Discrete Applied Mathematics 159 (2011) 1676–1688

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

The number of excellent discrete Morse functions on graphs

R. Ayala, D. Fernández-Ternero, J.A. Vilches *

Dpto. de Geometría y Topología, Universidad de Sevilla, 41080, Sevilla, Spain

ARTICLE INFO

Article history:
Available online 19 January 2011

Keywords:
Infinite locally finite graph
Critical simplex
Homological sequence
$\mathbb{Z}$-walk
Excellent discrete Morse function

ABSTRACT

In Nicolaescu (2008) [7] the number of non-homologically equivalent excellent Morse functions defined on $\mathbb{S}^2$ was obtained in the differentiable setting. We carried out an analogous study in the discrete setting for some kinds of graphs, including $\mathbb{S}^1$, in Ayala et al. (2009) [1]. This paper completes this study, counting excellent discrete Morse functions defined on any infinite locally finite graph.

1. Introduction

Since it was introduced, Morse theory has been a powerful tool in the study of smooth manifolds by means of differential geometry techniques. Basically, it allows us to describe the topology of a manifold in terms of the cellular decomposition generated by the critical points of a scalar smooth map defined on it.

At the end of the last century, Forman [3] developed a discrete version of Morse theory that turned out to provide a fruitful and efficient method for the study of the topology of discrete objects, such as simplicial and cellular complexes, which play a central role in many different fields of pure and applied mathematics.

Essentially, a discrete Morse function on a simplicial complex is a way to assign a real number to each simplex of a complex, without any continuity, in such a way that for each simplex the natural order given by the dimension simplices is respected, except in at most one (co)face of the given simplex. As in the smooth setting, changes in the topology of the level subcomplexes are deeply related to the presence of critical simplices of the function, and the analysis of the evolution of the homology of these complexes can be a very useful tool in computer vision for dealing with shape recognition problems by means of topological shape descriptors. In our opinion, there are many advantages of using Forman's theory. First, it can be applied to discrete objects more general than manifolds. In particular, for the one-dimensional case the smooth approach can only be applied essentially to circles and lines. However, the discrete version can be applied to any graph. Second, it is more suitable in the digital context for areas like pattern recognition, shape classification and recognition, and thinning 2D objects where usually discretized functions are used.

This paper completes the study of the size of the set of discrete Morse functions with a given number of critical simplices defined on a graph which was initiated by the authors in [1]. Our study is carried out by taking into account the rank evolution of the homology groups of the level sets corresponding to the critical values of the functions.

This paper is organized as follows. Section 2 contains the basic notions and results of discrete Morse theory on graphs which will be used later. In Section 3 we study some general properties of the homological sequences of a discrete Morse function on a graph and we establish links between them and certain kinds of walks in $\mathbb{Z}_{>0}$, whose number is obtained. Section 4 starts by giving two lemmas concerning the properties of the bridge components of a locally finite graph. Next, inspired by the results of Nicolaescu [7,6] on the number of smooth Morse functions on the 2-sphere, we prove the main result of the paper which establishes how many non-homologically equivalent discrete Morse functions with a given number of critical simplices exist on an infinite and locally finite graph G with $b_1(G) < +\infty$. It is worthwhile to mention that the

* Corresponding author.
E-mail addresses: rdayala@us.es (R. Ayala), desamfer@us.es (D. Fernández-Ternero), vilches@us.es (J.A. Vilches).

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doi:10.1016/j.dam.2010.12.011

proof is constructive in the sense that it indicates precisely how to define an excellent discrete Morse function from a pair of homological sequences satisfying certain conditions. Finally, we give examples which illustrate how this constructive procedure is carried out.

2. Preliminaries

Throughout this paper, we only consider infinite graphs which are locally finite. For general topics of graph theory we will follow [4]. Given such a graph G, a *bridge* is an edge whose deletion increases the number of connected components of G. A graph is said to be *bridgeless* if it contains no bridges. In the particular case of bridgeless graphs, we will consider non-trivial connected graphs, that is, connected bridgeless graphs not consisting of a unique vertex.

Let B be the set of all bridges of G. The *bridge components* of G are the connected components of $G - B$.

A graph G' is a *subdivision* of a graph G if G' can be obtained from G by introducing new vertices. Two graphs G_1 and G_2 are *combinatorially equivalent* if they have a common subdivision, that is, there is a graph G_3 which is a common subdivision of G_1 and G_2. Notice that the topological spaces $|G_1|$ and $|G_2|$ are the same.

We introduce here the basic notions of discrete Morse theory [3]. A *discrete Morse function* is a function $f : G \longrightarrow \mathbb{R}$ such that, for any p-simplex $\sigma \in G$:

(M1) $\operatorname{card}\{\tau^{(p+1)} > \sigma / f(\tau) \leq f(\sigma)\} \leq 1$.
(M2) $\operatorname{card}\{\upsilon^{(p-1)} < \sigma / f(\upsilon) \geq f(\sigma)\} \leq 1$.

A p-simplex $\sigma \in G$ is said to be *a critical simplex* with respect to f if:

(C1) $\operatorname{card}\{\tau^{(p+1)} > \sigma / f(\tau) \leq f(\sigma)\} = 0$.
(C2) $\operatorname{card}\{\upsilon^{(p-1)} < \sigma / f(\upsilon) \geq f(\sigma)\} = 0$.

A value of a discrete Morse function on a critical simplex is called a *critical value*.

A *ray* is an infinite sequence of simplices:

$$v_0, e_0, v_1, e_1, \ldots, v_r, e_r, v_{r+1} \ldots$$

such that the vertices v_i and v_{i+1} are faces of the edge e_i, for any $i \in \mathbb{N} \cup \{0\}$. Two rays contained in an infinite graph are said to be *equivalent* or *cofinal* if they coincide starting from a common 0-simplex.

If there is a discrete Morse function f defined on G, a *decreasing ray* is a ray such that

$$f(v_0) \geq f(e_0) > f(v_1) \geq f(e_1) > \cdots \geq f(e_r) > f(v_{r+1}) \geq \cdots.$$

A *critical element* of f on G is either a critical simplex or a decreasing ray.

Given $c \in \mathbb{R}$, the *level subcomplex* $G(c)$ is the subcomplex of G consisting of all simplices τ with $f(\tau) \leq c$, as well as all of their faces, that is,

$$G(c) = \bigcup_{f(\tau) \leq c} \bigcup_{\sigma \leq \tau} \sigma.$$

The next result is a special case of Theorem 3.1 in [2] pertaining to discrete Morse functions with no decreasing rays. It establishes links between the topology of a graph and the critical elements of a discrete Morse function defined on it.

Theorem 2.1. *Let G be a graph and let f be a discrete Morse function defined on G such that the numbers $m_i(f)$ of critical i-simplices of f with $i = 0, 1$ are finite and f has no decreasing rays. Then:*

(i) *$m_0(f) \geq b_0$ and $m_1(f) \geq b_1$, where b_i denotes the ith Betti number of G with $i = 0, 1$.*
(ii) *$b_0 - b_1 = m_0(f) - m_1(f)$.*

Given a discrete Morse function defined on G, we say that a pair of simplices $(\upsilon < e)$ is in the *gradient vector field* induced by f if and only if $f(\upsilon) \geq f(e)$.

Given a gradient vector field V on G, a *V-path* is a sequence of simplices

$$\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \ldots, \beta_r^{(p+1)}, \alpha_{r+1}^{(p)}, \ldots,$$

such that, for each $i \geq 0$, the pair $(\alpha_i^{(p)} < \beta_i^{(p+1)}) \in V$ and $\beta_i^{(p+1)} > \alpha_{i+1}^{(p)} \neq \alpha_i^{(p)}$.

Given a 0-critical simplex in G, we say that any vertex w of G is *rooted* in υ if there exists a finite V-path joining w and υ.

The next two results provide information about the structure of the set of all V-paths contained in a graph with a given gradient field. In particular, they state that this set is acyclic, that is, it is a forest.

Proposition 2.2 (*[1]*). *Let G be an infinite graph and let f be a discrete Morse function defined on G with no decreasing rays. It holds that:*

1. *Given w any vertex of G, there is a unique 0-critical simplex on which w is rooted.*
2. *Given any 0-critical simplex υ, the set of all V-paths rooted in it is a tree called* the tree rooted in *υ and denoted by T_υ.*
3. *Any two such rooted trees are disjoint.*

Theorem 2.3 (*[1]*). *Under the above definitions and notation, the forest F consisting of all rooted trees in G can be obtained by removing all critical edges of f on G.*

A discrete Morse function defined on a graph G is called *excellent* if all its critical values are different. It easy to prove that, by slightly modifying its critical values, every discrete Morse function with a finite number of critical simplices can be considered as an excellent one. Since the notion of critical value plays a central role in this paper and taking into account that it is not possible to get an analogous notion for decreasing rays, we will only deal with discrete Morse functions with no decreasing rays, that is, those whose critical elements are critical simplices.

Two excellent discrete Morse functions f and g defined on a graph G with critical values $a_0 < a_1 < \cdots < a_{m-1}$ and $c_0 < c_1 < \cdots < c_{m-1}$ respectively will be called *homologically equivalent* if for all $i = 0, \ldots, m-1$ the level subcomplexes $G(a_i)$ and $G(c_i)$ have the same Betti numbers.

3. Homological sequences on graphs

This section is devoted to the study of the properties of homological sequences of a discrete Morse function defined on a graph.

Let f be an excellent discrete Morse function defined on G with m critical simplices and critical values $a_0, \ldots, a_{m-1}$. Notice that the quantity $m - b_0(G) - b_1(G)$ is an even number, denoted by $2k$. We denote the level subcomplexes $G(a_i)$ by G_i for all $i = 0, \ldots, m-1$. The *homological sequences* of f are the two sequences $B_0, B_1 : \{0, 1, \ldots, m-1\} \to \mathbb{N}$ containing the homological information of the level subcomplexes $G_0, \ldots, G_{m-1}$, that is, $B_p(i) = b_p(G_i) = \dim(H_p(G_i))$ for each $i = 0, \ldots, m-1$ and $p = 0, 1$.

Notice that the homological sequences of f satisfy

$$\begin{aligned} &B_0(0) = B_0(m-1) = b_0 = 1, \quad B_0(i) > 0, \qquad |B_0(i+1) - B_0(i)| = 0 \text{ or } 1; \\ &B_1(0) = 0, \qquad B_1(m-1) = b_1, \quad B_1(i) \geq 0, \qquad B_1(i+1) - B_1(i) = 0 \text{ or } 1. \end{aligned}$$

Lemma 3.1 (*[1]*). *For each $i = 0, 1, \ldots, m-2$ exactly one of the following identities holds:*

(H1) $B_0(i) = B_0(i+1)$.
(H2) $B_1(i) = B_1(i+1)$.

Let f be an excellent discrete Morse function defined on a connected graph G with critical values $a_0 < \cdots < a_{n-1}$. We say that a critical vertex v is an *essential vertex* if $f(v)$ is the global minimum of f on G, that is, $f(v) = a_0$. A critical edge e_i with $f(e_i) = a_i$ is an *essential edge* if $B_1(i) - B_1(i-1) = 1$. Otherwise, if a critical simplex is not an essential one, we say that it is a *superfluous or cancellable simplex*. These kinds of simplices can be regarded as the "noise" generated by the discrete Morse function considered and so, the cancellation of superfluous critical simplices to obtain an optimal function can be interpreted as a denoising procedure. Notice that the set of superfluous simplices of a graph gives rise to a set of pairs $\mathcal{P}$ given by $(v, e) \in \mathcal{P} \Leftrightarrow$ both simplices are cancellable, there is a unique gradient path between them and v is the "youngest" vertex in the sense that it enters the filtration $\{G(a_i)\}$ at the latest stage (see [5]).

Notice that the identity (H1) in Lemma 3.1 holds exactly when a new 1-cycle of G appears at this stage in the process, and therefore it holds for exactly b_1 values of i. Thus, the homological sequences B_0 and B_1 obtained for a connected graph are as follows:

$$\begin{array}{lccccccccccc} B_0: & n_0 = 1, & \ldots, & n_{t_1}, & n_{t_1}, & n_{t_1+1}, & \ldots, & n_{t_{b_1}}, & n_{t_{b_1}}, & n_{t_{b_1}+1}, & \ldots, & n_{2k} = 1 \\ B_1: & 0, & \ldots, & 0, & 1, & 1, & \ldots, & b_1 - 1, & b_1, & b_1, & \ldots, & b_1 \end{array} \tag{1}$$

If $B_0(i) = B_0(i+1)$, then we remove $B_0(i+1)$ for these values of i in the sequence B_0. Hence, we obtain a walk

$$n_0 = 1, n_1, \ldots, n_{2k-1}, n_{2k} = 1$$

in $\mathbb{Z}_{>0}$ starting and ending at 1, with even length $2k$ and steps of size ± 1. The number of elements of the set D_k of such walks is the kth Catalan number $C_k = \frac{1}{k+1}\binom{2k}{k}$ (see [7]).

Now we can consider two kinds of walks in D_k:

(W1) : walks satisfying that $n_t \neq 1$, for every $t = 1, \ldots, 2k-1$;
(W2) : walks satisfying that there exist $t \in \{1, \ldots, 2k-1\}$ such that $n_t = 1$.

Notice that if we take a walk of type 2, since each step has size ± 1, $n_t = 1$ implies that t is even.

Lemma 3.2. *There are $C_j C_{k-j-1}$ different walks in D_k such that $n_{2j} = 1$ and $n_t \neq 1$ for all $2j < t < 2k$.*

Proof. For $j = 0$, the set of such walks satisfying the required property is the set of walks in D_k of type W1. Moreover, there is a bijection between these walks and the set D_{k-1}. Such a bijection can be obtained as follows: given a walk

$$n_0 = 1, n_1 = 2, \ldots, n_{2k-1} = 2, n_{2k} = 1$$

such that $n_t \neq 1$ for all $t = 1, \ldots, 2k-1$, we consider the following walk in D_{k-1}:

$$m_0 = n_1 - 1 = 1, m_1 = n_2 - 1, \ldots, m_{2k-2} = n_{2k-1} - 1 = 1.$$

Therefore, there are $C_{k-1} = C_0 C_{k-1}$ different walks in D_k of type 1.

Now, for $1 \leq j < k-1$, we divide the given walk $n_0, \ldots, n_{2k}$ in D_k into two walks:

- $n_0 = 1, n_1, \ldots, n_{2j-1}, n_{2j} = 1$, which is a walk in D_j, and
- $m_0 = n_{2j} = 1, m_1 = n_{2j+1}, \ldots, m_{2(k-j)-1} = n_{2k-1}, m_{2(k-j)} = n_{2k} = 1$, which is a walk in D_{k-j} of type W1.

Thus, we get a bijection between those walks in D_k satisfying the indicated property for certain $j > 0$ and those walks in D_k obtained by joining a walk in D_j with a walk in D_{k-j} of type 1. Therefore, the number of such walks is $C_j C_{k-j-1}$. □

As a direct consequence of Lemma 3.2, we determine the number of walks of types W1 and W2 respectively in the following result.

Proposition 3.3. *There are $C_0 C_{k-0-1} = C_{k-1}$ walks of type* (W1) *and $\sum_{j=1}^{k-1} C_j C_{k-j-1}$ walks of type* (W2) *respectively.*

Remark 3.4. Notice that it is easily follows from the above result that

$$C_{k-1} + \sum_{j=1}^{k-1} C_j C_{k-j-1} = \sum_{j=0}^{k-1} C_j C_{k-j-1} = C_k.$$

4. The set of excellent discrete Morse functions on a graph

In this section we will prove the main result of the paper, namely we count the number of non-equivalent excellent discrete Morse functions with a given number of critical simplices. The main tools used are the properties of the homological sequences of a function and putting these sequences in terms of certain kinds of reticular walks given in Section 3 and also the two following two lemmas which provide information about the structure of a connected graph in terms of its bridge components.

Lemma 4.1. *If G is a connected graph with at least one bridge and $b_1 < +\infty$, then $G = P_1 \cup P_2 \cup \cdots \cup P_p \cup F$, where $P_1, \ldots, P_p$ are the non-trivial bridge components of G, F is a forest and every tree in F intersects each P_i in at most one vertex. Moreover, if G is infinite, then F has at least an infinite tree.*

Proof. Let B be the set of all bridges of G. It is easy to prove that $b_1(G) < \infty$ implies that the number of connected components of $G - B$ is finite. Let $P_1, \ldots, P_p$ be the non-trivial bridge components of G and set

$$F = G - \left(P_1 \cup P_2 \cup \cdots \cup P_p\right).$$

Let us suppose that F is not a forest, that is, F has at least one cycle. If we delete an edge e in such a cycle, the number of connected components of F does not increase. Then e cannot be a bridge of G; however this is not possible since every edge in F is a bridge. Therefore F is a forest.

Now let us suppose that F has a tree T such that there exists P_t, $1 \leq t \leq p$, with $T \cap P_t$ containing at least two vertices u and v. Let e be an edge in the unique path in T joining u and v. Again the deletion of e does not increase the number of connected components of G; however, since $e \in T$, it is a bridge of G and this is a contradiction. Therefore every tree in F intersects each P_i in at most one vertex. □

Lemma 4.2. *Under the notation of the above lemma, if the degree of any vertex of G is greater than 1, then every edge in F is in at least one path joining two non-trivial bridge components of G. Moreover, given two non-trivial bridge components of G, there exists a unique path in G connecting them.*

Proof. Let $e = u_0 v_0$ be an edge in F. Then e is a bridge of G. If u_0 and v_0 are not in any P_i, for $i = 1, \ldots, p$, then we take edges $e_1 = u_0 u_1$ and $\mathrm{e}^1 = v_0 v_1$ in F obtaining the path $u_1, e_1, u_0, v_0, \mathrm{e}^1, v_1$ in F. If u_1 and v_1 are not in any P_i, we extend this path in the same way. If u_1 or v_1 is in some P_i, then we stop at this vertex. After several steps we obtain a path $u_r, e_r, \ldots, u_1, e_1, u_0, v_0, \mathrm{e}^1, v_1, \ldots, \mathrm{e}^s, v_s$ in F such that $u_r \in P_{i_r}$ and $v_s \in P_{i_s}$ for some $1 \leq i_r, i_s \leq p$. Notice that this process must finish at vertices in some non-trivial bridge component. This can be proved, taking into account that G does not contain leaves and hence every leaf of F must be in one non-trivial bridge component too. Since such a path is in F and, by the above lemma, every tree in F intersects each P_i in at most one vertex, P_{i_r} and P_{i_s} must be different non-trivial bridge components. We conclude that this path joins two non-trivial bridge components and contains the edge e.

Since G is connected, given two non-trivial bridge components P_i and P_j of G, there must exist paths joining each vertex of P_i with each vertex of P_j. Let us consider two such paths $\widetilde{P}_1$ and $\widetilde{P}_2$. Let $\widetilde{P}^1$ and $\widetilde{P}^2$ be the paths obtained by removing from $\widetilde{P}_1$ and $\widetilde{P}_2$ all the edges of P_i and P_j. Then $\widetilde{P}^1$ and $\widetilde{P}^2$ intersect P_i and P_j only at vertices v_i^1, v_i^2 and v_j^1, v_j^2, respectively, and both contain at least one edge in F. Now we take paths $\widetilde{P}^i$ and $\widetilde{P}^j$ joining v_i^1 and v_i^2 in P_i and v_j^1 and v_j^2 in P_j, respectively. Thus, if $\widetilde{P}^1$ and $\widetilde{P}^2$ are not the same path, then, by joining the paths $\widetilde{P}^1$, $\widetilde{P}^2$, $\widetilde{P}^i$ and $\widetilde{P}^j$, we obtain a cycle in G containing edges of F, which is a contradiction since P_i and P_j are different bridge components. Therefore there exists an unique path in G connecting any two non-trivial bridge components of G. □

The following theorem is the main result of this paper which essentially establishes how many classes of excellent discrete Morse functions with a given level of noise can be defined on a graph. This number not only depends on the homology of the graph, but also it is linked to certain aspects of its structure not encoded by the homology.

Theorem 4.3. *Given an infinite locally finite connected graph G with $b_1 < +\infty$, there exists a graph G' combinatorially equivalent to G such that the number of homology equivalence classes of excellent discrete Morse functions with $m = b_0 + b_1 + 2k$ critical simplices on G' is:*

1. $C_k \binom{m-1}{2k}$ *if G is infinite or has at least one vertex with degree 1.*
2. $C_k \binom{m-2}{2k}$ *if G is a non-trivial bridgeless graph.*
3. $\sum_{j=0}^{k-1} C_j C_{k-j-1} \left(\binom{m-1}{2k} - \binom{2j+b_{12}+1}{2j} \binom{2(k-j)+b_{11}-2}{2(k-j)-1} \right)$ *if G is finite and has at least one bridge, and the degree of any vertex of G is greater than 1, where $b_{11} = \min\{b_1(P_i) : F \cap P_i$ is a unique vertex $\}$ and $b_{12} = b_1 - b_{11}$, P_i and F being the subgraphs defined in Lemma 4.1.*

Remark 4.4. Throughout the proof of this theorem we will use the same notation for the initial graph G and any of its subdivisions. It will be clear from the context when a subdivision of G is considered.

Proof. 1. In this case G has at least one bridge and we can use Lemma 4.1.

If there are no non-trivial bridge components ($p = 0$), then G is a tree and by means of Theorem 6.1.1 of [1], the number of homology equivalence classes of excellent discrete Morse functions with m critical simplices is $C_k = C_k \binom{2k}{2k} = C_k \binom{m-1}{2k}$.

Now, let us consider $G = P_1 \cup P_2 \cup \cdots \cup P_p \cup F$, with $p \geq 1$, where $P_1, \ldots, P_p$ are the non-trivial bridge components of G. Moreover, if G is infinite, then F is a forest containing an infinite tree T or, if G is finite, then F is a forest containing a tree T with a leaf u which is a leaf in G.

We take into account that the homological sequences satisfy (1). If we remove the repeated copies of n_{t_i} for $i = 1, \ldots, b_1$ in the sequence B_0, we obtain any walk in D_k and $\sharp(D_k) = C_k$ (see Section 3). Now, we will consider all the possible positions of the increments in the sequence B_1 and we obtain that there are $\binom{m-1}{2k}$ different sequences B_1. Therefore the number of homology equivalence classes of excellent discrete Morse functions on G is less than or equal to $C_k \binom{m-1}{2k}$.

In order to prove the equality, given sequences B_0 and B_1 satisfying (1), we are going to construct an excellent discrete Morse function f on G whose homological sequences are the given ones. We start by choosing m simplices which will be the critical simplices of the Morse function. In particular, we select b_1 edges (which will play the role of essential edges) in $P_1 \cup P_2 \cup \cdots \cup P_p$ (one edge e_i for each basic cycle) and the remaining $2k + 1$ selected simplices in the forest F, in fact in the tree T. In order to obtain the edges $e_1, \ldots, e_{b_1}$ we consider a spanning tree $\widetilde{T}$ of G and take the b_1 edges of G not in $\widetilde{T}$. If T is infinite we can consider a ray $v_0, \widetilde{e}_0, v_1, \widetilde{e}_1, \ldots, v_r, \widetilde{e}_r, v_{r+1} \ldots$ in T and take the $2k + 1$ remaining simplices in the path $\widehat{v_0 v_k}$. Otherwise, if T is finite, we first subdivide the unique edge $v_0 u$ in T, where u is a vertex with degree 1, to obtain a path $v_0, \widetilde{e}_0, v_1, \widetilde{e}_1, \ldots, v_{k-1}, \widetilde{e}_{k-1}, v_k = u$ with $2k + 1$ simplices.

As in Theorem 6.1.1 in [1], we construct an excellent discrete Morse function g on the tree $\widetilde{T} = G - \{e_1, e_2, \ldots, e_{b_1}\}$ whose sequence B_0 is

$$n_0, n_1, \ldots, n_{t_1}, n_{t_1+1}, \ldots, n_{2k}$$

and we can suppose that g reaches its global minimum at v_0. Thus, g has critical values $c_0 < \cdots < c_{2k}$ where $c_j = g(q_j)$ for $i = 0, \ldots, 2k$ and $q_0 = v_0, \ldots, q_{2k}$ are its critical simplices (in fact, these simplices are $v_0, v_1, \ldots, v_{k-1}, v_k, \widetilde{e}_0, \widetilde{e}_1, \ldots, \widetilde{e}_{k-1}$ in T, ordered to obtain the given sequence B_0).

Next, starting from g, we can construct a new excellent function f on G having the given homological sequences. The critical simplices of f are $q_0, \ldots, q_{2k}$ and the edges $e_1, \ldots, e_{b_1}$, where every edge e_i is between q_{t_i} and $q_{t_{i+1}}$, that is, $c_i = f(q_{t_i}) < f(e_i) < f(q_{t_{i+1}}) = c_{i+1}$. Thus, we obtain an excellent discrete Morse function f on G with the given homological sequences.

Consequently, the number of homology equivalence classes of excellent discrete Morse function for graphs of this type is $C_k \binom{m-1}{2k}$.

2. If G is a non-trivial bridgeless graph, then the homological sequences satisfy

$$B_0(m-2) = B_0(m-1) = 1, \qquad B_1(m-1) = b_1$$

and

$$B_1(m-1) - B_1(m-2) = 1.$$

That is, every excellent discrete Morse function on G reaches its global maximum on a critical 1-simplex e, which completes one of the basic cycles of G.

If we take into account that two excellent discrete Morse functions f and g on G are homologically equivalent if and only if their restrictions to $G-\{e\}$ are homologically equivalent, we obtain that the number of homology equivalence classes of excellent discrete Morse functions on G with m critical simplices is equal to the number of equivalence classes on $G-\{e\}$ with $m-1$ critical simplices. By subdividing if necessary, we may assume that $G-\{e\}$ has at least one vertex with degree 1, and we conclude that the number of elements of this set is less than or equal to $C_k\binom{(m-1)-1}{2k} = C_k\binom{m-2}{2k}$.
To obtain the equality, we define an excellent discrete Morse function on $G-\{e\}$ as we did in case 1 and extend it to a function on G by assigning to e its global maximum.

3. In this case, since G has at least one bridge, by applying Lemma 4.1 we have $G = P_1 \cup P_2 \cup \cdots \cup P_p \cup F$, with $p \geq 1$, where $P_1, \ldots, P_p$ are the non-trivial bridge components of G and F is a finite forest. Moreover, since every leaf in F has degree greater than 1 in G, then every leaf in F is in exactly one non-trivial bridge component.
It is interesting to note that the proof of this case is rather more complicated than those of the above cases. As in the first case, we get that the number of homology equivalence classes is less than or equal to $C_k\binom{m-1}{2k}$, but now, given any sequences B_0 and B_1 satisfying (1), it is not always possible to construct an excellent discrete Morse function on G with the given homological sequences. Let us see which additional properties must satisfy B_0 and B_1 to be admissible sequences, that is, to be the homological sequences of an excellent discrete Morse function on G. Obviously, throughout this proof we will assume that all the homology sequences considered satisfy (1). Let us divide the proof into several steps:
Step 1
At this point we will consider sequences B_0 and B_1 satisfying

$$B_1(m-1) - B_1(m-2) = 1. \tag{2}$$

Then, we can construct an excellent discrete Morse function f on G with the given homological sequences. The property (2) implies that such a function f must reach its global maximum on an essential critical edge. We begin by selecting an edge e in a basic cycle of G. By subdividing if necessary, we may assume that $G-\{e\}$ has at least one vertex with degree 1 and we can construct, as we did in the first case, an excellent Morse function g on $G-\{e\}$ with homological sequences $B_0(i), B_1(i)$ with $i = 0, \ldots, m-2$. Next, let us set $f = g$ on $G-\{e\}$ and $f(e) = C+1$ where $C = \max\{g(\sigma)/\sigma \in G-\{e\}\}$. Thus f is an excellent discrete Morse function whose homological sequences are the given ones.
Now, let us consider those homological sequences not satisfying (2). In this case, we may find non-admissible sequences B_0, B_1 when for a certain t, we have that $B_1(t) = h > 0$ but the connected components corresponding to the non-trivial bridge components containing these h basic cycles have not been created yet in the level subcomplex G_t.
Step 2
Now, we are going to determine which homological sequences not satisfying (2) are not valid. Let us consider sequences B_0 and B_1 not satisfying (2) and let $n_0, \ldots, n_{2k}$ be the walk in D_k obtained by removing the repeated copies of n_{t_i}, for $i = 1, \ldots, b_1$, in B_0. Then, there exists $j \in \{0, 1, \ldots, k-1\}$ such that $n_{2j} = 1$ and $n_t \neq 1$, for every $t = 2j+1, \ldots, 2k-1$. If these sequences B_0 and B_1 satisfy the following additional property:

$$\text{there exists } h \text{ such that } B_0(2j+h) = 1 \quad \text{and} \quad B_1(2j+h) = h \geq b_{12}+1 \tag{3}$$

then such sequences are not admissible. Indeed, if there were to be an excellent discrete Morse function f on G whose homological sequences are the given ones, then property (3) would imply that the level subcomplex G_{2j+h} is a connected subgraph of G with $h > b_{12}$ cycles. Since $b_{11} + b_{12} = b_1$ and by Lemma 4.2 two bridgeless components are connected by a unique path, then every non-trivial bridge component $P_1, \ldots, P_p$ of G either is needed to connect two other non-trivial bridge components or has at least b_{11} cycles. Thus, G_{2j+h} must contain at least one cycle in every component P_i. By applying Lemma 4.2 again, we conclude that F must be included in G_{2j+h} since the paths joining $P_1, \ldots, P_p$ in G must be in G_{2j+h} too. Then the critical simplices not considered yet, that is, those with critical values greater than a_{2j+h}, must be in the $b_1 - h$ cycles which have not been completed in the subcomplex G_{2j+h}. On the other hand, since the homological sequences of f do not satisfy (2), f reaches its global maximum on a superfluous critical edge. Notice that all of the b_1 cycles of G have already been completed in the subcomplex G_{m-2}. However this is impossible since the cycle determined by the edge e cannot arise until G_{m-1} appears.
To count the sequences that we have just considered, we will divide them into two parts:

$$\text{(Left part)} \quad \begin{matrix} n_0 = 1, & \ldots, & n_{t_1}, & n_{t_1}, & \ldots, & n_{t_{b_{12}+1}}, & n_{t_{b_{12}+1}}, & \ldots, & n_{2j} = 1 \\ 0, & \ldots, & 0, & 1, & \ldots, & b_{12}, & b_{12}+1, & \ldots, & b_{12}+1 \end{matrix}$$

$$\text{(Right part)} \quad \begin{matrix} n_{2j} = 1, & \ldots, & n_{t_{b_{12}+2}}, & \ldots, & n_{t_{b_1}}, & \ldots, & n_{2k-1} = 2, & n_{2k} = 1 \\ b_{12}+1, & \ldots, & b_{12}+2, & \ldots, & b_1, & \ldots, & b_1, & b_1. \end{matrix}$$

We can observe that, since $B_1(0) = 0$, the left part contains $2j + b_{12} + 1$ possible positions of the $b_{12}+1$ increments of the sequence B_1. Taking into account that (2) is not satisfied, it follows that $B_0(m-2) \neq B_0(m-1)$ as we can see at the end of the right part. Thus there are $2(k-j) - 1 + b_{11} - 1$ possible positions of the $b_{11}-1$ increments of the sequence B_1. Moreover, by Lemma 3.2, there are C_jC_{k-j-1} walks $n_0, \ldots, n_{2k}$ for each $j \in \{0, \ldots, k-1\}$. Therefore we obtain that the number of non-admissible sequences is

$$C_jC_{k-j-1}\binom{2j+b_{12}+1}{b_{12}+1}\binom{2(k-j)-1+b_{11}-1}{b_{11}-1} = C_jC_{k-j-1}\binom{2j+b_{12}+1}{2j}\binom{2(k-j)+b_{11}-2}{2(k-j)-1}.$$

Step 3

Now let us see that, if B_0 and B_1 satisfy neither (2) nor (3), then they are admissible sequences.

Again, let $n_0, \ldots, n_{2k}$ be the walk in D_k obtained by removing from B_0 the repeated copies of n_{t_i}, with $i = 1, \ldots, b_1$. As we stated before, there exists $j \in \{0, 1, \ldots, k-1\}$ such that $n_{2j} = 1$ and $n_t \neq 1$, for every $t = 2j+1, \ldots, 2k-1$. Since (2) and (3) do not hold, it follows that $B_1(2j+h) = h \leq b_{12}$ for every $0 < h < b_1$ such that $B_0(2j+h) = 1$. Let $\widetilde{h}$ be the maximum of such h. Then we can divide the sequences B_0 and B_1 as follows:

$$\text{(Left part)} \quad \begin{matrix} n_0 = 1, & \ldots, & n_{t_1}, & n_{t_1}, & \ldots, & n_{t_{\widetilde{h}}}, & n_{t_{\widetilde{h}}}, & \ldots, & n_{2j} = 1 \\ 0, & \ldots, & 0, & 1, & \ldots, & \widetilde{h}-1, & \widetilde{h}, & \ldots, & \widetilde{h} \end{matrix}$$

$$\text{(Right part)} \quad \begin{matrix} n_{2j} = 1, & \ldots, & n_{t_{\widetilde{h}+1}}, & \ldots, & n_{t_{b_1}}, & \ldots, & n_{2k-1} = 2, & n_{2k} = 1 \\ \widetilde{h}, & \ldots, & \widetilde{h}+1, & \ldots, & b_1, & \ldots, & b_1, & b_1. \end{matrix}$$

In order to construct the function f, we first choose those simplices of G which will play the role of critical simplices. Let us suppose that P_1 is a non-trivial bridge component of G such that $P_1 \cap F$ is a unique vertex v and $b_{11} = b_1(P_1)$. In the unique tree of F that intersects P_1, let us take a path with length $2k+1$, beginning at v and satisfying that every vertex in such a path has degree at most 2. By subdividing the unique edge vu in F if necessary, we get a path γ:

$$\gamma = p_k, \widetilde{e}_k, p_{k-1}, \ldots, p_{j+1}, \widetilde{e}_{j+1}, p_j, \ldots, p_1, \widetilde{e}_1, p_0.$$

Notice that p_0 will be the essential critical vertex of the function that we will construct, and the remaining vertices of γ will be the superfluous critical simplices. To obtain the edges which will be essential, we consider a spanning tree $\widetilde{T}$ in G and we choose the edges $e_1, \ldots, e_{b_1}$ of G not in $\widetilde{T}$.

By removing the edge $\widetilde{e}_{j+1}$ of $\widetilde{T}$, we get a forest with two trees $\widetilde{T}_0$ and $\widetilde{T}_v$ containing the vertices p_0 and $v = p_k$ respectively. Then, we can construct an excellent discrete Morse function g_0 on the graph $\widetilde{G}_0 = \widetilde{T}_0 \cup \{e_1, \ldots, e_{\widetilde{h}}\}$ whose homological sequences are $B_0(i)$ and $B_1(i)$, $i = 0, \ldots, 2j+\widetilde{h}$, that is, they coincide with the left part of the given sequences B_0 and B_1. Such a function can be constructed as we did for graphs in the first part of this theorem. We may assume that p_0 is the essential critical vertex of g_0 with $g_0(p_0) = 0$. Let $q_0 = p_0, q_1 \ldots, q_{2j+\widetilde{h}}$ be the critical simplices of g_0 with critical values $c_i = g_0(q_i)$ for $i = 0, \ldots, 2j+\widetilde{h}$. Notice that these simplices are $p_0, \ldots, p_j, \widetilde{e}_1, \ldots, \widetilde{e}_j, e_1, \ldots, e_{\widetilde{h}}$ in a convenient order.

Also, we can construct an excellent discrete Morse function g_v on $\widetilde{T}_v$ whose homological sequence is

$$n_{2j+1} - 1, \ldots, n_{2k-2} - 1, n_{2k-1} - 1.$$

This sequence is a walk in D_{k-j-1} since $n_t \geq 2$ for every $t = 2j+1, \ldots, 2k-1$. In this way, we get that p_k is the essential critical vertex of g_v and we may assume that $g_v(p_k) = 0$. Let $w_0 = p_k, w_1, \ldots, w_{2(k-j-1)}$ be the critical simplices of g_v with critical values $a_i = g_v(w_i)$ for $i = 0, \ldots, 2(k-j-1)$.

Next, taking into account g_0 and g_v, we will construct a new excellent function f on G with the given homological sequences.

We define $f = g_0$ on $\widetilde{G}_0$; thus the first $2j+\widetilde{h}+1$ critical simplices of f are the critical simplices of g_0.

We continue constructing f by means of g_v in several steps. From the right part of the given sequences, it follows that we must get

$$B_1(2j+\widetilde{h}+1) = \cdots = B_1(t_{\widetilde{h}+1}+\widetilde{h}) = \widetilde{h},$$

that is, we must define f in such way that no cycle has been added in the process of construction of the level subcomplex $G_{t_{\widetilde{h}+1}+\widetilde{h}}$ from $G_{2j+\widetilde{h}}$, and hence only the number of connected components has been modified.

It holds that

$$B_0(2j+\widetilde{h}+1) = n_{2j+1}, \ldots, B_0(t_{\widetilde{h}+1}+\widetilde{h}) = n_{t_{\widetilde{h}+1}}.$$

So, if $A_1 = c_{2j+\widetilde{h}} + 1$, then we set $f = g_v + A_1$ on the graph $\widetilde{G}_1$, which is the level subcomplex of $\widetilde{T}_v$ corresponding to $a_{t_{\widetilde{h}+1}-2j-1}$. Now, the critical simplices of f are the critical simplices of g_0 together with $w_0, \ldots, w_{t_{\widetilde{h}+1}-2j-1}$ and its homological sequences are $B_0(i)$ and $B_1(i)$ with $i = 0, \ldots, t_{\widetilde{h}+1}+\widetilde{h}$. The new critical values are different, since we have defined f by adding a constant to g_v which is excellent. Moreover, $f(w_0) = f(p_k) = g_v(p_k) + c_{2j+\widetilde{h}} + 1 > c_{2j+\widetilde{h}}$ which is the maximum critical value of f on $\widetilde{G}_0$.

At this point, in order to obtain $B_1(t_{\widetilde{h}+1}+\widetilde{h}+1) = \widetilde{h}+1$, we need to complete a new cycle by taking as the next critical simplex of f one edge e_i with $i > \widetilde{h}$, so we take $e_{\widetilde{h}+1} = u_{\widetilde{h}+1}v_{\widetilde{h}+1}$ and we define $f(e_{\widetilde{h}+1}) = \max\{f(u_{\widetilde{h}+1}), f(v_{\widetilde{h}+1}), C_1\} + 1$ where C_1 is the maximum critical value of f on $\widetilde{G}_1$. On the graph $\widetilde{G}_2$ whose simplices are $w_{t_{\widetilde{h}+1}-2j}, \ldots, w_{t_{\widetilde{h}+2}-2j-1}$ we put $f = g_v + A_2$ where $A_2 = A_1 + f(e_{\widetilde{h}+1}) - C_1$. Now, the function f is defined on $\widetilde{G}_0 \cup \widetilde{G}_1 \cup \widetilde{G}_2 \cup \{e_{\widetilde{h}+1}\}$, the new critical simplices of f are $e_{\widetilde{h}+1}$ and w_{t-2j-1} for $t_{\widetilde{h}+1}+1 \leq t \leq t_{\widetilde{h}+2}$ and its homological sequences are $B_0(i)$ and $B_1(i)$ with $i = 0, \ldots, t_{\widetilde{h}+2}+\widetilde{h}+1$.

Now, let us check that the critical values of f are different. In fact, we have $f(w_{t_{\widetilde{h}+1}-2j-1}) < f(e_{\widetilde{h}+1}) < f(w_{t_{\widetilde{h}+1}-2j})$ since

$$f(e_{\widetilde{h}+1}) > C_1 = f(w_{t_{\widetilde{h}+1}-2j-1})$$

and

$$\begin{aligned} f(w_{t_{\widetilde{h}+1}-2j}) - f(e_{\widetilde{h}+1}) &= \Big(g_v(w_{t_{\widetilde{h}+1}-2j}) + A_1 + f(e_{\widetilde{h}+1}) - C_1\Big) - f(e_{\widetilde{h}+1}) \\ &= g_v(w_{t_{\widetilde{h}+1}-2j}) - g_v(w_{t_{\widetilde{h}+1}-2j-1}) > 0. \end{aligned}$$

We continue defining f in a similar way, until we get an excellent discrete Morse function on $G-\{\widetilde{e}_{j+1}\}$ whose homological sequences are B_0 and B_1 with $i = 0, \ldots, m-2$.
Finally, in order to obtain $B_0(m-1) = 1$, we define

$$f(e_{j+1}) = \max\{f(p_j), f(p_{j+1}), C\} + 1$$

where C is the maximum critical value of f on $G - \{\widetilde{e}_{j+1}\}$. Thus, we obtain an excellent discrete Morse function f on G whose homological sequences are the given ones.

We conclude this proof by counting the number of admissible sequences. As we obtained before, there are

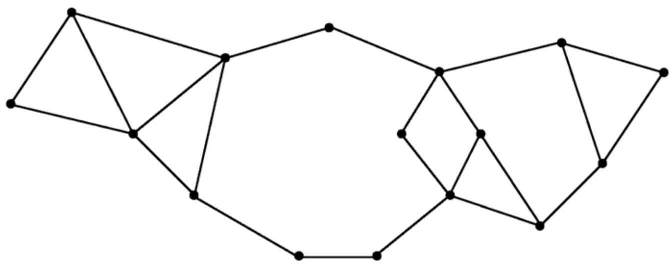
$$\sum_{j}^{k-1} C_j C_{k-j-1} \binom{2j + b_{12} + 1}{2j} \binom{2(k-j) + b_{11} - 2}{2(k-j) - 1}$$

non-admissible sequences. Since there are $C_k \binom{m-1}{2k}$ possible sequences, then the number of homology classes of excellent discrete Morse functions for graphs of this type is

$$\begin{aligned} & C_k \binom{m-1}{2k} - \sum_{j=0}^{k-1} C_j C_{k-j-1} \binom{2j + b_{12} + 1}{2j} \binom{2(k-j) + b_{11} - 2}{2(k-j) - 1} \\ &= \sum_{j=0}^{k-1} C_j C_{k-j-1} \left(\binom{m-1}{2k} - \binom{2j + b_{12} + 1}{2j} \binom{2(k-j) + b_{11} - 2}{2(k-j) - 1} \right). \quad \square \end{aligned}$$

The following examples clarify the constructions carried out in the proof of the above theorem. Notice that since examples corresponding to case 1 are provided in [1], then we mainly focus our attention on the cases 2 and 3:

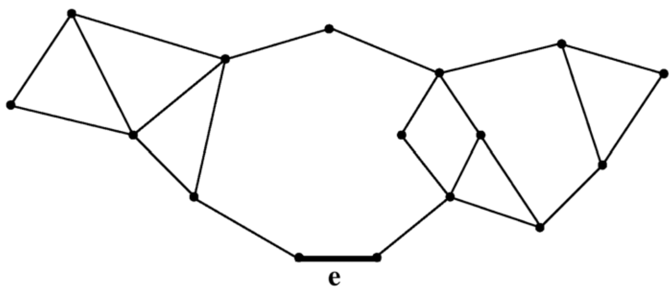
Example 4.5. Let G be the graph given by the figure below:



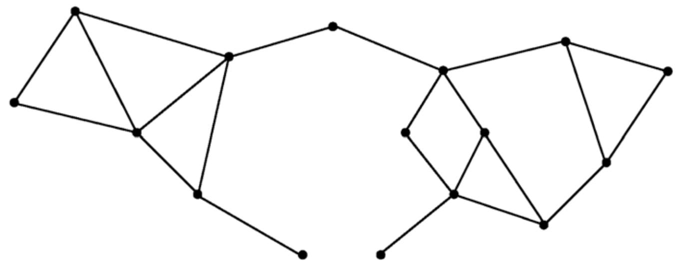
Now, let us consider an excellent discrete Morse function on G with 13 critical simplices and whose homological sequences are

$$\begin{array}{lccccccccccccc} B_0: & 1, & 1, & 1, & 2, & 2, & 3, & 2, & 2, & 2, & 2, & 2, & 1, & 1 \\ B_1: & 0, & 1, & 2, & 2, & 3, & 3, & 3, & 4, & 5, & 6, & 7, & 7, & 8. \end{array}$$

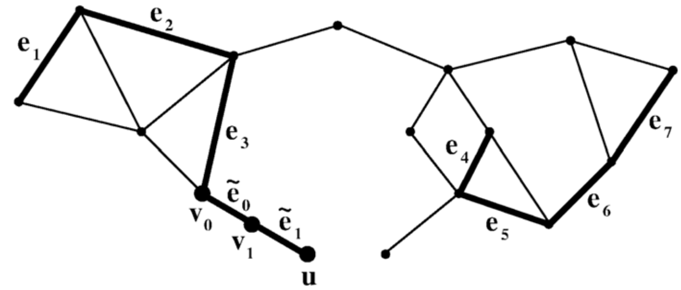
Notice that G is a bridgeless graph and the given pair of sequences is admissible since $B_1(m-1) - B_1(m-2) = 1$. That is, the function f will reach its maximum on an edge e (which can be any edge of G).



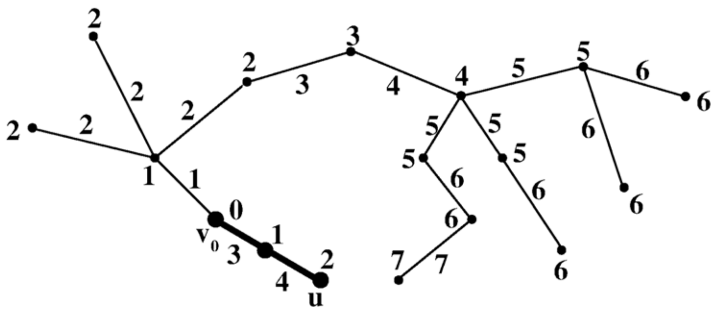
Now, on the graph $G - \{e\}$, which has at least one vertex with degree 1, we are going to define an excellent discrete Morse function g.



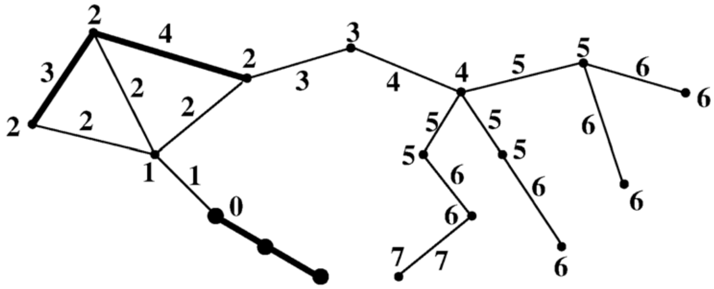
First, we select the critical simplices of g: we start by subdividing the edge v_0u obtaining the path with five simplices $v_0, \widetilde{e}_0, v_1, \widetilde{e}_1, v_2 = u$. Next, we choose the edges $e_1, \ldots, e_7$ corresponding to each cycle in $G - \{e\}$ which are obtained by considering a spanning tree $\widetilde{T}$ in G and taking those edges of G not in $\widetilde{T}$.



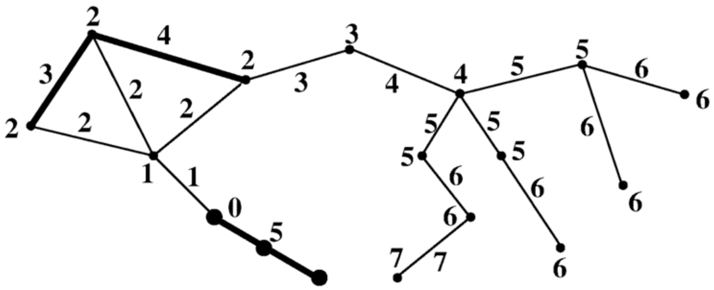
In the first step, we define an excellent discrete Morse function h on $\widetilde{T}$, whose essential vertex is v_0 and whose homological sequence B_0 is 1, 2, 3, 2, 1 (obtained by removing the repeated values in B_0):



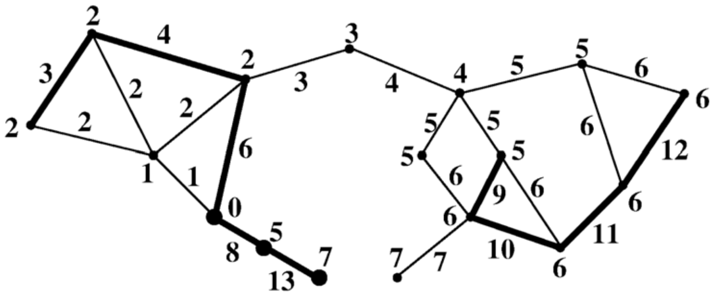
Now, we are going to extend h in several steps by assigning values to the edges $e_1, \ldots, e_7$ which will play the role of essential critical edges. Taking into account the sequences B_0 and B_1, we see that the first critical simplex is v_0 and the next two critical simplices must be essential critical edges. Thus, we take $g = h$ in $\widetilde{T} - \{v_1, v_2, \widetilde{e}_0, \widetilde{e}_1\}$ and we define g on e_1 and e_2 in such a way that they became the next critical simplices.



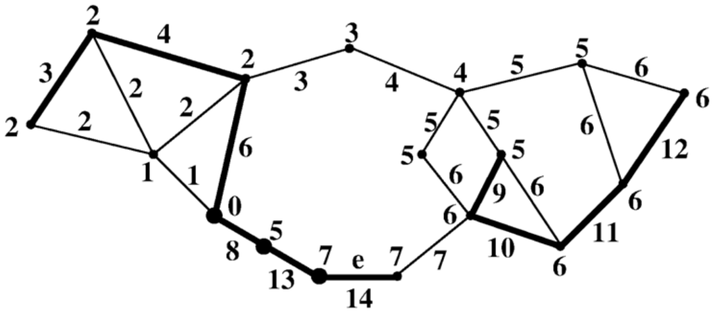
We continue defining g on v_1 to be the next critical simplex by adding a suitable value to h on v_1:



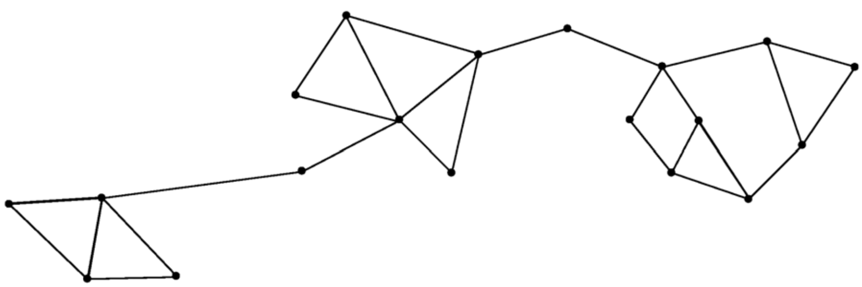
Repeating the same arguments, we obtain an excellent discrete Morse function g on $G - \{e\}$ whose homological sequences are $B_0(i)$ and $B_1(i)$ with $i = 0, \ldots, m - 2 = 12$:



Finally, we put $f = g$ on $G - \{e\}$ and $f(e) = C + 1$ where $C = \max\{g(\sigma)/\sigma \in G - \{e\}\}$. Thus f is an excellent discrete Morse function whose homological sequences are the given ones.



Example 4.6. Let us define an excellent discrete Morse function on the graph G in the figure below:



with 14 critical simplices and whose homological sequences are

B_0 :	1,	1,	1,	1,	2,	2,	2,	1,	1,	1,	2,	2,	2,	1
B_1 :	0,	1,	2,	3,	3,	4,	5,	5,	6,	7,	7,	8,	9,	9.

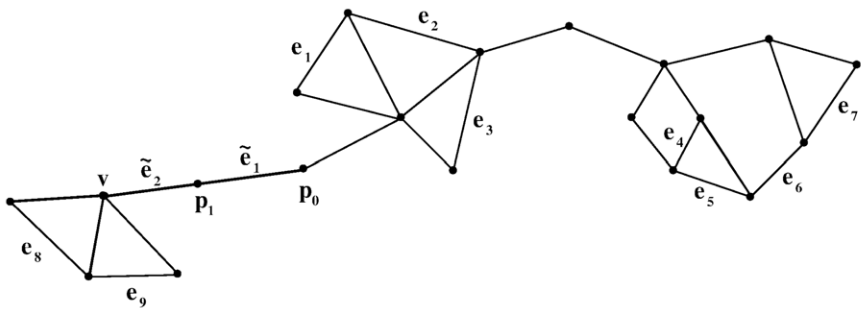
Notice that G is a graph in the third case of Theorem 4.3 with

$$b_{11} = \min\{b_1(P_i) : F \cap P_i \text{ is a unique vertex }\} = 2.$$

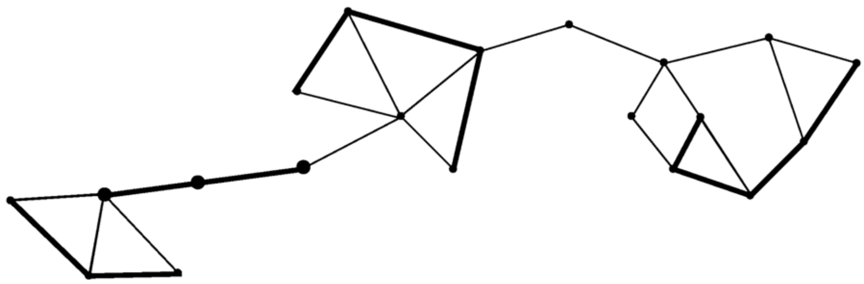
Taking into account the given sequences, we get that:

- if we remove the repeated copies of B_0, we get the walk 1, 2, 1, 2, 1 in D_2 of type W2 with $j = 1$ and
- $\widetilde{h} = 7$ is the maximum value of those h such that $B_0(2j + h) = 1$ and $B_1(2j + h) = h$.

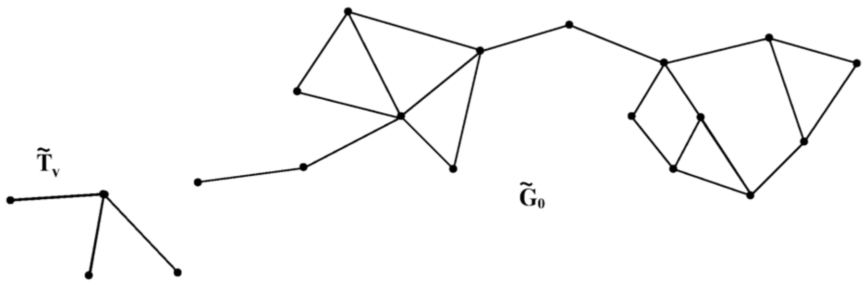
First, we need to choose those simplices which will play the role of critical simplices. Let P_1 be a non-trivial bridgeless component such that $P_1 \cap F$ is a unique vertex v and $b_1(P_1) = b_{11}$. We subdivide the unique edge vu in F and we obtain a path $v = p_2, \widetilde{e}_2, p_1, \widetilde{e}_1, p_0 = u$ such that p_1 has degree 2. Moreover, we consider a spanning tree $\widetilde{T}$ in G to obtain the edges e_i with $i = 1, \ldots, 9$ corresponding of the basic cycles of G.



At this point, we are going to select the critical simplices: we take $p_0 = u$, the edges $e_1, \ldots, e_9$ in $G - \widetilde{T}$ and the remaining simplices of the path $\widehat{uv}$ (shown as the thicker lines in the picture below):

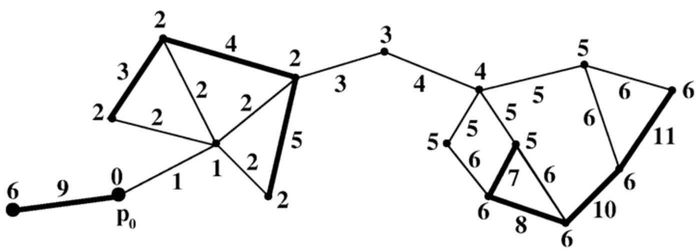


Now, by removing the edges $\widetilde{e}_2$, e_8 and e_9, we obtain the graph $\widetilde{G}_0$ and the tree $\widetilde{T}_v$:

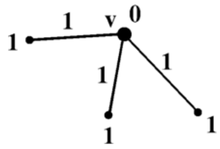


Then, since $\widetilde{G}_0$ is a graph in the first case of Theorem 4.3, we define an excellent discrete Morse function g_0 on the graph $\widetilde{G}_0$ with essential critical vertex p_0 and whose homological sequences are

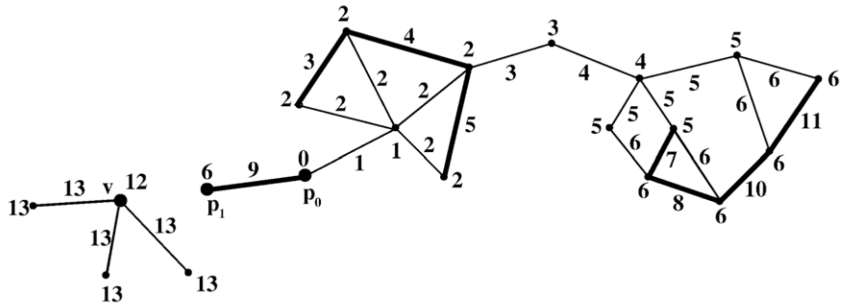
$$\begin{matrix} 1, & 1, & 1, & 1, & 2, & 2, & 2, & 1, & 1, & 1 \\ 0, & 1, & 2, & 3, & 3, & 4, & 5, & 5, & 6, & 7. \end{matrix}$$



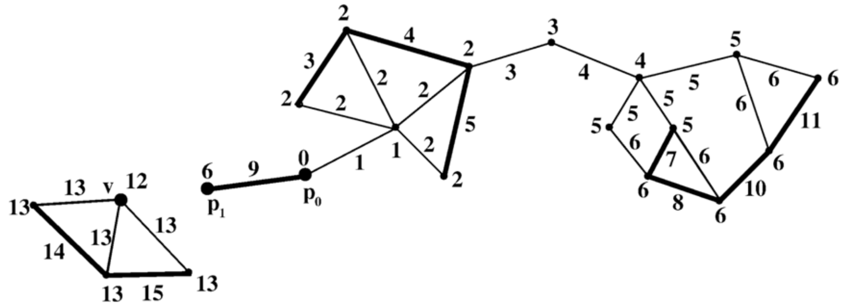
Also, on $\widetilde{T}_v$ we define the following excellent discrete Morse function g_v:



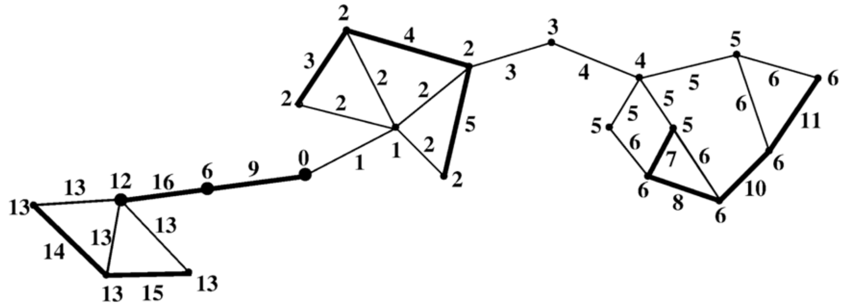
Now, we define f on $\widetilde{G}_0 \sqcup \widetilde{T}_v$ starting from g_0 and g_v just by adding a suitable value to g_v, for instance, $f = g_v + 12$ on $\widetilde{T}_v$ and $f = g_0$ on $\widetilde{G}_0$.



Next, we define f on the edges e_8 and e_9 to be the critical simplices after to $p_2 = v$. We put $f(e_8) = \max\{f(u_8), f(v_8), f(p_2)\} + 1$ and $f(e_9) = \max\{f(u_9), f(v_9), f(e_8)\} + 1$ where $e_8 = u_8v_8$ and $e_9 = u_9v_9$:



Finally, we define f on the edge $\widetilde{e}_2$ as the greatest critical value of f.



As we can see in the last picture, the excellent Morse function f has the given homological sequences.

Remark 4.7. It interesting to point out that the following homological sequences are not admissible for the graph of the above example:

$$B_0: \quad 1, \quad 1, \quad 1, \quad 1, \quad 1, \quad 1, \quad 1, \quad 1, \quad 1, \quad 2, \quad 3, \quad 2, \quad 2, \quad 1$$
$$B_1: \quad 0, \quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad 8, \quad 8, \quad 8, \quad 9, \quad 9.$$

Otherwise, if we suppose that such a function exists, then the level subcomplex G_8 is connected and $b_1(G_8) = 8 > b_{12}$. Thus G_8 contains the forest F and all the basic cycles of G but one. In consequence, all the remaining critical simplices which have not been considered yet will be in the last basic cycle, but we need to complete it before the last level subcomplex arises, which is a contradiction.

Acknowledgements

We would like to thank the referees for their valuable comments that led to great improvements of this paper.

The first and third author was partially supported by the Plan Nacional de Investigación 2.010, Project MTM2010-20445, España, 2010, and FQM 189, Junta de Andalucía.

The second author was partially supported by FQM 326, Junta de Andalucía.

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