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A note on property of the Mittag-Leffler function ☆

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ABSTRACT

Recently the authors have found in some publications that the following property (0.1) of Mittag-Leffler function is taken for granted and used to derive other properties.

$$E_\alpha\big(a(t+s)^\alpha\big) = E_\alpha\big(at^\alpha\big)E_\alpha\big(as^\alpha\big), \quad t, s \geqslant 0, \tag{0.1}$$

where a is a real constant and $\alpha > 0$. In this note it is proved that the above property is unavailable unless $\alpha = 1$ or $a = 0$. Moreover, a new equality on $E_\alpha(at^\alpha)$ is developed, whose limit state as $\alpha \uparrow 1$ is just the property (0.1).

1. Introduction

The Mittag-Leffler function is such a one-parameter function defined in the complex plane $\mathbb{C}$ by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \tag{1.1}$$

where $\alpha > 0$ is the parameter and Γ the Gamma function [2]. It was originally introduced by G.M. Mittag-Leffler in 1902 [1]. Obviously, the exponential function e^z is a particular Mittag-Leffler function with the specified parameter $\alpha = 1$, or in other words, the Mittag-Leffler function is the parameterized exponential function.

In recent years the Mittag-Leffler function has caused extensive interest among scientists, engineers and applied mathematicians, due to its role played in investigations of fractional differential equations (see, for example, [2,6,7,9–11]). A large of its properties have been proved (see, e.g., [2,3,6,8]), among which the following one will perhaps receive considerable interests from the society of dynamical systems: the function $t \mapsto E_\alpha(at^\alpha)$ solves the fractional differential equation of order α

$${}_0^C D_t^\alpha x(t) = ax(t), \quad t \geqslant 0, \tag{1.2}$$

where ${}_0^C D_t^\alpha \cdot$ denotes the Caputo's derivative operator of order α, that is,

$${}_0^C D_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} x^{(n)}(\tau)\, d\tau, \tag{1.3}$$

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where n is the minimum integer not less than α, and $x^{(n)}(t)$ the traditional n-order derivative of $x(t)$. Since the particular Mittag-Leffler function e^{at} possesses the semigroup property (i.e., $e^{a(t+s)} = e^{at}e^{as}$ for all $t, s \geqslant 0$), it seems reasonable to assume that the function $E_\alpha(at^\alpha)$ also enjoys the semigroup property

$$E_\alpha(a(t+s)^\alpha) = E_\alpha(at^\alpha)E_\alpha(as^\alpha), \quad \forall t, s \geqslant 0. \tag{1.4}$$

Recently, we have found in some existing publications that the semigroup property of $E_\alpha(at^\alpha)$ is taken for granted and used to derive other properties of the Mittag-Leffler function (see, e.g., [3, formula (3.10)], [4, formula (5.1)]).

The purpose of this note is to prove that the function $E_\alpha(at^\alpha)$ cannot satisfy the semigroup property unless $\alpha = 1$ or $a = 0$, and to further develop a new equality relationship involving $E_\alpha(at^\alpha)$, $E_\alpha(as^\alpha)$ and $E_\alpha(a(t+s)^\alpha)$. To this end, the following properties of Mittag-Leffler function and Caputo's fractional derivative are needed:

(P1) (cf. [2, formula (2.140)]) The Laplace transform of Caputo's derivative is given by

$$\widehat{{}_0^C D_t^\alpha f(t)}(\lambda) = \lambda^\alpha \hat{f}(\lambda) - \sum_{k=0}^{n-1} \lambda^{\alpha-k-1} f^{(k)}(0), \tag{1.5}$$

where $n-1 < \alpha \leqslant n$, $\widehat{{}_0^C D_t^\alpha f(t)}(\lambda)$ and $\hat{f}(\lambda)$ denote the Laplace transforms of ${}_0^C D_t^\alpha f(t)$ and $f(t)$, respectively.

(P2) (cf. [12, p.287]) The Laplace transform of the function $E_\alpha(at^\alpha)$ is given by

$$\widehat{E_\alpha\big(at^\alpha\big)}(\lambda) = \frac{\lambda^{\alpha-1}}{\lambda^\alpha - a}, \quad \operatorname{Re}\lambda > |a|^{1/\alpha}. \tag{1.6}$$

2. Counterexample and disproof

According to [2, formula (1.65), p. 16], the Mittag-Leffler function $E_\alpha(z)$ for $\alpha = \frac{1}{2}$ is computed by

$$E_{\frac{1}{2}}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\frac{k}{2}+1)} = e^{z^2} \cdot \operatorname{erfc}(-z), \tag{2.1}$$

where $\operatorname{erfc}(z)$ is the complementary error function, which is defined by

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\,dt. \tag{2.2}$$

Let $a = 1$ and $t = s = 1$. Then, we have that

$$E_{\frac{1}{2}}\big(a(t+s)^{\frac{1}{2}}\big) = E_{\frac{1}{2}}(\sqrt{2}) = e^2 \cdot \operatorname{erfc}(-\sqrt{2}), \tag{2.3}$$

$$E_{\frac{1}{2}}\big(at^{\frac{1}{2}}\big)E_{\frac{1}{2}}\big(as^{\frac{1}{2}}\big) = \big(E_{\frac{1}{2}}(1)\big)^2 = e^2 \cdot \big(\operatorname{erfc}(-1)\big)^2. \tag{2.4}$$

Using the software Matlab to compute $\operatorname{erfc}(z)$ with 0.1% precision, we get the result that $\operatorname{erfc}(-1) \approx 1.8427$ and $\operatorname{erfc}(-\sqrt{2}) \approx 1.9545$. Which shows that $E_{\frac{1}{2}}(a(t+s)^{\frac{1}{2}}) \neq E_{\frac{1}{2}}(at^{\frac{1}{2}})E_{\frac{1}{2}}(as^{\frac{1}{2}})$ for $a = 1$ and $t = s = 1$.

The above example shows that the function $E_\alpha(at^\alpha)$ does not possess the semigroup property (1.4) for the specified $\alpha = \frac{1}{2}$ and $a = 1$. In fact, it can be further proved that the function $E_\alpha(at^\alpha)$ cannot possess the semigroup property unless $\alpha = 1$ or $a = 0$. Indeed, if the semigroup property (1.4) is available, then, as a direct result of the well-known fact that the exponential functions are the only non-zero anywhere-continuous functions with the semigroup property (cf. [5, p. 197]), there exists a real constant c such that $E_\alpha(at^\alpha) = e^{ct}$ for all $t \in \mathbb{R}$. By the Laplace transform formula (1.6), it follows that

$$\frac{\lambda^{\alpha-1}}{\lambda^\alpha - a} = \frac{1}{\lambda - c}, \quad \forall \operatorname{Re}\lambda > \max\big\{c, |a|^{\alpha^{-1}}\big\}. \tag{2.5}$$

It is clear to see that the above equality holds only when $\alpha = 1$ or $a = c = 0$.

3. A new equality relationship

By the definition (1.3) it is clear that the Caputo's fractional derivative operator is nonlocal in the case of non-integer order α. The memory character of Caputo's derivative operator is perhaps the cause leading to the result that $E_\alpha(at^\alpha)$, as an eigenfunction of Caputo's derivative operator (see Eq. (1.2)), does not possess semigroup property that is non-memory. This seems to tell us that any equality relationship involving $E_\alpha(at^\alpha)$, $E_\alpha(as^\alpha)$ and $E_\alpha(a(t+s)^\alpha)$ should be of memory and hence be characterized with integrals. The equality relationship stated in the following theorem is a result of the above idea. Without loss of generality, the following discussion is restricted to the case that $0 < \alpha < 1$.

Theorem 1. *For every real a there holds that*

$$\int_0^{t+s} \frac{E_\alpha(a\tau^\alpha)}{(t+s-\tau)^\alpha}\,d\tau - \int_0^{t} \frac{E_\alpha(a\tau^\alpha)}{(t+s-\tau)^\alpha}\,d\tau - \int_0^{s} \frac{E_\alpha(a\tau^\alpha)}{(t+s-\tau)^\alpha}\,d\tau$$
$$= \alpha \int_0^t \int_0^s \frac{E_\alpha(ar_1^\alpha)E_\alpha(ar_2^\alpha)}{(t+s-r_1-r_2)^{1+\alpha}}\,dr_1\,dr_2, \quad t,s \geqslant 0. \tag{3.1}$$

Proof. Denote $E_\alpha(at^\alpha)$ by $f(t)$ for convenience. Then, by the definition (1.3) we have that, for all $t, s \geqslant 0$,

$$\begin{aligned}
{}_0^C D_t^\alpha f(t+s) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{df(\tau+s)}{d\tau}\,d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \int_s^{t+s} (t+s-\tau)^{-\alpha} \frac{df(\tau)}{d\tau}\,d\tau \\
&= {}_0^C D_r^\alpha f(r)\big|_{r=t+s} - \frac{1}{\Gamma(1-\alpha)} \int_0^s (t+s-\tau)^{-\alpha} \frac{df(\tau)}{d\tau}\,d\tau \\
&= af(t+s) - \frac{1}{\Gamma(1-\alpha)} \int_0^s (t+s-\tau)^{-\alpha} \frac{df(\tau)}{d\tau}\,d\tau.
\end{aligned} \tag{3.2}$$

In the last equality the fact that $T(t) = E_\alpha(at^\alpha)$ solves Eq. (1.2) is used. Making Laplace transform with respect to t in both sides of (3.2), we get by the property (1.5) that

$$\lambda^\alpha \hat{f}_s(\lambda) - \lambda^{\alpha-1} f(s) = a\hat{f}_s(\lambda) - \frac{1}{\Gamma(1-\alpha)} \int_0^s \widehat{\big((t+s-\tau)^{-\alpha}\big)}(\lambda) \frac{df(\tau)}{d\tau}\,d\tau \tag{3.3}$$

where $\hat{f}_s(\lambda)$ and $\widehat{((t+s-\tau)^{-\alpha})}(\lambda)$ represent respectively the Laplace transforms of $f(t+s)$ and $(t+s-\tau)^{-\alpha}$ with respect to t. For the integral term in (3.3), by integrating by parts we have that

$$\begin{aligned}
\int_0^s \widehat{\big((t+s-\tau)^{-\alpha}\big)}(\lambda) \frac{df(\tau)}{d\tau}\,d\tau &= \widehat{(t^{-\alpha})}(\lambda) f(s) - \widehat{\big((t+s)^{-\alpha}\big)}(\lambda) - \int_0^s \frac{d(\widehat{(t+s-\tau)^{-\alpha})}(\lambda))}{d\tau} f(\tau)\,d\tau \\
&= \Gamma(1-\alpha)\lambda^{\alpha-1} f(s) - \widehat{\big((t+s)^{-\alpha}\big)}(\lambda) - \alpha \int_0^s \widehat{\big((t+s-\tau)^{-1-\alpha}\big)}(\lambda) f(\tau)\,d\tau,
\end{aligned}$$

which, combining with the quality (3.3), leads to that

$$\Gamma(1-\alpha)\lambda^{\alpha-1}\hat{f}_s(\lambda) = \frac{\lambda^{\alpha-1}}{\lambda^\alpha - a} \widehat{\big((t+s)^{-\alpha}\big)}(\lambda) + \alpha \int_0^s \frac{\lambda^{\alpha-1}}{\lambda^\alpha - a} \widehat{\big((t+s-\tau)^{-1-\alpha}\big)}(\lambda) f(\tau)\,d\tau. \tag{3.4}$$

So, making the inverse Laplace transform in both sides and using the convolution property of Laplace transform, we get that

$$\int_0^t (t-\tau)^{-\alpha} f(\tau+s)\,d\tau = \int_0^t (t+s-\tau)^{-\alpha} f(\tau)\,d\tau + \alpha \int_0^s \left(\int_0^t (t+s-\tau-r)^{-\alpha-1} f(r)\,dr \right) f(\tau)\,d\tau.$$

Replacing the integral variable τ with $\tau + s$ in the left term yields directly equality (3.1). The proof is therefore completed. □

Remark 1. It should be noted that for $\alpha = 1$, the integrals in (3.1) are divergent and hence equality (3.1) is not available. However, it can be shown that the semigroup property of $E_1(at)$ is just the limit state of equality (3.1) as $\alpha \uparrow 1$. Indeed, if we multiply both sides of (3.1) with $1-\alpha$ and integrate by parts, then, letting $\alpha \uparrow 1$ we get that the limit state of the left is just $E_1(a(t+s))$ and that of the right is $E_1(at)E_1(as)$.

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