC (the so-called GPC–BDU [19]) to improve the system robustness, the CRHPC can also be tuned via BDU (CRHPC–BDU) with the same aim.

Assuming the true matrices $\mathbf{G}_1$, $\mathbf{G}_2$ and vectors $\mathbf{e}_1$, $\mathbf{e}_2$ to be $\mathbf{G}_1+\boldsymbol{\delta}\mathbf{G}_1$, $\mathbf{G}_2+\boldsymbol{\delta}\mathbf{G}_2$ and $\mathbf{e}_1+\boldsymbol{\delta}\mathbf{e}_1$, $\mathbf{e}_2+\boldsymbol{\delta}\mathbf{e}_2$, respectively, where $\boldsymbol{\delta}\mathbf{G}_1$, $\boldsymbol{\delta}\mathbf{G}_2$, $\boldsymbol{\delta}\mathbf{e}_1$, $\boldsymbol{\delta}\mathbf{e}_2$ are unknown, but a bound of them is known η_{G_1}, η_{G_2}, η_{e_1} and η_{e_2}, the control law can be obtained by solving the following BDU problem subject to the terminal constraint for the worst-case

$$\min_{\mathbf{\Delta u}}\max_{\substack{\|\boldsymbol{\delta}\mathbf{G}_1\|_2\leqslant\eta_{G_1}\\ \|\boldsymbol{\delta}\mathbf{e}_1\|_2\leqslant\eta_{e_1}}}\left[\left\|[\mathbf{G}_1+\boldsymbol{\delta}\mathbf{G}_1]\mathbf{\Delta u}-[\mathbf{e}_1+\boldsymbol{\delta}\mathbf{e}_1]\right\|_2^2+\rho\|\mathbf{\Delta u}\|_2^2\right]$$
$$\text{s.t.}\quad\max_{\substack{\|\boldsymbol{\delta}\mathbf{G}_2\|_2\leqslant\eta_{G_2}\\ \|\boldsymbol{\delta}\mathbf{e}_2\|_2\leqslant\eta_{e_2}}}\left\|[\mathbf{G}_2+\boldsymbol{\delta}\mathbf{G}_2]\mathbf{\Delta u}-[\mathbf{e}_2+\boldsymbol{\delta}\mathbf{e}_2]\right\|_2=\mathbf{0}, \tag{26}$$

which can be stated as the solution to two linear equation systems. The terminal constraint for the worst-case, as a BDU problem

$$\min_{\mathbf{\Delta u}}\max_{\substack{\|\boldsymbol{\delta}\mathbf{G}_2\|_2\leqslant\eta_{G_2}\\ \|\boldsymbol{\delta}\mathbf{e}_2\|_2\leqslant\eta_{e_2}}}\left\|[\mathbf{G}_2+\boldsymbol{\delta}\mathbf{G}_2]\mathbf{\Delta u}-[\mathbf{e}_2+\boldsymbol{\delta}\mathbf{e}_2]\right\|_2, \tag{27}$$

and the following cost function (BDU problem) is optimized for the remaining degrees of freedom

$$\min_{\mathbf{\Delta u}}\max_{\substack{\|\boldsymbol{\delta}\mathbf{G}_1\|_2\leqslant\eta_{G_1}\\ \|\boldsymbol{\delta}\mathbf{e}_1\|_2\leqslant\eta_{e_1}}}\left[\left\|[\mathbf{G}_1+\boldsymbol{\delta}\mathbf{G}_1]\mathbf{\Delta u}-[\mathbf{e}_1+\boldsymbol{\delta}\mathbf{e}_1]\right\|_2^2+\rho\|\mathbf{\Delta u}\|_2^2\right]. \tag{28}$$

4.1. Obtaining the particular solution

First of all, the BDU problem (27) is solved for $\mathbf{\Delta u} = \mathbf{\Delta u}_p$, resulting in

$$\mathbf{\Delta\hat{u}}_p = \mathbf{G}_2^T\big[\mathbf{G}_2\mathbf{G}_2^T + \lambda_{G_2}\mathbf{I}\big]^{-1}\mathbf{e}_2, \tag{29}$$

$$\lambda_{G_2} = \frac{\eta_{G_2}\|\mathbf{G}_2\mathbf{\Delta\hat{u}}_p - \mathbf{e}_2\|_2}{\|\mathbf{\Delta\hat{u}}_p\|_2}. \tag{30}$$

By having the solution, it is possible to express the uncertainty for the worst-case $\hat{\boldsymbol{\delta}}\mathbf{G}_2$ and $\hat{\boldsymbol{\delta}}\mathbf{e}_2$ as [21]

$$\hat{\boldsymbol{\delta}}\mathbf{G}_2 = \eta_{G_2}\frac{[\mathbf{G}_2\mathbf{\Delta\hat{u}}_p - \mathbf{e}_2]}{\|\mathbf{G}_2\mathbf{\Delta\hat{u}}_p - \mathbf{e}_2\|_2}\frac{\mathbf{\Delta\hat{u}}_p^T}{\|\mathbf{\Delta\hat{u}}_p\|_2}, \tag{31}$$

$$\hat{\boldsymbol{\delta}}\mathbf{e}_2 = -\eta_{e_2}\frac{[\mathbf{G}_2\mathbf{\Delta\hat{u}}_p - \mathbf{e}_2]}{\|\mathbf{G}_2\mathbf{\Delta\hat{u}}_p - \mathbf{e}_2\|_2}, \tag{32}$$

in such a way that the Min–Max problem subject to constraints (BDU) is stated as a minimization problem without constraints

$$\min_{\mathbf{\Delta u}_p}\big\|[\mathbf{G}_2 + \hat{\boldsymbol{\delta}}\mathbf{G}_2]\mathbf{\Delta u}_p - [\mathbf{e}_2 + \hat{\boldsymbol{\delta}}\mathbf{e}_2]\big\|_2. \tag{33}$$

On the other hand, the general solution $\mathbf{\Delta\hat{u}}$ can contain all the possible solutions as

$$\big\{\mathbf{\Delta\hat{u}} \in \mathbb{R}^{N_u} \bigm| [\mathbf{G}_2 + \hat{\boldsymbol{\delta}}\mathbf{G}_2]\mathbf{\Delta\hat{u}} = [\mathbf{e}_2 + \hat{\boldsymbol{\delta}}\mathbf{e}_2]\big\} = \big\{\mathbf{\Delta\hat{u}}_p + \mathbf{z} \bigm| \mathbf{z} \in \mathcal{N}\big([\mathbf{G}_2 + \hat{\boldsymbol{\delta}}\mathbf{G}_2]\big)\big\}, \tag{34}$$

where

- $\mathbf{\Delta\hat{u}}_p$ is a particular solution to the equations system.
- $\mathbf{z}$ is the general solution to the homogeneous equations system

$$\mathcal{N}\big([\mathbf{G}_2 + \hat{\boldsymbol{\delta}}\mathbf{G}_2]\big) = \big\{\mathbf{z} \in \mathbb{R}^{N_u} \bigm| [\mathbf{G}_2 + \hat{\boldsymbol{\delta}}\mathbf{G}_2]\mathbf{z} = \mathbf{0}\big\} \tag{35}$$

  being $\mathcal{N}([\mathbf{G}_2 + \hat{\boldsymbol{\delta}}\mathbf{G}_2])$ the null space of matrix $[\mathbf{G}_2 + \hat{\boldsymbol{\delta}}\mathbf{G}_2]$.

So, if $[\mathbf{e}_2 + \hat{\boldsymbol{\delta}}\mathbf{e}_2] = [\mathbf{G}_2 + \hat{\boldsymbol{\delta}}\mathbf{G}_2]\mathbf{\Delta\hat{u}}_p$ and $\mathbf{z} \in \mathcal{N}([\mathbf{G}_2 + \hat{\boldsymbol{\delta}}\mathbf{G}_2])$, then

$$[\mathbf{e}_2 + \hat{\boldsymbol{\delta}}\mathbf{e}_2] = [\mathbf{G}_2 + \hat{\boldsymbol{\delta}}\mathbf{G}_2][\mathbf{\Delta\hat{u}}_p + \mathbf{z}] \tag{36}$$

so $\mathbf{z}$ represents the remaining degrees of freedom in the solution to (27). If the singular value decomposition (SVD) of matrix $[\mathbf{G}_2 + \hat{\boldsymbol{\delta}}\mathbf{G}_2]$ is taken into account $[\mathbf{G}_2 + \hat{\boldsymbol{\delta}}\mathbf{G}_2] = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$, it is possible to calculate the null space $\mathcal{N}([\mathbf{G}_2 + \hat{\boldsymbol{\delta}}\mathbf{G}_2])$ from the vectors $[\mathbf{v}_{m+1}, \ldots, \mathbf{v}_{N_u}]$ of matrix $\mathbf{V}_{N_u \times N_u}$ (being $\mathbf{v}_i$ the ith vector of matrix $\mathbf{V}$), which are related to the null singular values. That is to say, any control action vector $\mathbf{\Delta u}$ which satisfies $[\mathbf{G}_2 + \hat{\boldsymbol{\delta}}\mathbf{G}_2]\mathbf{\Delta u} = \mathbf{0}$ can be expressed as the linear combination of vectors $\mathbf{v}_i$, $[\mathbf{G}_2 + \hat{\boldsymbol{\delta}}\mathbf{G}_2]\mathbf{v}_i = \sigma_i\mathbf{u}_i = \mathbf{0}$, $\forall i \in [m+1, \ldots, N_u]$. So, assuming $\mathbf{H} = [\mathbf{v}_{m+1}, \ldots, \mathbf{v}_{N_u}]$ the null space defined in (35) can be stated as $\mathbf{z} = \mathbf{H}\mathbf{\Delta u}_f$ where the vector $\mathbf{\Delta u}_f$ represents the remaining degrees of freedom from the solution to (27), which can be used for minimizing the cost function (28).

4.2. Obtaining the homogeneous solution

Once both the particular solution $\mathbf{\Delta\hat{u}}_p$ and matrix $\mathbf{H}$ are obtained, the general solution $(\mathbf{\Delta\hat{u}}_p + \mathbf{H}\mathbf{\Delta u}_f)$ is substituted in the index (28)

$$\min_{\mathbf{\Delta u}_f}\max_{\substack{\|\boldsymbol{\delta}\mathbf{G}_1\|_2 \leqslant \eta_{G_1}\\ \|\boldsymbol{\delta}\mathbf{e}_1\|_2 \leqslant \eta_{e_1}}}\big[\big\|[\mathbf{G}_1 + \boldsymbol{\delta}\mathbf{G}_1][\mathbf{\Delta\hat{u}}_p + \mathbf{H}\mathbf{\Delta u}_f] - [\mathbf{e}_1 + \boldsymbol{\delta}\mathbf{e}_1]\big\|_2^2 + \rho\|\mathbf{\Delta\hat{u}}_p + \mathbf{H}\mathbf{\Delta u}_f\|_2^2\big] \tag{37}$$

and assuming the change of variables

$$\begin{aligned} &\mathbf{G} \to \mathbf{G}_1\mathbf{H}, && \mathbf{e} \to \mathbf{e}_1 - \mathbf{G}_1\boldsymbol{\Delta}\hat{\mathbf{u}}_p, \\ &\boldsymbol{\delta}\mathbf{G} \to \boldsymbol{\delta}\mathbf{G}_1\mathbf{H}, && \boldsymbol{\delta}\mathbf{e} \to \boldsymbol{\delta}\mathbf{e}_1 - \boldsymbol{\delta}\mathbf{G}_1\boldsymbol{\Delta}\hat{\mathbf{u}}_p, \\ &\eta_G \to \eta_{G_1}\|\mathbf{H}\|_2, && \eta_e \to \eta_{e_1} + \eta_{G_1}\|\boldsymbol{\Delta}\hat{\mathbf{u}}_p\|_2 \end{aligned}$$

results in

$$\min_{\boldsymbol{\Delta}\mathbf{u}_f} \max_{\substack{\|\boldsymbol{\delta}\mathbf{G}\|_2 \leqslant \eta_G \\ \|\boldsymbol{\delta}\mathbf{e}\|_2 \leqslant \eta_e}} \left[\left\|[\mathbf{G}+\boldsymbol{\delta}\mathbf{G}]\boldsymbol{\Delta}\mathbf{u}_f - [\mathbf{e}+\boldsymbol{\delta}\mathbf{e}]\right\|_2^2 + \rho\|\boldsymbol{\Delta}\hat{\mathbf{u}}_p + \mathbf{H}\boldsymbol{\Delta}\mathbf{u}_f\|_2^2\right]. \tag{38}$$

Substituting the maximization by using (8), the problem is stated as

$$\min_{\boldsymbol{\Delta}\mathbf{u}_f}\left[\left[\|\mathbf{G}\boldsymbol{\Delta}\mathbf{u}_f - \mathbf{e}\|_2 + \eta_G\|\boldsymbol{\Delta}\mathbf{u}_f\|_2 + \eta_e\right]^2 + \rho\|\boldsymbol{\Delta}\hat{\mathbf{u}}_p + \mathbf{H}\boldsymbol{\Delta}\mathbf{u}_f\|_2^2\right], \tag{39}$$

where the minimization provides the $\boldsymbol{\Delta}\mathbf{u}_f$ variables, which assuming scalar values λ_1, λ_2 (see Appendix A), and through the changes of variables results in

$$\boldsymbol{\Delta}\hat{\mathbf{u}}_f = \left[\mathbf{H}^T\left[\mathbf{G}_1^T\mathbf{G}_1 + \lambda_1\mathbf{I}\right]\mathbf{H}\right]^{-1}\left[\mathbf{H}^T\mathbf{G}_1^T[\mathbf{e}_1 - \mathbf{G}_1\boldsymbol{\Delta}\hat{\mathbf{u}}_p] - \lambda_2\mathbf{H}^T\boldsymbol{\Delta}\hat{\mathbf{u}}_p\right]. \tag{40}$$

4.3. Obtaining the control law

The final control law can be stated as

$$\boldsymbol{\Delta}\hat{\mathbf{u}} = \boldsymbol{\Delta}\hat{\mathbf{u}}_p + \mathbf{H}\boldsymbol{\Delta}\hat{\mathbf{u}}_f = \boldsymbol{\Delta}\hat{\mathbf{u}}_p + \mathbf{H}\left[\mathbf{H}^T\left[\mathbf{G}_1^T\mathbf{G}_1 + \lambda_1\mathbf{I}\right]\mathbf{H}\right]^{-1}\mathbf{H}^T\left[\mathbf{G}_1^T\mathbf{e}_1 - \left[\mathbf{G}_1^T\mathbf{G}_1 + \lambda_2\mathbf{I}\right]\boldsymbol{\Delta}\hat{\mathbf{u}}_p\right] \tag{41}$$

which results in equations from (42) to (48) by taking into account $\boldsymbol{\Delta}\hat{\mathbf{u}}_p$ (29) and the changes of variables for λ_1 and λ_2 in $\mathbf{G}$, $\mathbf{e}$, η_G and η_e

$$\boldsymbol{\Delta}\hat{\mathbf{u}} = \overset{*}{\mathbf{M}}\mathbf{G}_1^T\mathbf{e}_1 + \left[\mathbf{I} - \overset{*}{\mathbf{M}}\left[\mathbf{G}_1^T\mathbf{G}_1 + \lambda_2\mathbf{I}\right]\right]\boldsymbol{\Delta}\hat{\mathbf{u}}_p, \tag{42}$$

$$\boldsymbol{\Delta}\hat{\mathbf{u}}_p = \mathbf{G}_2^T\left[\mathbf{G}_2\mathbf{G}_2^T + \lambda_{G_2}\mathbf{I}\right]^{-1}\mathbf{e}_2, \tag{43}$$

$$\lambda_{G_2} = \frac{\eta_{G_2}\|\mathbf{G}_2\boldsymbol{\Delta}\hat{\mathbf{u}}_p - \mathbf{e}_2\|_2}{\|\boldsymbol{\Delta}\hat{\mathbf{u}}_p\|_2}, \tag{44}$$

$$\overset{*}{\mathbf{M}} = \mathbf{H}\left[\mathbf{H}^T\left[\mathbf{G}_1^T\mathbf{G}_1 + \lambda_1\mathbf{I}\right]\mathbf{H}\right]^{-1}\mathbf{H}^T, \tag{45}$$

$$\lambda_1 = \frac{\eta_{G_1}\|\mathbf{H}\|_2\|\mathbf{G}_1\mathbf{H}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - [\mathbf{e}_1 - \mathbf{G}_1\boldsymbol{\Delta}\hat{\mathbf{u}}_p]\|_2}{\|\boldsymbol{\Delta}\hat{\mathbf{u}}_f\|_2} + \frac{\rho\|\mathbf{G}_1\mathbf{H}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - [\mathbf{e}_1 - \mathbf{G}_1\boldsymbol{\Delta}\hat{\mathbf{u}}_p]\|_2}{\left[\begin{array}{c}\|\mathbf{G}_1\mathbf{H}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - [\mathbf{e}_1 - \mathbf{G}_1\boldsymbol{\Delta}\hat{\mathbf{u}}_p]\|_2 \\ + \eta_{G_1}\|\mathbf{H}\|_2\|\boldsymbol{\Delta}\hat{\mathbf{u}}_f\|_2 + [\eta_{e_1} + \eta_{G_1}\|\boldsymbol{\Delta}\hat{\mathbf{u}}_p\|_2]\end{array}\right]}, \tag{46}$$

$$\lambda_2 = \frac{\rho\|\mathbf{G}_1\mathbf{H}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - [\mathbf{e}_1 - \mathbf{G}_1\boldsymbol{\Delta}\hat{\mathbf{u}}_p]\|_2}{\left[\begin{array}{c}\|\mathbf{G}_1\mathbf{H}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - [\mathbf{e}_1 - \mathbf{G}_1\boldsymbol{\Delta}\hat{\mathbf{u}}_p]\|_2 \\ + \eta_{G_1}\|\mathbf{H}\|_2\|\boldsymbol{\Delta}\hat{\mathbf{u}}_f\|_2 + [\eta_{e_1} + \eta_{G_1}\|\boldsymbol{\Delta}\hat{\mathbf{u}}_p\|_2]\end{array}\right]}, \tag{47}$$

$$\boldsymbol{\Delta}\hat{\mathbf{u}}_f = \left[\mathbf{H}^T\left[\mathbf{G}_1^T\mathbf{G}_1 + \lambda_1\mathbf{I}\right]\mathbf{H}\right]^{-1}\left[\mathbf{H}^T\mathbf{G}_1^T[\mathbf{e}_1 - \mathbf{G}_1\boldsymbol{\Delta}\hat{\mathbf{u}}_p] - \lambda_2\mathbf{H}^T\boldsymbol{\Delta}\hat{\mathbf{u}}_p\right]. \tag{48}$$

Again, if condition $N_u \geqslant m$ holds as $N_u = m$, then $\mathbf{G}_2$ results in a square matrix, and therefore the solution can be stated as

$$\boldsymbol{\Delta}\hat{\mathbf{u}} = \boldsymbol{\Delta}\hat{\mathbf{u}}_p = \mathbf{G}_2^T\left[\mathbf{G}_2\mathbf{G}_2^T + \lambda_{G_2}\mathbf{I}\right]^{-1}\mathbf{e}_2, \tag{49}$$

because the N_u degrees of freedom are used for satisfying the terminal constraint and there are no additional degrees for the cost function minimization.

It is noticeable that if uncertainty is not present, the CRHPC–BDU control law is transformed into the CRHPC one. So, assuming $\eta_{G_1} = \eta_{G_2} = \eta_{e_1} = \eta_{e_2} = 0$ the following values are obtained $\lambda_{G_2} = 0$, $\lambda_1 = \lambda_2 = \rho$ and equations from (42) to (48) coincide with Eqs. (24) and (25).

Table 1
GPC and CRHPC tuning parameters

Controller	N_1	N_2	m	N_u	$\mathbf{T}(z^{-1})$	ρ
GPC	1	6	–	4	1	1
CRHPC	1	6	3	4	1	1

If $m = 0$ then $\boldsymbol{\Delta}\hat{\mathbf{u}}_p = \mathbf{0}$ and $\mathbf{H} = \mathbf{I}$, and the control law coincides with the GPC–BDU one [19]

$$\boldsymbol{\Delta}\hat{\mathbf{u}} = \overset{*}{\mathbf{M}}\mathbf{G}_1^T\mathbf{e}_1, \tag{50}$$

$$\overset{*}{\mathbf{M}} = \left[\mathbf{G}_1^T\mathbf{G}_1 + \lambda_1\mathbf{I}\right]^{-1}, \tag{51}$$

$$\lambda_1 = \frac{\eta_{G_1}\|\mathbf{G}_1\boldsymbol{\Delta}\hat{\mathbf{u}} - \mathbf{e}_1\|_2}{\|\boldsymbol{\Delta}\hat{\mathbf{u}}\|_2} + \frac{\rho\|\mathbf{G}_1\boldsymbol{\Delta}\hat{\mathbf{u}} - \mathbf{e}_1\|_2}{\|\mathbf{G}_1\boldsymbol{\Delta}\hat{\mathbf{u}} - \mathbf{e}_1\|_2 + \eta_{G_1}\|\boldsymbol{\Delta}\hat{\mathbf{u}}\|_2 + \eta_{e_1}}. \tag{52}$$

5. Examples

5.1. GPC *versus* CRHPC

This example shows the stabilizing effect of the terminal constraint. A GPC will not be able to stabilize a particular process when process and model coincide, but the CRHPC will. Assuming the non-minimum phase discrete time system from [32], where the value of one pole has been changed, and the rest have been slightly modified

$$G(z) = \frac{z - 1.4}{(z - 0.8)(z - 0.7)}, \tag{53}$$

and the GPC tuning parameters of Table 1, the system becomes unstable (see Fig. 2). On the other hand, a CRHPC with the tuning parameters shown in Table 1 (being $n_a = \deg(\mathbf{A}(z^{-1})\boldsymbol{\Delta}) = 3 = m$, $n_b = \deg(\mathbf{B}(z^{-1})) = 1$) stabilizes the system (see Fig. 2).

5.2. CRHPC *versus* CRHPC–BDU

This example shows how the stabilizing effect of the CRHPC can vanish when a mismatch between process and model appears, due to the fact that the CRHPC only ensures nominal stability. In comparison, the CRHPC–BDU increases the system robustness and can provide a better performance. Assuming the same model $G(z)$ as the previous example, but in this case, the process results slightly different

$$G_p(z) = \frac{z - 1.4}{(z - 0.8)(z - 0.75)}, \tag{54}$$

when the CRHPC tuned as in the previous example, the system becomes unstable (see Fig. 3). For the CRHPC–BDU controller, a bound for uncertainty $\boldsymbol{\delta}\mathbf{G}_1$ and $\boldsymbol{\delta}\mathbf{G}_2$ must be obtained. From the unitary step response of both model and process, matrices $\mathbf{G}_1$ and $\mathbf{G}_2$ (which are considered for the controller tuning) and matrices $\mathbf{G}_{1_p}$ and $\mathbf{G}_{2_p}$ (from the real process) are obtained, and $\boldsymbol{\delta}\mathbf{G}_1$ and $\boldsymbol{\delta}\mathbf{G}_2$ can be calculated as the difference $\boldsymbol{\delta}\mathbf{G}_1 = \mathbf{G}_{1_p} - \mathbf{G}_1$ and $\boldsymbol{\delta}\mathbf{G}_2 = \mathbf{G}_{2_p} - \mathbf{G}_2$. Their bounds η_{G_1} and η_{G_2} are calculated as $\eta_{G_1} = \|\boldsymbol{\delta}\mathbf{G}_1\|_2$ and $\eta_{G_2} = \|\boldsymbol{\delta}\mathbf{G}_2\|_2$, resulting in $\eta_{G_1} = 0.11$ and $\eta_{G_2} = 0.25$, assuming $\eta_{e_1} = \eta_{e_2} = 0$.

With the CRHPC–BDU the stable response of Fig. 3 is obtained. Fig. 4 shows the adaptive tuning of the regularization parameters λ_{G_2}, λ_1 and λ_2, for facing the mismatch.

6. Conclusions and future work

The penalization parameter in MPC avoids control actions that are too abrupt, and at the same time improves system robustness. The main drawback is the fact that it is tuned by empirical criteria, without the objective of improving robustness.

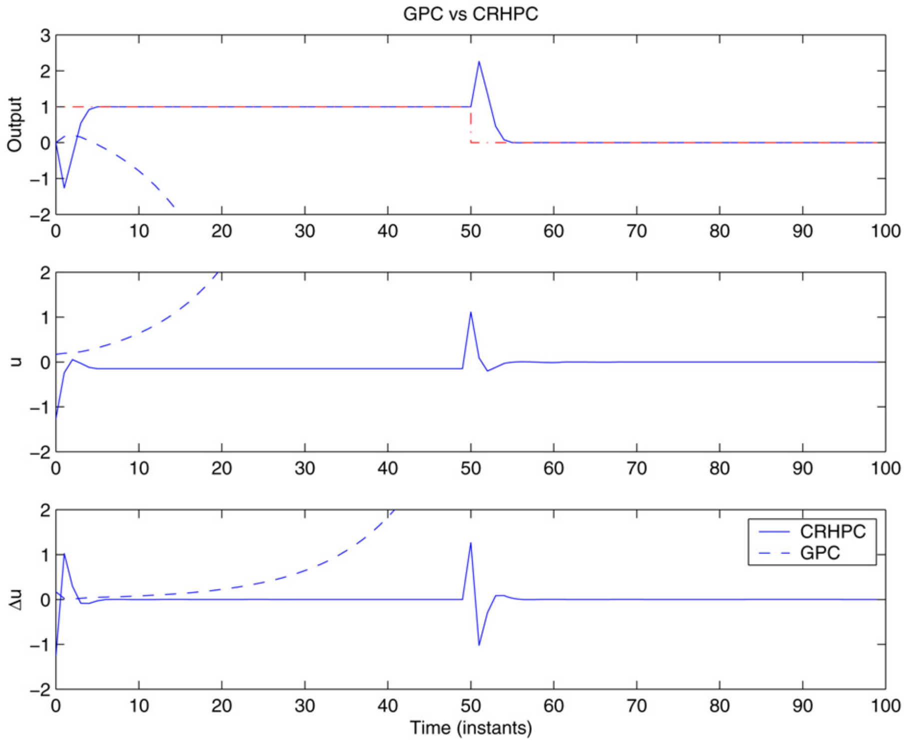

Fig. 2. System response with GPC and CRHPC when process and model coincide.

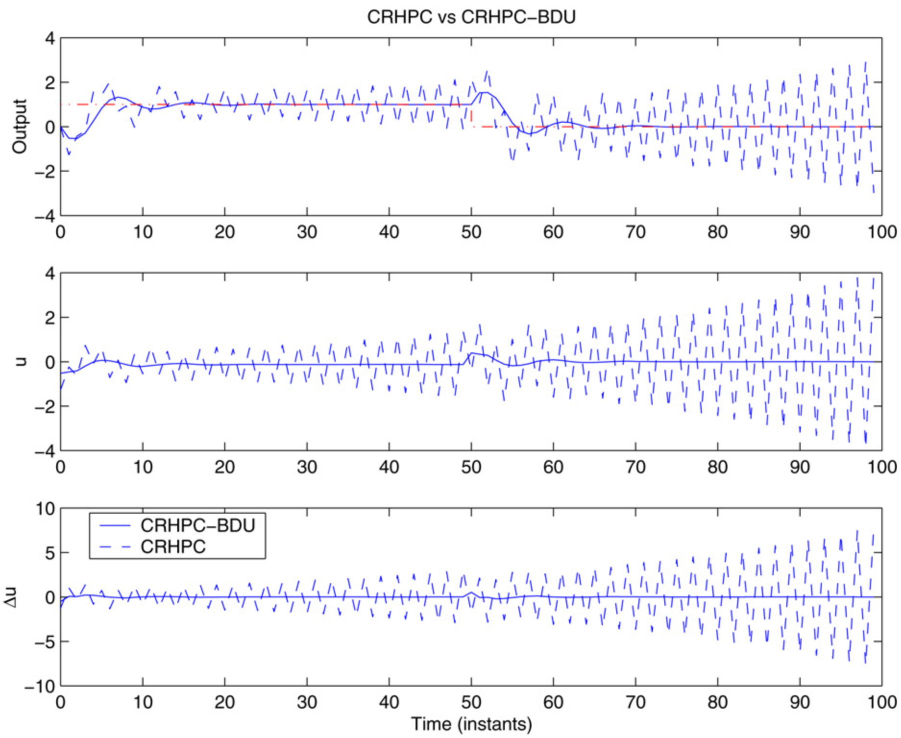

Fig. 3. System response with CRHPC and CRHPC–BDU when mismatch is present.

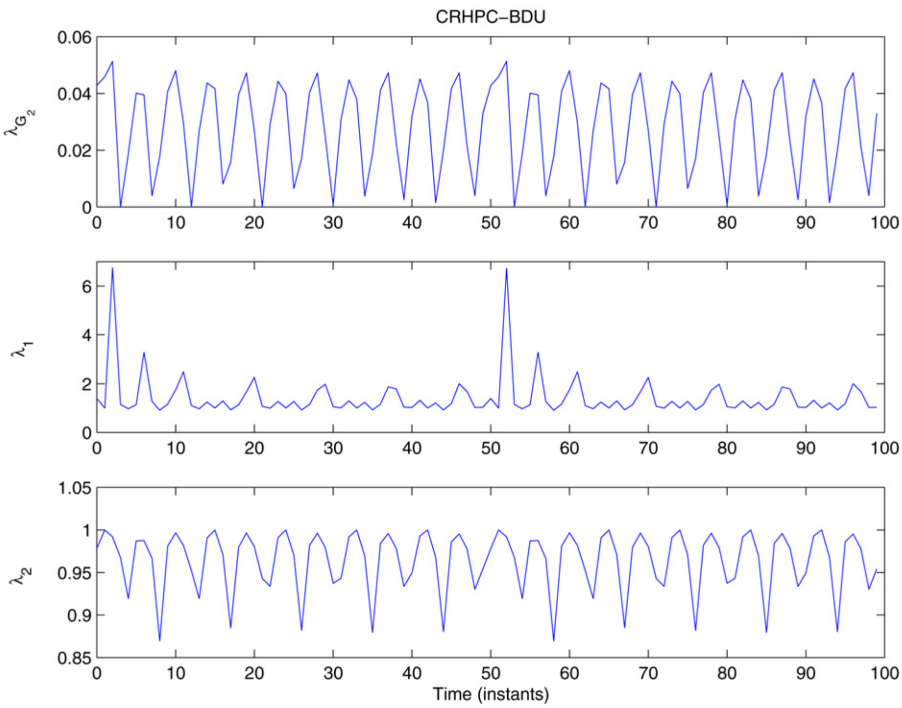

Fig. 4. Adaptive tuning of λ_{G_2}, λ_1 and λ_2.

From another point of view, MPC can be stated as a least-squares problem, but its main drawback is that the method is sensitive to data errors (ill-conditioning) which can be improved by the empirically tuned regularization parameter λ (which is similar to the penalization parameter for the control effort in MPC).

The BDU (Bounded Data Uncertainties) is a regularization technique for least-squares problems, which designs the regularization parameter λ taking into account the bound of the system uncertainty, and the problem is stated as a Min–Max optimization. It is possible to establish an analogy between BDU and the Min–Max problem in robust MPC, and the main objective consists in using BDU for tuning λ in a guided way to improve system robustness. Focusing on the GPC algorithm, the GPC–BDU can be stated in which λ is chosen automatically depending on the desired uncertainty bound. Another objective is to ensure stability. From the stabilizing GPC version (CRHPC or Constrained Receding-Horizon Predictive Control), which ensures nominal stability, the CRHPC–BDU is stated, and improves system robustness when discrepancies between model and process are present. Therefore, the CRHPC–BDU is a stable and robust GPC which constitutes the objective of this work.

Future work consists of constraints consideration in the MPC problem via LMIs, as well as application to non-linear systems and a real process.

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Appendix A. The homogeneous solution

From the problem (39) the minimization provides the $\mathbf{\Delta u}_f$ variables,

$$\nabla J = 2\big[\|\mathbf{G}\mathbf{\Delta u}_f - \mathbf{e}\|_2 + \eta_G\|\mathbf{\Delta u}_f\|_2 + \eta_e\big]\left[\frac{\mathbf{G}^T[\mathbf{G}\mathbf{\Delta u}_f - \mathbf{e}]}{\|\mathbf{G}\mathbf{\Delta u}_f - \mathbf{e}\|_2} + \frac{\eta_G\mathbf{\Delta u}_f}{\|\mathbf{\Delta u}_f\|_2}\right] + 2\rho\mathbf{H}^T[\mathbf{\Delta\hat{u}}_p + \mathbf{H}\mathbf{\Delta u}_f]. \tag{A.1}$$

The equation $\nabla J(\boldsymbol{\Delta}\hat{\mathbf{u}}_f) = \mathbf{0}$ results in

$$\big[\|\mathbf{G}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - \mathbf{e}\|_2 + \eta_G\|\boldsymbol{\Delta}\hat{\mathbf{u}}_f\|_2 + \eta_e\big]\left[\frac{\mathbf{G}^T[\mathbf{G}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - \mathbf{e}]}{\|\mathbf{G}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - \mathbf{e}\|_2} + \frac{\eta_G\boldsymbol{\Delta}\hat{\mathbf{u}}_f}{\|\boldsymbol{\Delta}\hat{\mathbf{u}}_f\|_2}\right] + \rho\mathbf{H}^T[\boldsymbol{\Delta}\hat{\mathbf{u}}_p + \mathbf{H}\boldsymbol{\Delta}\hat{\mathbf{u}}_f] = \mathbf{0}, \tag{A.2}$$

$$\frac{\mathbf{G}^T\mathbf{G}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - \mathbf{G}^T\mathbf{e}}{\|\mathbf{G}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - \mathbf{e}\|_2} + \eta_G\frac{\boldsymbol{\Delta}\hat{\mathbf{u}}_f}{\|\boldsymbol{\Delta}\hat{\mathbf{u}}_f\|_2} + \frac{\rho\mathbf{H}^T\boldsymbol{\Delta}\hat{\mathbf{u}}_p}{\|\mathbf{G}\Delta\hat{\mathbf{u}}_f - \mathbf{e}\|_2 + \eta_G\|\boldsymbol{\Delta}\hat{\mathbf{u}}_f\|_2 + \eta_e}$$
$$+ \frac{\rho\mathbf{H}^T\mathbf{H}\boldsymbol{\Delta}\hat{\mathbf{u}}_f}{\|\mathbf{G}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - \mathbf{e}\|_2 + \eta_G\|\boldsymbol{\Delta}\hat{\mathbf{u}}_f\|_2 + \eta_e} = \mathbf{0}, \tag{A.3}$$

$$\mathbf{G}^T\mathbf{G}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - \mathbf{G}^T\mathbf{e} + \frac{\eta_G\|\mathbf{G}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - \mathbf{e}\|_2}{\|\boldsymbol{\Delta}\hat{\mathbf{u}}_f\|_2}\boldsymbol{\Delta}\hat{\mathbf{u}}_f + \frac{\rho\mathbf{H}^T\mathbf{H}\|\mathbf{G}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - \mathbf{e}\|_2\boldsymbol{\Delta}\hat{\mathbf{u}}_f}{\|\mathbf{G}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - \mathbf{e}\|_2 + \eta_G\|\boldsymbol{\Delta}\hat{\mathbf{u}}_f\|_2 + \eta_e}$$
$$+ \frac{\rho\mathbf{H}^T\|\mathbf{G}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - \mathbf{e}\|_2\boldsymbol{\Delta}\hat{\mathbf{u}}_p}{\|\mathbf{G}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - \mathbf{e}\|_2 + \eta_G\|\boldsymbol{\Delta}\hat{\mathbf{u}}_f\|_2 + \eta_e} = \mathbf{0}. \tag{A.4}$$

Assuming the scalar values λ_1 and λ_2

$$\lambda_1 = \frac{\eta_G\|\mathbf{G}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - \mathbf{e}\|_2}{\|\boldsymbol{\Delta}\hat{\mathbf{u}}_f\|_2} + \frac{\rho\|\mathbf{G}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - \mathbf{e}\|_2}{\|\mathbf{G}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - \mathbf{e}\|_2 + \eta_G\|\boldsymbol{\Delta}\hat{\mathbf{u}}_f\|_2 + \eta_e}, \tag{A.5}$$

$$\lambda_2 = \frac{\rho\|\mathbf{G}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - \mathbf{e}\|_2}{\|\mathbf{G}\boldsymbol{\Delta}\hat{\mathbf{u}}_f - \mathbf{e}\|_2 + \eta_G\|\boldsymbol{\Delta}\hat{\mathbf{u}}_f\|_2 + \eta_e}, \tag{A.6}$$

and by taking into account $\mathbf{H}^T\mathbf{H} = \mathbf{I}$ the solution is obtained

$$\boldsymbol{\Delta}\hat{\mathbf{u}}_f = \big[\mathbf{G}^T\mathbf{G} + \lambda_1\mathbf{I}\big]^{-1}\big[\mathbf{G}^T\mathbf{e} - \lambda_2\mathbf{H}^T\boldsymbol{\Delta}\hat{\mathbf{u}}_p\big], \tag{A.7}$$

which through the changes of variables results in (40).

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