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Blow up of positive initial energy solutions for a wave equation with fractional boundary dissipation[☆]

Liqing Lu [a,*], Shengjia Li [b]

[a] School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi 030006, PR China

[b] Research Institute of Mathematics and Applied Mathematics, Shanxi University, Taiyuan, Shanxi 030006, PR China

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ABSTRACT

In this paper, we consider a strongly damped wave equation with fractional damping on part of its boundary and also with an internal source. Under some appropriate assumptions on the parameters, and with certain initial data, a blow-up result with positive initial energy is established.

1. Introduction

Consider the following wave equation with a fractional derivative term on part of its boundary:

$$\begin{cases} u_{tt} - \Delta u_t = \Delta u + f(u), & \text{in } (0,\infty)\times\Omega, \\ u = 0, & \text{on } [0,\infty)\times\Gamma_1, \\ \dfrac{\partial u_t}{\partial \nu} + \dfrac{\partial u}{\partial \nu} = -\displaystyle\int_0^t k_{\alpha,\beta}(t-s)u_t(s)\mathrm{d}s - bu_t, & \text{on } [0,\infty)\times\Gamma_0, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & \text{in } \bar{\Omega}. \end{cases} \tag{1.1}$$

Here, $0 < \alpha < 1, \beta, b > 0$, Ω is a bounded open subset of $R^n (n \geq 1)$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, and $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$, where Γ_0 and Γ_1 are measurable over $\partial\Omega$, endowed with the (n-1)-dimensional Lebesgue measure $\lambda_{n-1}(\Gamma_i)$, $i = 0, 1$. ν is the unit outward normal to $\partial\Omega$. The function $f(u) = |u|^{p-2}u$ is a polynomial source. The convolution term in (1.1) represents a modified fractional derivative (in the sense of Caputo) of u, and the weakly singular kernel $k_{\alpha,\beta}(t)$ is equal to $t^{\alpha-1}\mathrm{e}^{-\beta t}/\Gamma(\alpha)$. We can see that we are in the presence of a "weak fractional dissipation" on part of the boundary and an internal source term.

In the case of an undamped wave equation, the presence of a nonlinear source term will destabilize the solution, and force the solution to blow up in finite time. In the case of damping of order one, the source term will compete with the damping term. In the case of a "weak fractional dissipation" and a strong singular kernel $k_1(t) = t^{-\alpha}/\Gamma(1-\alpha)$, Tatar and Kirane [1] proved an exponential growth result for the solutions of a wave equation with a nonlinear polynomial source. Tatar [2] extended this result to a larger class of initial data. A blow-up result for sufficiently large data has been proved in [3]. In the

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* Corresponding author. Tel.: +86 351 7010555; fax: +86 351 7010979.
E-mail address: lulq@sxu.edu.cn (L. Lu).

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case of a "weak fractional dissipation" and a weak singular kernel, Tatar [4] considered a strongly damped wave equation with an internal polynomial source and frictional damping on part of the boundary; a blow-up result has been established with certain initial data and an appropriate source term. For more related results, we refer the reader to [5–20].

Inspired by [4,13], we intend to study the blow up of positive initial energy solutions for problem (1.1). By combining the concavity method with the potential well method introduced by [21,22], we prove that, under some appropriate assumptions on the parameters, and with certain initial data, solutions to problem (1.1) blow up in finite time.

The present work is organized as follows. In Section 2, we present some notation and material needed for our work and state our main result. Section 3 is devoted to proving our main result.

2. Notation and main results

In this section, we present some material needed in the proof of our result. Without loss of generality, we take $b = 1$. The following notation will be used throughout the paper.

$$\|\cdot\|_q = \|\cdot\|_{L^q(\Omega)}, \qquad \|\cdot\|_{q,\Gamma_0} = \|\cdot\|_{L^q(\Gamma_0)}, \quad 1 \le q \le \infty,$$

and the Hilbert space

$$H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega) : u|_{\Gamma_1} = 0\},$$

($u|_{\Gamma_1}$ is in the trace sense). When $\lambda_{n-1}(\Gamma_1) > 0$, the Poincaré inequality holds, so that $\|\nabla u\|_2$ is an equivalent norm on $H^1_{\Gamma_1}(\Omega)$, and we shall use the Sobolev embedding frequently,

$$H^1_{\Gamma_1}(\Omega) \hookrightarrow L^q(\Omega), \quad \text{for } 1 \le q < \frac{2n}{n-2} (n \ge 3).$$

Let C_* be the best constant satisfying the trace-Sobolev embedding

$$\|u\|_{2,\Gamma_0} \le C_* \|\nabla u\|_2, \quad \forall u \in H^1_{\Gamma_1}(\Omega).$$

We set

$$C(\alpha, \beta) = \frac{\sqrt{\Gamma(2\alpha-1)}}{\Gamma(\alpha)\sqrt{\beta^{2\alpha-1}}} \left(\frac{1}{2} < \alpha < 1\right), \qquad a = B^{-\frac{p}{p-2}}, \quad \text{and} \quad d = \frac{p-2}{2p} B^{-\frac{2p}{p-2}},$$

where $\Gamma(\alpha)$ is the usual Gamma function, and B is the optimal constant of Sobolev embedding $H^1_{\Gamma_1}(\Omega) \hookrightarrow L^p(\Omega)$ given by

$$\|u\|_p \le B\|\nabla u\|_2, \quad \forall u \in H^1_{\Gamma_1}(\Omega). \tag{2.1}$$

By a formal calculation we define the energy functional of (1.1) as

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p. \tag{2.2}$$

A simple computation yields

$$E'(t) = -\frac{1}{\Gamma(\alpha)} \int_{\Gamma_0} u_t \int_0^t (t-s)^{\alpha-1} e^{-\beta(t-s)} u_s \mathrm{d}s\mathrm{d}\sigma - \|\nabla u_t\|_2^2 - \|u_t\|_{2,\Gamma_0}^2, \tag{2.3}$$

integrating (2.3) from 0 to t, we have

$$E(t) - E(0) = -\int_0^t \|\nabla u_s\|_2^2 \mathrm{d}s - \int_0^t \|u_s\|_{2,\Gamma_0}^2 \mathrm{d}s - \frac{1}{\Gamma(\alpha)} \int_0^t \int_{\Gamma_0} u_s \int_0^s (s-z)^{\alpha-1} e^{-\beta(s-z)} u_z \mathrm{d}z\mathrm{d}\sigma\mathrm{d}s. \tag{2.4}$$

Noting that our kernel is positive definite in the sense of Definition 1 (see [23]), it follows that

$$E(t) \le E(0). \tag{2.5}$$

We are now in a position to state our main result.

Theorem 2.1. *Assume that $p > 2$ if $n = 1, 2$, and that $2 < p \le \frac{2n}{n-2}$ if $n > 2$. Suppose that the parameters α, β, p satisfy*

$$C(\alpha, \beta) C_* < p - 2. \tag{2.6}$$

Let u be a solution of (1.1) with initial data satisfying

$$E(0) \le \delta d, \qquad \|\nabla u_0\|_2 > B^{-\frac{p}{p-2}}, \tag{2.7}$$

where $0 < \delta < \frac{1}{2} - \frac{\Gamma(2\alpha-1)C_*^2}{2(p-2)^2\Gamma^2(\alpha)\beta^{2\alpha-1}}$. Then the solution of (1.1) blows up in finite time in the sense of (3.14).

Remark 2.1. By utilizing (2.6), we can easily check that $0 < \delta < 1$.

3. Proof of main result

The proof of Theorem 2.1 relies on the following lemmas.

Lemma 3.1. *Assume that* $u_0 \in H^1_{\Gamma_1}(\Omega),\ u_1 \in L^2(\Omega)$, *and suppose that* (2.6), (2.7) *hold. Then, for any* $T > 0$,

$$E(t) \le \delta d, \qquad \|\nabla u(t)\|_2 > B^{-\frac{p}{p-2}}, \tag{3.1}$$

for all $t \in [0, T)$.

Proof. We first note that, by exploiting (2.2), we have

$$E(t) \ge \frac{1}{2}\|\nabla u\|_2^2 - \frac{B^p}{p}\|\nabla u\|_2^p = h(\|\nabla u\|_2), \tag{3.2}$$

where the function $h(\xi) = \frac{1}{2}\xi^2 - \frac{B^p}{p}\xi^p$ $(\xi \ge 0)$. It is easy to verify that h is increasing for $0 < \xi < a$, decreasing for $\xi > a$, and that h attains its maximum d at $\xi = a$. By using (2.5), (2.7) and (3.2), we have

$$d > \delta d \ge E(0) \ge E(t) \ge h(\|\nabla u\|_2);$$

therefore there is no $t \in (0, T)$ such that $\|\nabla u(t)\|_2 = a$. Since the function $t \to \|\nabla u(t)\|_2$ is continuous, and by using (2.7), it follows that

$$\|\nabla u(t)\|_2 > B^{-\frac{p}{p-2}}, \quad \forall\, t \in [0, T). \quad \blacksquare$$

Remark 3.1. From the proof of Lemma 3.1, we can conclude that the solution u of (1.1) is global when the initial data satisfies the following condition:

$$E(0) < \delta d, \qquad \|\nabla u_0\|_2 < B^{-\frac{p}{p-2}}.$$

Lemma 3.2 ([24]). *Assume that* $P(t) \in C^2$, $P(t) \ge 0$, *satisfies the inequality*

$$P(t)P''(t) - (1+\gamma)P'^2(t) \ge 0,$$

for a certain real number $\gamma > 0$, *and that* $P(0) > 0$, $P'(0) > 0$. *Then there exists a real number* $\tilde{T}$ *such that* $0 < \tilde{T} \le P(0)/\gamma P'(0)$ *and*

$$P(t) \to \infty \quad \text{as } t \to \tilde{T}^-.$$

Proof of Theorem 2.1. Assume by contradiction that the solution u of (1.1) is global. Then we consider a function $g : [0, T_0] \longrightarrow R^+$ defined by

$$g(t) = \|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 ds + (T_0 - t)\|\nabla u_0\|_2^2 + \int_0^t \|u(s)\|_{2,\Gamma_0}^2 ds + (T_0 - t)\|u_0\|_{2,\Gamma_0}^2 + \mu(t + t_0)^2, \quad t \le T_0,$$

where the parameters μ, t_0, T_0 are positive constants to be determined later.

Notice that $g(t) > 0$ for all $t \in [0, T_0]$; hence, since g is continuous, there exists a constant $\rho > 0$ such that

$$g(t) \ge \rho \quad \text{for all } t \in [0, T_0]. \tag{3.3}$$

Furthermore,

$$\begin{aligned} g'(t) &= 2\int_\Omega u u_t dx + \|\nabla u(t)\|_2^2 - \|\nabla u_0\|_2^2 + \|u(t)\|_{2,\Gamma_0}^2 - \|u_0\|_{2,\Gamma_0}^2 + 2\mu(t + t_0) \\ &= 2\int_\Omega u u_t dx + 2\int_0^t \int_\Omega \nabla u(s) \cdot \nabla u_s(s) dx ds + 2\int_0^t \int_{\Gamma_0} u(s) u_s(s) d\sigma ds + 2\mu(t + t_0), \end{aligned}$$

and, consequently, utilizing Eq. (1.1), we obtain

$$g''(t) = 2\|u_t\|_2^2 - 2\|\nabla u\|_2^2 + 2\|u\|_p^p - \frac{2}{\Gamma(\alpha)}\int_{\Gamma_0} u \int_0^t (t-s)^{\alpha-1} e^{-\beta(t-s)} u_s ds d\sigma + 2\mu. \tag{3.4}$$

Therefore, using the Hölder inequality, we get

$$\begin{aligned}
g'(t)^2 &= 4\left(\int_\Omega uu_t\mathrm{d}x+\int_0^t\int_\Omega \nabla u(s)\cdot\nabla u_s(s)\mathrm{d}x\mathrm{d}s+\int_0^t\int_{\Gamma_0} u(s)u_s(s)\mathrm{d}\sigma\mathrm{d}s+\mu(t+t_0)\right)^2\\
&=4\Bigg[\left(\int_\Omega uu_t\mathrm{d}x\right)^2+\left(\int_0^t\int_\Omega \nabla u(s)\cdot\nabla u_t(s)\mathrm{d}x\mathrm{d}s\right)^2+\left(\int_0^t\int_{\Gamma_0} u(s)u_s(s)\mathrm{d}\sigma\mathrm{d}s\right)^2+\mu^2(t+t_0)^2\\
&\quad+2\left(\int_\Omega uu_t\mathrm{d}x\right)\left(\int_0^t\int_\Omega \nabla u(s)\cdot\nabla u_s(s)\mathrm{d}x\mathrm{d}s\right)+2\left(\int_\Omega uu_t\mathrm{d}x\right)\left(\int_0^t\int_{\Gamma_0} u(s)u_s(s)\mathrm{d}\sigma\mathrm{d}s\right)\\
&\quad+2\left(\int_0^t\int_\Omega \nabla u(s)\cdot\nabla u_t(s)\mathrm{d}x\mathrm{d}s\right)\left(\int_0^t\int_{\Gamma_0} u(s)u_s(s)\mathrm{d}\sigma\mathrm{d}s\right)+2\mu(t+t_0)\\
&\quad\times\left(\int_\Omega uu_t\mathrm{d}x+\int_0^t\int_\Omega \nabla u(s)\cdot\nabla u_s(s)\mathrm{d}x\mathrm{d}s+\int_0^t\int_{\Gamma_0} u(s)u_s(s)\mathrm{d}\sigma\mathrm{d}s\right)\Bigg]\\
&\le 4\Bigg[\|u_t\|_2^2\|u\|_2^2+\left(\int_0^t\|\nabla u(s)\|_2^2\mathrm{d}s\right)\left(\int_0^t\|\nabla u_s(s)\|_2^2\mathrm{d}s\right)+\left(\int_0^t\|u(s)\|_{2,\Gamma_0}^2\mathrm{d}s\right)\left(\int_0^t\|u_s(s)\|_{2,\Gamma_0}^2\mathrm{d}s\right)\\
&\quad+\mu^2(t+t_0)^2+\|u_t\|_2^2\left(\int_0^t\|\nabla u(s)\|_2^2\mathrm{d}s\right)+\|u\|_2^2\left(\int_0^t\|\nabla u_s(s)\|_2^2\mathrm{d}s\right)+\|u_t\|_2^2\left(\int_0^t\|u(s)\|_{2,\Gamma_0}^2\mathrm{d}s\right)\\
&\quad+\|u\|_2^2\left(\int_0^t\|u_s\|_{2,\Gamma_0}^2\mathrm{d}s\right)+\left(\int_0^t\|\nabla u(s)\|_2^2\mathrm{d}s\right)\left(\int_0^t\|u_s\|_{2,\Gamma_0}^2\mathrm{d}s\right)+\left(\int_0^t\|\nabla u_s(s)\|_2^2\mathrm{d}s\right)\left(\int_0^t\|u(s)\|_{2,\Gamma_0}^2\mathrm{d}s\right)\\
&\quad+\mu(t+t_0)^2\left(\|u_t\|_2^2+\int_0^t\|\nabla u_s(s)\|_2^2\mathrm{d}s+\int_0^t\|u_s(s)\|_{2,\Gamma_0}^2\mathrm{d}s\right)\\
&\quad+\mu\left(\|u\|_2^2+\int_0^t\|\nabla u(s)\|_2^2\mathrm{d}s+\int_0^t\|u(s)\|_{2,\Gamma_0}^2\mathrm{d}s\right)\Bigg]\\
&=4\left(\|u\|_2^2+\int_0^t\|\nabla u(s)\|_2^2\mathrm{d}s+\int_0^t\|u(s)\|_{2,\Gamma_0}^2\mathrm{d}s+\mu(t+t_0)^2\right)\\
&\quad\times\left(\|u_t\|_2^2+\int_0^t\|\nabla u_s(s)\|_2^2\mathrm{d}s+\int_0^t\|u_s(s)\|_{2,\Gamma_0}^2\mathrm{d}s+\mu\right)\\
&\le 4g(t)\left(\|u_t\|_2^2+\int_0^t\|\nabla u_s(s)\|_2^2\mathrm{d}s+\int_0^t\|u_s(s)\|_{2,\Gamma_0}^2\mathrm{d}s+\mu\right). \qquad (3.5)
\end{aligned}$$

Using (2.2) and (2.4), we can have

$$\begin{aligned}
g''(t)g(t)-\frac{p+2}{4}g'(t)^2 &\ge \left(g''(t)-(p+2)(\|u_t\|_2^2+\int_0^t\|\nabla u_s(s)\|_2^2\mathrm{d}s+\int_0^t\|u_s(s)\|_{2,\Gamma_0}^2\mathrm{d}s+\mu)\right)g(t)\\
&=\Bigg[-p\|u_t\|_2^2-2\|\nabla u\|_2^2+2\|u\|_p^p-(p+2)\int_0^t\|\nabla u_s(s)\|_2^2\mathrm{d}s-(p+2)\int_0^t\|u_s(s)\|_{2,\Gamma_0}^2\mathrm{d}s\\
&\quad-p\mu-\frac{2}{\Gamma(\alpha)}\int_{\Gamma_0}u\int_0^t(t-s)^{\alpha-1}\mathrm{e}^{-\beta(t-s)}u_s\mathrm{d}s\mathrm{d}\sigma\Bigg]g(t)\\
&=\Bigg[-2pE(0)-p\mu+(p-2)\|\nabla u\|_2^2+(p-2)\int_0^t\|\nabla u_s(s)\|_2^2\mathrm{d}s+(p-2)\\
&\quad\times\int_0^t\|u_s(s)\|_{2,\Gamma_0}^2\mathrm{d}s+\frac{2p}{\Gamma(\alpha)}\int_0^t\int_{\Gamma_0}u_s\int_0^s(s-z)^{\alpha-1}\mathrm{e}^{-\beta(s-z)}u_z\mathrm{d}z\mathrm{d}\sigma\mathrm{d}s\\
&\quad-\frac{2}{\Gamma(\alpha)}\int_{\Gamma_0}u\int_0^t(t-s)^{\alpha-1}\mathrm{e}^{-\beta(t-s)}u_s\mathrm{d}s\mathrm{d}\sigma\Bigg]g(t). \qquad (3.6)
\end{aligned}$$

We now estimate the term $\frac{2}{\Gamma(\alpha)}\int_{\Gamma_0} u\int_0^t (t-s)^{\alpha-1}\mathrm{e}^{-\beta(t-s)}u_s\mathrm{d}s\mathrm{d}\sigma$ in the right-hand side of (3.6). By exploiting the Hölder inequality and the Young inequality, we have

$$\begin{aligned}\frac{2}{\Gamma(\alpha)}\int_{\Gamma_0} u\int_0^t (t-s)^{\alpha-1}\mathrm{e}^{-\beta(t-s)}u_s\mathrm{d}s\mathrm{d}\sigma &\le \frac{2}{\Gamma(\alpha)}\|u\|_{2,\Gamma_0}\left(\int_{\Gamma_0}\left|\int_0^t (t-s)^{\alpha-1}\mathrm{e}^{-\beta(t-s)}u_s\mathrm{d}s\right|^2\mathrm{d}\sigma\right)^{\frac{1}{2}}\\ &\le \frac{2}{\Gamma(\alpha)}\|u\|_{2,\Gamma_0}\left(\int_0^t (t-s)^{2(\alpha-1)}\mathrm{e}^{-2\beta(t-s)}\mathrm{d}s\right)^{\frac{1}{2}}\left[\int_{\Gamma_0}\int_0^t |u_s(s)|^2\mathrm{d}s\mathrm{d}\sigma\right]^{\frac{1}{2}}\\ &\le C(\alpha,\beta)\left(\epsilon\|u\|_{2,\Gamma_0}^2+\frac{1}{\epsilon}\int_0^t \|u_s(s)\|_{2,\Gamma_0}^2\mathrm{d}s\right)\\ &\le C(\alpha,\beta)\left(\epsilon C_*^2\|\nabla u\|_2^2+\frac{1}{\epsilon}\int_0^t \|u_s(s)\|_{2,\Gamma_0}^2\mathrm{d}s\right),\end{aligned} \tag{3.7}$$

for any $\epsilon > 0$.

Since our kernel is positive definite, it follows that

$$\int_0^t\int_{\Gamma_0} u_s\int_0^s (s-z)^{\alpha-1}\mathrm{e}^{-\beta(s-z)}u_z\mathrm{d}z\mathrm{d}\sigma\mathrm{d}s \ge 0. \tag{3.8}$$

Combining (3.6)–(3.8), we can get

$$\begin{aligned}&g''(t)g(t)-\frac{p+2}{4}g'(t)^2\\ &\ge\left[-2pE(0)-p\mu+(p-2-\epsilon C_*^2C(\alpha,\beta))\|\nabla u\|_2^2+(p-2-C(\alpha,\beta)/\epsilon)\int_0^t\|u_s(s)\|_{2,\Gamma_0}^2\mathrm{d}s\right]g(t).\end{aligned} \tag{3.9}$$

Now, choosing $\mu=\frac{(p-2)\delta}{p}B^{-\frac{2p}{p-2}}$, and using (2.6), (2.7) and (3.1), we can get

$$\begin{aligned}-2pE(0)-p\mu+(p-2-\epsilon C_*^2C(\alpha,\beta))\|\nabla u\|_2^2 &\ge -2\delta(p-2)B^{\frac{-2p}{p-2}}+(p-2-\epsilon C_*^2C(\alpha,\beta))\|\nabla u\|_2^2\\ &\ge [(p-2)(1-2\delta)-\epsilon C_*^2C(\alpha,\beta)]\|\nabla u\|_2^2.\end{aligned} \tag{3.10}$$

So we take ϵ to be

$$C(\alpha,\beta)/(p-2)<\epsilon<(p-2)(1-2\delta)/C_*^2C(\alpha,\beta) \tag{3.11}$$

such that

$$\begin{cases}(p-2)(1-2\delta)-\epsilon C_*^2C(\alpha,\beta)>0,\\ p-2-C(\alpha,\beta)/\epsilon>0.\end{cases}$$

Then, combining (3.3), (3.9)–(3.11), there exists a positive constant C such that

$$g''(t)g(t)-\frac{p+2}{4}g'(t)^2\ge C\rho,\quad \text{for } t\in[0,T_0].$$

Moreover, $g(0)>0$, and we can choose t_0 large enough so that

$$(p-2)\left(\int_\Omega u_0u_1\mathrm{d}x+\mu t_0\right)>2(\|\nabla u_0\|_2^2+\|u_0\|_{2,\Gamma_0}^2). \tag{3.12}$$

From (3.12), we can check that $g'(0)>0$, and we can select T_0 satisfying

$$\|u_0\|_2^2+\mu t_0^2<T_0\left[\frac{p-2}{2}\left(\int_\Omega u_0u_1\mathrm{d}x+\mu t_0\right)-\|\nabla u_0\|_2^2-\|u_0\|_{2,\Gamma_0}^2\right]. \tag{3.13}$$

According to Lemma 3.2, there exists a real number T^* such that $T^*<g(0)/\gamma g'(0)$ and $T^*<T_0$, and we have

$$\lim_{t\to T^{*-}} g(t)=\infty;$$

i.e.,

$$\lim_{t\to T^{*-}}\left(\|u(t)\|_2^2+\int_0^t\|\nabla u(s)\|_2^2\mathrm{d}s+\int_0^t\|u(s)\|_{2,\Gamma_0}^2\mathrm{d}s\right)=\infty. \tag{3.14}$$

This completes the proof of Theorem 2.1. ∎

Remark 3.2. The inequality (3.13) implies that $T^* < g(0)/\gamma g'(0) < T_0$.

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