



On gravity, torsion and the spectral action principle

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Received 10 February 2011; accepted 10 November 2011

Available online 17 November 2011

Communicated by Alain Connes

Abstract

We consider compact Riemannian spin manifolds without boundary equipped with orthogonal connections. We investigate the induced Dirac operators and the associated commutative spectral triples. In case of dimension four and totally anti-symmetric torsion we compute the Chamseddine–Connes spectral action, deduce the equations of motions and discuss critical points.

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Keywords: Orthogonal connections with torsion; Spectral triples; Commutative geometries; Chamseddine–Connes spectral action

1. Introduction

Einstein's field equations can be deduced as equations of motion of the Einstein–Hilbert functional. In the classical context one considers (Lorentzian or Riemannian) manifolds equipped only with the Levi-Civita connection.

In the 1920s É. Cartan investigated general orthogonal connections i.e. connections which are compatible with the metric. The difference of such a connection and the Levi-Civita connection is called torsion. In his seminal articles [6–8] Cartan observed that in general the torsion tensor splits into three components: the vectorial torsion, the totally anti-symmetric one and the one of Cartan-type. Taking the scalar curvature of orthogonal connections as the Lagrangian one attains the Einstein–Cartan–Hilbert functional. Its critical points are exactly Einstein manifolds,

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in particular the torsion of a critical point is zero. (Physics literature refers to this fact as the Palatini formalism.)

Aiming for a unified theory of gravity and the other forces Chamseddine and Connes introduced the spectral action principle [9]. It states that any reasonable physical action is determined only by the spectrum of a Dirac operator. Specifically, the Chamseddine–Connes spectral action comprises the Einstein–Hilbert action and the full bosonic part of the action of the Standard Model of Particle Physics, if one considers suitable twisted Dirac operators based on Levi-Civita connections. It even predicts the correct Higgs potential necessary for the electro-weak symmetry breaking and allows to put constraints on the Higgs mass. Into this framework orthogonal connections with totally anti-symmetric torsion have been incorporated in [23], and by Iochum, Levy and Vassilevich in [25] for the purely gravitational action on manifolds with boundary. Restricting to connections with totally anti-symmetric torsion was geometrically justified by the fact that the geodesics of such connections coincide with those of the Levi-Civita connection.

Throughout the present article we consider closed manifolds, i.e. compact ones without boundary. For these we deal with the full class of orthogonal connections. We review Cartan’s classification and Einstein–Cartan theory in Section 2, and we compute some curvature quantities in the case of totally anti-symmetric torsion in Section 3. In Section 4 we describe the Dirac operators constructed from orthogonal connections, and we notice that the vectorial component of the torsion has to be zero to assure that the Dirac operator is symmetric. This follows from a result by Friedrich and Sulanke [17]. We show that the Cartan-type component of the torsion has no effect on the Dirac operator (even pointwise) which provides another good reason to consider only anti-symmetric torsion.

In this setting many examples for commutative geometries in the sense of Connes’ spectral triples [12] can be supplied. Given the Reconstruction Theorem [14], we remark that in the even-dimensional case anti-symmetric torsion is reconstructable from the spectral data. In four dimensions we calculate the purely gravitational part of the Chamseddine–Connes spectral action in some detail. We find that some terms of the action given in [23] actually vanish, thus confirming the result by [25]. For this action we derive the equations of motion in Theorem 5.7. One of them is a Proca equation for the torsion 3-form which suggests an interpretation of the torsion as massive vector boson. The set of critical points of the action, i.e. the solutions of the equations of motion, contains all Einstein manifolds (with zero torsion). Furthermore, in Lemmas 5.10 and 5.13 we exclude critical points which are warped products and carry special choices of non-zero torsion.

We tried to keep this text elementary and accessible, and we hope that it may also serve as an introduction for non-experts.

2. Orthogonal connections on Riemannian manifolds

We consider an n -dimensional manifold M equipped with some Riemannian metric g . Let ∇^g denote the Levi-Civita connection on the tangent bundle. For any affine connection ∇ on the tangent bundle there exists a $(2, 1)$ -tensor field A such that

$$\nabla_X Y = \nabla_X^g Y + A(X, Y) \quad (1)$$

for all vector fields X, Y .

In this article we will require all connections ∇ to be *orthogonal*, i.e. for all vector fields X, Y, Z one has

$$\partial_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle, \tag{2}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product given by the Riemannian metric g . For any tangent vector X one gets from (1) and (2) that the endomorphism $A(X, \cdot)$ is skew-adjoint:

$$\langle A(X, Y), Z \rangle = -\langle Y, A(X, Z) \rangle. \tag{3}$$

Next, we want to express some curvature quantities for ∇ in terms of A and curvature quantities for ∇^g . To that end we fix some point $p \in M$, and we extend any tangent vectors $X, Y, Z, W \in T_p M$ to vector fields again denoted by X, Y, Z, W being *synchronous* in p , which means

$$\nabla_V^g X = \nabla_V^g Y = \nabla_V^g Z = \nabla_V^g W = 0 \quad \text{for any tangent vector } V \in T_p M.$$

Furthermore, we choose a local orthogonal frame of vector fields E_1, \dots, E_n on a neighbourhood of p , all being synchronous in p . Then the Lie bracket $[X, Y] = \nabla_X^g Y - \nabla_Y^g X = 0$ vanishes in p , and synchronicity in p implies

$$\nabla_X \nabla_Y Z = \nabla_X^g \nabla_Y^g Z + (\nabla_X^g A)(Y, Z) + A(X, A(Y, Z)).$$

Hence, in p the Riemann tensor of ∇ reads as

$$\begin{aligned} \text{Riem}(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z \\ &= \text{Riem}^g(X, Y)Z + (\nabla_X^g A)(Y, Z) - (\nabla_Y^g A)(X, Z) \\ &\quad + A(X, A(Y, Z)) - A(Y, A(X, Z)) \end{aligned} \tag{4}$$

where Riem^g denotes the Riemann tensor of ∇^g . We note that $\text{Riem}(X, Y)Z$ is anti-symmetric in X and Y . And by differentiation of (3) we get that $(\nabla_{E_i}^g A)(E_j, \cdot)$ and $(\nabla_{E_j}^g A)(E_i, \cdot)$ are skew-adjoint, and therefore we have

$$\langle \text{Riem}(E_i, E_j)E_k, E_l \rangle = -\langle \text{Riem}(E_i, E_j)E_l, E_k \rangle. \tag{5}$$

In general, Riem does not satisfy the Bianchi identity. The Ricci curvature of ∇ is defined as

$$\text{ric}(X, Y) = \text{tr}(V \mapsto \text{Riem}(V, X)Y),$$

by (4) this can be expressed as

$$\begin{aligned} \text{ric}(X, Y) &= \sum_{i=1}^n \langle \text{Riem}(E_i, X)Y, E_i \rangle \\ &= \text{ric}^g(X, Y) + \sum_{i=1}^n (\langle (\nabla_{E_i}^g A)(X, Y), E_i \rangle - \langle (\nabla_X^g A)(E_i, Y), E_i \rangle) \\ &\quad + \sum_{i=1}^n (-\langle A(X, Y), A(E_i, E_i) \rangle + \langle A(E_i, Y), A(X, E_i) \rangle) \end{aligned} \tag{6}$$

where ric^g is the Ricci curvature of ∇^g . We have used that $A(E_i, \cdot)$ and $A(X, \cdot)$ are skew-adjoint.

One obtains the scalar curvature R of ∇ by taking yet another trace, in p it is given as $R = \sum_{j=1}^n \text{ric}(E_j, E_j)$. For the following calculation we use that $(\nabla_V^g A)(X, \cdot)$ is skew-adjoint for any tangent vectors V, X , and we get:

$$\begin{aligned}
 R &= R^g + \sum_{i,j=1}^n (\langle (\nabla_{E_i}^g A)(E_j, E_j), E_i \rangle + \langle E_j, (\nabla_{E_j}^g A)(E_i, E_i) \rangle) \\
 &\quad + \sum_{i,j=1}^n (-\langle A(E_j, E_j), A(E_i, E_i) \rangle + \langle A(E_i, E_j), A(E_j, E_i) \rangle) \\
 &= R^g + 2 \sum_{i,j=1}^n \langle (\nabla_{E_i}^g A)(E_j, E_j), E_i \rangle - \left\| \sum_{i=1}^n A(E_i, E_i) \right\|^2 \\
 &\quad + \sum_{i,j=1}^n \langle A(E_i, E_j), A(E_j, E_i) \rangle \tag{7}
 \end{aligned}$$

where R^g denotes the scalar curvature of ∇^g .

The classification of orthogonal connections with torsions traces back to [8, Chapter VIII]. Here we adopt the notations of [32, Chapter 3] (see also [1]). From (3) we know that the torsion tensor $A(X, \cdot)$ is skew-adjoint on the tangent space T_pM . Any torsion tensor A induces a $(3, 0)$ -tensor by setting

$$A_{XYZ} = \langle A(X, Y), Z \rangle \quad \text{for any } X, Y, Z \in T_pM.$$

We define the space of all possible torsion tensors on T_pM by

$$\mathcal{T}(T_pM) = \left\{ A \in \bigotimes^3 T_p^*M \mid A_{XYZ} = -A_{XZY} \ \forall X, Y, Z \in T_pM \right\}.$$

This vector space carries a scalar product

$$\langle A, A' \rangle = \sum_{i,j,k=1}^n A_{E_i E_j E_k} A'_{E_i E_j E_k}, \tag{8}$$

and the orthogonal group $O(T_pM)$ acts on $\mathcal{T}(T_pM)$ via $(\alpha A)_{XYZ} = A_{\alpha^{-1}(X)\alpha^{-1}(Y)\alpha^{-1}(Z)}$.

For $A \in \mathcal{T}(T_pM)$ and $Z \in T_pM$ one denotes the trace over the first two entries by

$$c_{12}(A)(Z) = \sum_{i=1}^n A_{E_i E_i Z}. \tag{9}$$

The space of quadratic invariants on $\mathcal{T}(T_pM)$ with respect to the $O(T_pM)$ -representation is spanned by the three quadratic forms

$$\|A\|^2 = \langle A, A \rangle, \tag{10}$$

$$\langle A, \widehat{A} \rangle = \sum_{i,j,k=1}^n A_{E_i E_j E_k} A_{E_j E_i E_k}, \tag{11}$$

$$\|c_{12}(A)\|^2 = \sum_{i,j,k=1}^n A_{E_i E_i E_k} A_{E_j E_j E_k}. \tag{12}$$

Here \widehat{A} denotes the $(3, 0)$ -tensor obtained from A by interchanging the first two slots, i.e. $\widehat{A}_{XYZ} = A_{YXZ}$, for all tangent vectors X, Y, Z .

Theorem 2.1. For $\dim(M) \geq 3$ one has the following decomposition of $\mathcal{T}(T_p M)$ into irreducible $O(T_p M)$ -subrepresentations:

$$\mathcal{T}(T_p M) = \mathcal{T}_1(T_p M) \oplus \mathcal{T}_2(T_p M) \oplus \mathcal{T}_3(T_p M).$$

This decomposition is orthogonal with respect to $\langle \cdot, \cdot \rangle$, and it is given by

$$\mathcal{T}_1(T_p M) = \{A \in \mathcal{T}(T_p M) \mid \exists V \text{ s.t. } \forall X, Y, Z: A_{XYZ} = \langle X, Y \rangle \langle V, Z \rangle - \langle X, Z \rangle \langle V, Y \rangle\},$$

$$\mathcal{T}_2(T_p M) = \{A \in \mathcal{T}(T_p M) \mid \forall X, Y, Z: A_{XYZ} = -A_{YXZ}\},$$

$$\mathcal{T}_3(T_p M) = \{A \in \mathcal{T}(T_p M) \mid \forall X, Y, Z: A_{XYZ} + A_{YZX} + A_{ZXY} = 0 \text{ and } c_{12}(A)(Z) = 0\}.$$

For $\dim(M) = 2$ the $O(T_p M)$ -representation

$$\mathcal{T}(T_p M) = \mathcal{T}_1(T_p M)$$

is irreducible. \square

The above theorem is just Theorem 3.1 from [32]. The connections whose torsion tensor is contained in $\mathcal{T}_1(T_p M) \cong T_p M$ are called *vectorial*. Those whose torsion tensor is in $\mathcal{T}_2(T_p M) = \bigwedge^3 T_p^* M$ are called *totally anti-symmetric*, and those with torsion tensor in $\mathcal{T}_3(T_p M)$ are called *of Cartan-type*.

We note that any Cartan-type torsion tensor $A \in \mathcal{T}_3(T_p M)$ is trace-free in any pair of entries, i.e. for any Z one has

$$\sum_{i=1}^n A_{E_i E_i Z} = 0, \quad \sum_{i=1}^n A_{E_i Z E_i} = 0, \quad \sum_{i=1}^n A_{Z E_i E_i} = 0.$$

The second equality holds as $A \in \mathcal{T}(T_p M)$, and the third one follows from the cyclic identity $A_{XYZ} + A_{YZX} + A_{ZXY} = 0$.

Remark 2.2. The invariant quadratic form given in (12) has the null space $\mathcal{T}_2(T_p M) \oplus \mathcal{T}_3(T_p M)$. More precisely, one has $A \in \mathcal{T}_2(T_p M) \oplus \mathcal{T}_3(T_p M)$ if and only if $c_{12}(A)(Z) = 0$ for any $Z \in T_p M$. \square

Remark 2.3. The decomposition given in Theorem 2.1 is orthogonal with respect to the bilinear form given in (11), i.e. for $\alpha, \beta \in \{1, 2, 3\}$, $\alpha \neq \beta$, and $A_\alpha \in \mathcal{T}_\alpha(T_pM)$, $A_\beta \in \mathcal{T}_\beta(T_pM)$ one gets $\langle A_\alpha, \hat{A}_\beta \rangle = 0$. \square

Varying the base point $p \in M$, the decomposition in Theorem 2.1 is parallel with respect to the Levi-Civita connection ∇^g (induced on $(3, 0)$ -tensor fields). And from Theorem 2.1 one gets immediately:

Corollary 2.4. For any orthogonal connection ∇ on some Riemannian manifold of dimension $n \geq 3$ there exist a vector field V , a 3-form T and a $(3, 0)$ -tensor field S with $S_p \in \mathcal{T}_3(T_pM)$ for any $p \in M$ such that $\nabla_X Y = \nabla_X^g Y + A(X, Y)$ takes the form

$$A(X, Y) = \langle X, Y \rangle V - \langle V, Y \rangle X + T(X, Y, \cdot)^\sharp + S(X, Y, \cdot)^\sharp,$$

where $T(X, Y, \cdot)^\sharp$ and $S(X, Y, \cdot)^\sharp$ are the unique vectors with

$$T(X, Y, Z) = \langle T(X, Y, \cdot)^\sharp, Z \rangle \quad \text{and} \quad S(X, Y, Z) = \langle S(X, Y, \cdot)^\sharp, Z \rangle \quad \text{for all } Z. \quad (13)$$

For any orthogonal connection these V, T, S are unique. \square

Lemma 2.5. The scalar curvature of an orthogonal connection is given by

$$R = R^g + 2(n - 1) \operatorname{div}^{\nabla^g}(V) - (n - 1)(n - 2) \|V\|^2 - \|T\|^2 + \frac{1}{2} \|S\|^2$$

with V, T, S as in Corollary 2.4, and $\operatorname{div}^{\nabla^g}(V)$ is the divergence of the vector field V taken with respect to the Levi-Civita connection.

Proof. With the notations from (9)–(12) we rewrite (7) as

$$R = R^g + 2 \sum_{i=1}^n c_{12}(\nabla_{E_i}^g A)(E_i) - \|c_{12}(A)\|^2 + \langle A, \hat{A} \rangle. \quad (14)$$

By Remark 2.2 only the vectorial part of the torsion contributes to the c_{12} -terms, and therefore one gets

$$\begin{aligned} \sum_{i=1}^n c_{12}(\nabla_{E_i}^g A)(E_i) &= \sum_{i,j=1}^n (\langle E_j, E_j \rangle \langle \nabla_{E_i}^g V, E_i \rangle - \langle \nabla_{E_i}^g V, E_j \rangle \langle E_i, E_j \rangle) \\ &= (n - 1) \operatorname{div}^{\nabla^g}(V), \\ \|c_{12}(A)\|^2 &= \left\| \sum_{j=1}^n c_{12}(A)(E_j) E_j \right\|^2 \\ &= \left\| \sum_{i,j=1}^n (\langle E_i, E_i \rangle \langle V, E_j \rangle E_j - \langle V, E_i \rangle \langle E_i, E_j \rangle E_j) \right\|^2 \end{aligned} \quad (15)$$

$$\begin{aligned} &= \left\| \sum_{i=1}^n (V - \langle V, E_i \rangle E_i) \right\|^2 \\ &= \|(n - 1)V\|^2. \end{aligned} \tag{16}$$

In order to compute the last term in (14) we decompose $A = A_1 + A_2 + A_3$ with $A_\alpha \in \mathcal{T}_\alpha(T_p M)$. From Remark 2.3 we get

$$\langle A, \widehat{A} \rangle = \sum_{\alpha=1}^3 \langle A_\alpha, \widehat{A}_\alpha \rangle.$$

For the vectorial part we get

$$\begin{aligned} \langle A_1, \widehat{A}_1 \rangle &= \sum_{i,j,k=1}^n (\delta_{ij} \langle V, E_k \rangle - \delta_{ik} \langle V, E_j \rangle) \cdot (\delta_{ji} \langle V, E_k \rangle - \delta_{jk} \langle V, E_i \rangle) \\ &= \sum_{i,j,k=1}^n (\delta_{ij} \langle V, E_k \rangle^2 - \delta_{ij} \delta_{jk} \langle V, E_k \rangle \langle V, E_i \rangle - \delta_{ik} \delta_{ji} \langle V, E_j \rangle \langle V, E_k \rangle \\ &\quad + \delta_{ik} \delta_{jk} \langle V, E_j \rangle \langle V, E_i \rangle) \\ &= (n - 1)\|V\|^2. \end{aligned} \tag{17}$$

For the totally anti-symmetric part we get

$$\langle A_2, \widehat{A}_2 \rangle = \sum_{i,j,k=1}^n T_{E_i E_j E_k} T_{E_j E_i E_k} = - \sum_{i,j,k=1}^n T_{E_i E_j E_k} T_{E_i E_j E_k} = -\|T\|^2. \tag{18}$$

Finally, for the Cartan-type part we get

$$\begin{aligned} \langle A_3, \widehat{A}_3 \rangle &= \sum_{i,j,k=1}^n S_{E_i E_j E_k} S_{E_j E_i E_k} \\ &= - \sum_{i,j,k=1}^n (S_{E_i E_j E_k} S_{E_i E_k E_j} + S_{E_i E_j E_k} S_{E_k E_j E_i}) \end{aligned} \tag{19}$$

$$= \sum_{i,j,k=1}^n S_{E_i E_j E_k} S_{E_i E_j E_k} - \sum_{i,j,k=1}^n S_{E_i E_k E_j} S_{E_k E_i E_j} \tag{20}$$

$$= \frac{1}{2}\|S\|^2, \tag{21}$$

where (19) is due to the cyclic identity for S , (20) follows from the anti-symmetry in the last two entries, and $\sum_{i,j,k=1}^n S_{E_i E_k E_j} S_{E_k E_i E_j} = \langle A_3, \widehat{A}_3 \rangle$ implies (21). Plugging (15)–(21) into (14) finishes the proof. \square

Corollary 2.6. *Let M be a closed manifold of dimension $n \geq 3$ with Riemannian metric g and orthogonal connection ∇ . Let dvol denote the Riemannian volume measure taken with respect to g . Then the Einstein–Cartan–Hilbert functional is*

$$\int_M R \, \text{dvol} = \int_M R^g \, \text{dvol} - (n - 1)(n - 2) \int_M \|V\|^2 \, \text{dvol} - \int_M \|T\|^2 \, \text{dvol} + \frac{1}{2} \int_M \|S\|^2 \, \text{dvol}.$$

Considering variations over all Riemannian metrics, for which the volume $\text{vol}_g(M)$ stays fixed, and all orthogonal connections (i.e. over all torsion tensors), we get that (M, g, ∇) is a critical point of the Einstein–Cartan–Hilbert functional if and only if (M, g) is an Einstein manifold and $\nabla = \nabla^g$ is the Levi-Civita connection (i.e. $V \equiv 0, T \equiv 0$ and $S \equiv 0$).

For an in depth treatment of the physical consequences of Einstein–Cartan–Hilbert theory in Lorentzian geometry we refer to the classical review [24] and the more recent overview [30] and references therein.

3. Curvature calculations in case of totally anti-symmetric torsion in four dimensions

Let us collect now some equalities involving curvature tensors and the totally anti-symmetric torsion. To keep the main part of this paper as readable as possible the proofs of the following theorems and lemmata have been allocated to Appendix A.

We consider a 4-dimensional manifold M equipped with a Riemannian metric g . Let ∇^g denote the Levi-Civita connection on the tangent bundle. We fix some 3-form T on M and some $s \in \mathbb{R}$, and we are studying the connection ∇ which is given by

$$\nabla_X Y = \nabla_X^g Y + sT(X, Y, \cdot)^\sharp \tag{22}$$

for any vector fields X and Y on M and $T(X, Y, \cdot)^\sharp$ is defined as in (13). Hence ∇ is an orthonormal connection with totally anti-symmetric torsion.

3.1. Pointwise equalities

In the following we want to express some curvature quantities for ∇ in terms of T and curvature quantities for ∇^g . For the Riemann curvature and the scalar curvature we will calculate the norms explicitly in terms of T , the Levi-Civita connection and its curvatures. The norm of the Ricci curvature is given in Appendix A.

As in Section 2 we fix some point $p \in M$, and we extend any tangent vectors $X, Y, Z, W \in T_p M$ to vector fields again denoted by X, Y, Z, W being synchronous in p . Hence we obtain from (4) the Riemann curvature of ∇

$$\begin{aligned} \langle \text{Riem}(X, Y)Z, W \rangle &= \langle \text{Riem}^g(X, Y)Z, W \rangle + s((\nabla_X^g T)(Y, Z, W) - (\nabla_Y^g T)(X, Z, W)) \\ &\quad + s^2(T(X, T(Y, Z, \cdot)^\sharp), W) - T(Y, T(X, Z, \cdot)^\sharp), W) \\ &= \langle \text{Riem}^g(X, Y)Z, W \rangle + s((\nabla_X^g T)(Y, Z, W) - (\nabla_Y^g T)(X, Z, W)) \\ &\quad + s^2(\langle T(X, Z, \cdot)^\sharp, T(Y, W, \cdot)^\sharp \rangle - \langle T(Y, Z, \cdot)^\sharp, T(X, W, \cdot)^\sharp \rangle). \end{aligned} \tag{23}$$

We used the identity $T(X, T(Y, Z, \cdot)^\sharp, W) = -\langle T(X, W, \cdot)^\sharp, T(Y, Z; \cdot)^\sharp \rangle$, which follows from (13).

From (6) we conclude that the Ricci curvature of ∇ for any orthonormal, synchronous frame E_1, \dots, E_n defined on some neighbourhood of p is

$$\text{ric}(X, Y) = \text{ric}^g(X, Y) + s \sum_{i=1}^n (\nabla_{E_i}^g T)(X, Y, E_i) - s^2 \sum_{i=1}^n \langle T(E_i, X, \cdot)^\sharp, T(E_i, Y, \cdot)^\sharp \rangle. \tag{24}$$

This formula shows that in general the Ricci curvature is not symmetric in X and Y .

From Lemma 2.5 we get for the scalar curvature R of ∇ that

$$R = R^g - s^2 \|T\|^2. \tag{25}$$

Next, we are aiming at finding an expression for the norm of the Riemann tensor. As the vector fields X, Y, Z, W are synchronous in p we get for the differential dT and the codifferential δT of the 3-form T :

$$\begin{aligned} dT(X, Y, Z, W) &= (\nabla_X^g T)(Y, Z, W) - (\nabla_Y^g T)(X, Z, W) \\ &\quad + (\nabla_Z^g T)(X, Y, W) - (\nabla_W^g T)(X, Y, Z), \end{aligned} \tag{26}$$

$$\delta T(X, Y) = - \sum_{i=1}^n (\nabla_{E_i}^g T)(X, Y, E_i). \tag{27}$$

We define the $(4, 0)$ -tensors riem and riem^g by

$$\text{riem}^{(g)}(X, Y, Z, W) = \langle \text{Riem}^{(g)}(X, Y)Z, W \rangle. \tag{28}$$

We decompose the Riemann tensor into its symmetric and anti-symmetric component

$$\text{riem}(X, Y, Z, W) = \text{riem}^S(X, Y, Z, W) + \text{riem}^A(X, Y, Z, W). \tag{29}$$

The symmetric part of riem is

$$\text{riem}^S(X, Y, Z, W) = \frac{1}{2} (\text{riem}(X, Y, Z, W) + \text{riem}(Z, W, X, Y))$$

and the anti-symmetric part of riem is then given by

$$\text{riem}^A(X, Y, Z, W) = \frac{1}{2} (\text{riem}(X, Y, Z, W) - \text{riem}(Z, W, X, Y)).$$

Since riem^S and riem^A are orthogonal with respect to the scalar product of $(4, 0)$ -tensors, as defined in (57), we find

$$\|\text{Riem}\|^2 = \|\text{riem}\|^2 = \|\text{riem}^S\|^2 + \|\text{riem}^A\|^2. \tag{30}$$

We also decompose the Ricci curvature into its symmetric and its anti-symmetric components

$$\begin{aligned}
 \text{ric}(X, Y) &= \text{ric}^S(X, Y) + \text{ric}^A(X, Y), \\
 \text{with } \text{ric}^S(X, Y) &= \frac{1}{2}(\text{ric}(X, Y) + \text{ric}(Y, X)) \text{ and} \\
 \text{ric}^A(X, Y) &= \frac{1}{2}(\text{ric}(X, Y) - \text{ric}(Y, X)).
 \end{aligned}
 \tag{31}$$

Now we give an explicit formula for $\|\text{riem}\|^2$ in the case of M being 4-dimensional.

Theorem 3.1. *Let M be a 4-dimensional manifold with Riemannian metric g and connection ∇ as given in (22). Then the norm of the Riemann tensor of ∇ is given by*

$$\begin{aligned}
 \|\text{Riem}\|^2 &= \|\text{riem}^g\|^2 + \frac{1}{3}s^4\|T\|^4 + \frac{1}{4}s^2\|dT\|^2 - \frac{1}{3}s^2R^g\|T\|^2 \\
 &\quad + 4s^2\text{B}(T) + \|\text{riem}^A\|^2
 \end{aligned}
 \tag{32}$$

with

$$\text{B}(T) = \sum_{i,j,k} \text{ric}^g(E_i, E_k) \langle T(E_i, E_j, \cdot)^\sharp, T(E_j, E_k, \cdot)^\sharp \rangle + \frac{1}{4}R^g\|T\|^2.$$

Proof. See Appendix A. \square

We notice that the term $\text{B}(T)$ couples the torsion to the Ricci curvature, and the term $\|\text{riem}^A\|^2$ is being computed in Lemma A.3.

3.2. Integral formulas

In this section (M, g) will be a closed, 4-dimensional Riemannian manifold. We will exploit the topological invariance of the Euler characteristic to deduce integral formulas for 3-forms defined on M .

Definition 3.2. Let ∇ be an orthogonal connection on M and let $\text{Riem}_{ij}(X, Y) := \langle \text{Riem}(E_i, E_j)X, Y \rangle$ be its curvature 2-form defined by Eq. (4). Define the 4-form

$$\mathbf{K} = \frac{1}{32\pi^2} \sum_{i,j,k,l=1}^4 \epsilon_{ijkl} \text{Riem}_{ij} \wedge \text{Riem}_{kl},$$

where ϵ_{ijkl} is the totally anti-symmetric tensor with normalisation $\epsilon_{1234} = +1$.

One obtains the classical result for the interplay between the topological invariant Euler characteristic $\chi(M)$ and the curvature 2-form of ∇ :

Theorem 3.3. *The Euler characteristic of M is*

$$\chi(M) = \int_M \mathbf{K}.$$

Proof. For a proof of this theorem we refer to [26, Vol. II, Chapter XII, Theorem 5.1]. \square

In four dimensions the Euler characteristic can be expressed in a particularly convenient form in terms of squares of the Riemann, Ricci and scalar curvature of ∇ .

Theorem 3.4. *Let ∇ be an orthogonal connection on M and let $\text{riem} = \text{riem}^S + \text{riem}^A$, $\text{ric} = \text{ric}^S + \text{ric}^A$ and R be the Riemann curvature, the Ricci curvature and the scalar curvature of ∇ decomposed into their symmetric and anti-symmetric parts according to (29) and (31). Then the Euler characteristic $\chi(M)$ is*

$$\chi(M) = \frac{1}{8\pi^2} \int_M (R^2 - 4\|\text{ric}^S\|^2 + 4\|\text{ric}^A\|^2 + \|\text{riem}^S\|^2 - \|\text{riem}^A\|^2) \text{dvol}.$$

Proof. See Appendix A. \square

The classical result for the Euler characteristic in terms of the curvatures of the Levi-Civita connection is due to Berger [3], and it follows immediately from the above theorem:

Corollary 3.5. *Let M be a 4-dimensional manifold with Riemannian metric g and Levi-Civita connection ∇^g . Then the Euler characteristic $\chi(M)$ is given by*

$$\chi(M) = \frac{1}{8\pi^2} \int_M ((R^g)^2 - 4\|\text{ric}^g\|^2 + \|\text{riem}^g\|^2) \text{dvol}.$$

Proof. For ∇^g we have $\text{riem}^S = \text{riem}^g$, $\text{ric}^S = \text{ric}^g$ and hence $\text{riem}^A \equiv 0$ and $\text{ric}^A \equiv 0$. \square

The fact that the Euler characteristic does not depend on the connection allows us to deduce a useful integral formula for 3-forms on closed Riemannian 4-manifolds.

Lemma 3.6. *Let M be a closed 4-dimensional manifold with Riemannian metric g and T any 3-form on M . Let R^g denote the scalar curvature of the Levi-Civita connection ∇^g . Then*

$$\int_M 4\|\delta T\|^2 \text{dvol} = \int_M \left(\frac{1}{3} R^g \|T\|^2 - \frac{1}{4} \|dT\|^2 + 4B(T) + \frac{1}{s^2} \|\text{riem}^A\|^2 \right) \text{dvol} \quad (33)$$

with $B(T)$ as defined in Theorem 3.1 and riem^A is the anti-symmetric component of the Riemann curvature of ∇ with sT as torsion 3-form.

Proof. See Appendix A. \square

4. Dirac operators associated to orthogonal connections

In this section we consider an n -dimensional oriented Riemannian manifold (M, g) equipped with some spin structure. Let ∇ be an orthogonal connection given as in (1)–(3). Then the connection ∇ acting on vector fields induces a connection acting on spinor fields. Next, we will briefly discuss the construction of this connection (compare [1, p. 17f], or see [27, Chapter II.4] for more details). Again, we write $\nabla_X Y = \nabla_X^g Y + A(X, Y)$ with the Levi-Civita connection ∇^g . For any $X \in T_p M$ the endomorphism $A(X, \cdot)$ is skew-adjoint and hence it is an element of $\mathfrak{so}(T_p M) \cong \wedge^2 T_p M$, we can express it as

$$A(X, \cdot) = \sum_{i < j} \alpha_{ij} E_i \wedge E_j. \tag{34}$$

Here $E_i \wedge E_j$ is meant as the endomorphism of $T_p M$ defined by $E_i \wedge E_j(Z) = \langle E_i, Z \rangle E_j - \langle E_j, Z \rangle E_i$. For any $X \in T_p M$ one determines the coefficients in (34) by

$$\alpha_{ij} = \langle A(X, E_i), E_j \rangle = A_{XE_i E_j}. \tag{35}$$

Each $E_i \wedge E_j$ lifts to $\frac{1}{2} E_i \cdot E_j$ in $\mathfrak{spin}(n)$, and the spinor connection induced by ∇ is locally given by

$$\nabla_X \psi = \nabla_X^g \psi + \frac{1}{2} \sum_{i < j} \alpha_{ij} E_i \cdot E_j \cdot \psi = \nabla_X^g \psi + \frac{1}{2} \sum_{i < j} A_{XE_i E_j} E_i \cdot E_j \cdot \psi. \tag{36}$$

Remark 4.1. The connection given by (36) is compatible with the metric on spinors and with Clifford multiplication (see e.g. [1, Lemma 2.1]).

Remark 4.2. For totally anti-symmetric torsion, given by a 3-form T as in Corollary 2.4, one can rewrite (36) as

$$\nabla_X \psi = \nabla_X^g \psi + \frac{1}{2} (X \lrcorner T) \cdot \psi,$$

where $X \lrcorner T$ is the 2-form defined by $X \lrcorner T(X, Z) = T(X, Y, Z)$. We recall that a k -form $\omega \in \wedge^k T_p M$, given as $\omega = \sum_{i_1, \dots, i_k} \omega_{i_1, \dots, i_k} E_{i_1}^\flat \wedge \dots \wedge E_{i_k}^\flat$, acts on the spinor space as $\omega \cdot \psi = \sum_{i_1, \dots, i_k} \omega_{i_1, \dots, i_k} E_{i_1} \cdot \dots \cdot E_{i_k} \cdot \psi$.

Remark 4.3. Not any connection on spinor fields is induced by an orthogonal connection on tangent vector fields. For example, for the connection $\nabla_X \psi = \nabla_X^g \psi + X \cdot \psi$ the endomorphism α of spinors defined by

$$\alpha(\psi) = \nabla_X(Y \cdot \psi) - Y \cdot (\nabla_X \psi) \tag{37}$$

is given by multiplication by the Clifford element $\nabla_X^g Y + X \cdot Y - Y \cdot X$, which does not equal to the Clifford multiplication by any tangent vector. This consideration applies in any dimension $n \geq 2$.

Remark 4.4. If we assume that for a spinor connection ∇ for any vector fields X, Y the endomorphism α defined in (37) is the Clifford multiplication by a tangent vector $V_{X,Y}$, i.e. $\alpha(\psi) = V_{X,Y} \cdot \psi$ for all spinors ψ , then it can be shown that the assignment $\nabla_X Y = V_{X,Y}$ defines an orthogonal connection on tangent vector fields such that the spinor connection is compatible with the Clifford multiplication. In that case, physics literature occasionally refers to (37) as the *tetrad postulate*.

The Dirac operator associated to the spinor connection from (36) is defined as

$$\begin{aligned}
 D\psi &= \sum_{i=1}^n E_i \cdot \nabla_{E_i} \psi \\
 &= D^g \psi + \frac{1}{2} \sum_{i=1}^n \sum_{j < k} A_{E_i E_j E_k} E_i \cdot E_j \cdot E_k \cdot \psi \\
 &= D^g \psi + \frac{1}{4} \sum_{i,j,k=1}^n A_{E_i E_j E_k} E_i \cdot E_j \cdot E_k \cdot \psi,
 \end{aligned}
 \tag{38}$$

where D^g is the Dirac operator induced by the Levi-Civita connection.

The next theorem tells us when the Dirac operator D is formally selfadjoint (i.e. symmetric on the space of compactly supported smooth spinor fields as domain), it is provided as Satz 2 in [17]:

Theorem 4.5. *The Dirac operator D is formally selfadjoint if and only if the divergence of ∇ coincides with the divergence of ∇^g , i.e. for any vector field Z one has*

$$\sum_{i=1}^n \langle \nabla_{E_i} Z, E_i \rangle = \sum_{i=1}^n \langle \nabla_{E_i}^g Z, E_i \rangle
 \tag{39}$$

in any point p and for any orthonormal basis E_1, \dots, E_n of $T_p M$. \square

Taking the specific form of $\nabla_X Y = \nabla_X^g Y + A(X, Y)$ into account, we see that (39) is equivalent to

$$c_{12}(A)(Z) = \sum_{i=1}^n \langle A(E_i, E_i), Z \rangle = - \sum_{i=1}^n \langle A(E_i, Z), E_i \rangle = 0.$$

Hence, we can conclude from Remark 2.2:

Corollary 4.6. *The Dirac operator D associated to an orthogonal connection is formally selfadjoint if and only if the $(3, 0)$ -torsion tensor A does not have any vectorial component, i.e. one has*

$$A_p \in \mathcal{T}_2(T_p M) \oplus \mathcal{T}_3(T_p M)$$

in any point $p \in M$. \square

The next lemma states that for the Dirac operator the Cartan-type component of the torsion is invisible:

Lemma 4.7. *On a Riemannian spin manifold we consider some vector field V , some 3-form T and some $(3, 0)$ -tensor field S with $S_p \in \mathcal{T}_3(T_pM)$ for any $p \in M$. Let ∇_1 and ∇_2 be the orthogonal connections determined by*

$$A_1(X, Y) = \langle X, Y \rangle V - \langle V, Y \rangle X + T(X, Y, \cdot)^\sharp + S(X, Y, \cdot)^\sharp \quad \text{and}$$

$$A_2(X, Y) = \langle X, Y \rangle V - \langle V, Y \rangle X + T(X, Y, \cdot)^\sharp,$$

respectively (compare Corollary 2.4). Denote the associated Dirac operators by D_1 and D_2 . Then, for any spinor field ψ one has

$$D_1\psi = D_2\psi.$$

Proof. By (38) the difference of the two Dirac operators is

$$D_1\psi - D_2\psi = \frac{1}{4} \sum_{i,j,k=1}^n S_{E_i E_j E_k} E_i \cdot E_j \cdot E_k \cdot \psi. \tag{40}$$

We use the cyclic identity for S , the fact that S is trace-free in any pair of entries and the Clifford relations $E_i \cdot E_j = -E_j \cdot E_i$ for $i \neq j$ as well, in order to obtain:

$$\begin{aligned} \sum_{i,j,k=1}^n S_{E_i E_j E_k} E_i \cdot E_j \cdot E_k &= - \sum_{i,j,k=1}^n S_{E_j E_k E_i} E_i \cdot E_j \cdot E_k - \sum_{i,j,k=1}^n S_{E_k E_i E_j} E_i \cdot E_j \cdot E_k \\ &= - \sum_{i,j,k=1}^n S_{E_j E_k E_i} E_j \cdot E_k \cdot E_i - \sum_{i,j,k=1}^n S_{E_k E_i E_j} E_k \cdot E_i \cdot E_j \\ &= -2 \sum_{i,j,k=1}^n S_{E_i E_j E_k} E_i \cdot E_j \cdot E_k. \end{aligned}$$

Therefore we get $\sum_{i,j,k=1}^n S_{E_i E_j E_k} E_i \cdot E_j \cdot E_k = 0$, and the right hand side of (40) is zero. \square

One should note that the above lemma applies pointwise.

Remark 4.8. In the Lorentzian case it is known that torsion of Cartan-type does not contribute to the Dirac action under the integral [30, Chapter 2.3]. It is also known that the Dirac action is not real if the torsion has a non-vanishing vectorial component [19, Chapter 11.6]. Therefore only totally anti-symmetric torsion is considered to couple to fermions reasonably.

The spinor connection in (36) is the connection which is induced by the tangent vector connection ∇ given in (1), hence one expects that their curvatures are related. Let (E_1, \dots, E_n) be an arbitrary local orthonormal frame. For i, j the curvature endomorphism w.r.t. this frame is defined as

$$\Omega_{ij}\psi = \nabla_{E_i}\nabla_{E_j}\psi - \nabla_{E_j}\nabla_{E_i}\psi - \nabla_{[E_i, E_j]}\psi.$$

These curvature endomorphisms for spinors can be naturally determined by the Riemann tensor for tangent vectors, compare with formula (4.37) in Theorem 4.15 of [27, Chapter II].

Lemma 4.9. *For the spinor connection ∇ defined in (36) the curvature endomorphisms in p are given by*

$$\Omega_{ij}\psi = \frac{1}{4} \sum_{a,b} \langle \text{Riem}(E_i, E_j)E_a, E_b \rangle E_a \cdot E_b \cdot \psi$$

with Riemann tensor as defined in (4). \square

Corollary 4.10. *Let tr denote the trace over the spinor space over some footpoint p . Then one has*

$$\sum_{i,j=1}^n \text{tr}(\Omega_{ij}\Omega_{ij}) = -\frac{1}{8} \cdot 2^{[n/2]} \cdot \|\text{Riem}\|^2$$

where Riem is the Riemann tensor of the vector connection ∇ .

Proof. The spinor space has dimension $2^{[n/2]}$. If $a \neq b$ and $c \neq d$ the Clifford relations imply

$$\text{tr}(E_a \cdot E_b \cdot E_c \cdot E_d) = 2^{[n/2]}(\delta_{bc}\delta_{ad} - \delta_{bd}\delta_{ac}).$$

From Lemma 4.9 we derive

$$\begin{aligned} & \sum_{i,j} \text{tr}(\Omega_{ij}\Omega_{ij}) \\ &= \frac{1}{16} \sum_{i,j} \sum_{a \neq b} \sum_{c \neq d} \langle \text{Riem}(E_i, E_j)E_a, E_b \rangle \langle \text{Riem}(E_i, E_j)E_c, E_d \rangle \text{tr}(E_a \cdot E_b \cdot E_c \cdot E_d) \\ &= \frac{1}{16} \cdot 2^{[n/2]} \sum_{i,j} \sum_{a \neq b} (\langle \text{Riem}(E_i, E_j)E_a, E_b \rangle \langle \text{Riem}(E_i, E_j)E_b, E_a \rangle \\ &\quad - \langle \text{Riem}(E_i, E_j)E_a, E_b \rangle \langle \text{Riem}(E_i, E_j)E_a, E_b \rangle) \\ &= -\frac{1}{8} \cdot 2^{[n/2]} \sum_{i,j} \sum_{a \neq b} (\langle \text{Riem}(E_i, E_j)E_a, E_b \rangle)^2 \\ &= -\frac{1}{8} \cdot 2^{[n/2]} \cdot \|\text{Riem}\|^2, \end{aligned}$$

where we have used the anti-symmetry of $\text{Riem}(E_i, E_j)E_a, E_b$ in the indices a and b , which holds due to (5). \square

5. Commutative geometries and the spectral action principle

In this section we want to discuss torsion connections within the framework of Connes' non-commutative geometry (see [11]). Let M be a closed Riemannian manifold of dimension n with some fixed spin structure. We denote the algebra of smooth functions by $\mathcal{A} = C^\infty(M)$, and we denote the Hilbert space of square integrable spinor fields by \mathcal{H} . The Dirac operator D^g associated to the Levi-Civita connection is a selfadjoint operator in \mathcal{H} . The triple $(\mathcal{A}, \mathcal{H}, D^g)$ forms a *canonical spectral triple*, and it satisfies all *axioms for commutative geometry* (see [12], or [21] for more details).

Now, let ∇ be an orthogonal connection on the tangent bundle of M , and let D denote the associated Dirac operator. By Corollary 4.6 we know that D is symmetric if and only if the vectorial component of the torsion of ∇ is zero. In that case D is the sum of a selfadjoint operator and a bounded symmetric one, and thus selfadjoint. We notice that D has the same principal symbol and the same Weyl asymptotics as D^g . Furthermore, we note that any natural algebraic structure on the spinor space such as a real structure or the Clifford multiplication with the volume element is parallel with respect to any spinor connection which is induced by an orthogonal connection on the tangent bundle. Therefore D commutes or anti-commutes with such a structure exactly if D^g does. Following the details of the proof of [21, Theorem 11.1] we see that these observations suffice to verify all axioms for commutative geometry and we conclude:

Lemma 5.1. *If the vectorial component of the torsion of ∇ is zero, the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ satisfies the axioms for commutative geometry. \square*

Connes' Reconstruction Theorem (conjectured in [12], proved in [14]) states that, given a commutative geometry $(\mathcal{A}, \mathcal{H}, D)$, one can construct a differentiable spin manifold such that \mathcal{A} coincides with the smooth functions on it. Then, from the data $(\mathcal{A}, \mathcal{H}, D)$ one also gets a Riemannian metric and one obtains that \mathcal{H} is isomorphic to the square integrable spinor fields (for some spin structure) (see [13, Théorème 6]), and the natural Dirac operator we can always construct is the one induced by the Levi-Civita connection.

In the above situation of Lemma 5.1, one can algebraically recover the totally anti-symmetric torsion component from the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ if the underlying manifold M has even dimension. This can be done by considering the endomorphism of spinors given as difference of D and the Levi-Civita Dirac operator D^g , see (38). In even dimensions the complex Clifford algebra and the space of endomorphisms of the spinor space are identical (see [16, Proposition on p. 13]), and hence the endomorphism $D - D^g$ can be uniquely determined as a 3-form. In odd dimensions this argument does not apply, as one easily sees by noticing that e.g. in the 3-dimensional case the volume form acts as multiple of the identity on the spinor space.

By Lemma 4.7 the Cartan-type component of the torsion is invisible for the Dirac operator D , and therefore it cannot be recovered from $(\mathcal{A}, \mathcal{H}, D)$. This can be interpreted as some sort of gauge freedom, which is schematically illustrated in Fig. 1.

Remark 5.2. For some even-dimensional compact Riemannian manifold M with a given spin structure we fix $\mathcal{A} = C^\infty(M)$ and \mathcal{H} the space of square integrable spinor fields. The above considerations show that one has a family of Dirac operators parametrized by the 3-forms $T \in \Omega^3(M)$ such that the associated spectral triples are pairwise distinct commutative geometries. Notice that all these Dirac operators are in the same K -homology class since they all have the

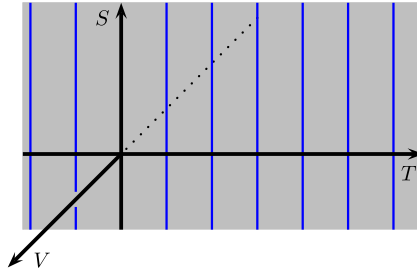


Fig. 1. Cartan-type component S of torsion is invisible for Dirac operator (gauge freedom).

same principal symbol. We leave it open how big the class of first order operators D in \mathcal{H} is for which $(\mathcal{A}, \mathcal{H}, D)$ forms a commutative geometry.

Remark 5.3. For spectral triples of odd KO -dimension it has recently been shown in [31, Proposition 1.2] that one can modify the Dirac operator by adding a term induced by a selfadjoint element of the algebra \mathcal{A} and still finds the axioms of spectral triples satisfied. In the case of $\mathcal{A} = C^\infty(M)$ and KO -dimensions 3 or 7 modulo 8, i.e. M is of dimension 3 or 7 modulo 8, this modification is realised by adding a real-valued function $\Phi \in \mathcal{A}$ to the Dirac operator, see [31, Remark 1.3].

In the following we will only consider orthogonal connections ∇ with zero vectorial component to ensure selfadjointness of the induced Dirac operator D . For the computation of the Chamseddine–Connes spectral action (see [9]) we need the Seeley–deWitt coefficients $a_{2k}(D^2)$ of the heat trace asymptotics [18]

$$\text{Tr}(e^{-tD^2}) \sim \sum_{k \geq 0} t^{k-n/2} a_{2k}(D^2) \quad \text{as } t \rightarrow 0.$$

Proposition 5.4. *The first two Seeley–deWitt coefficients are*

$$a_0(D^2) = \frac{1}{(4\pi)^{n/2}} 2^{[n/2]} \int_M \text{dvol},$$

$$a_2(D^2) = \frac{1}{(4\pi)^{n/2}} 2^{[n/2]} \int_M \left(\frac{3}{4} \|T\|^2 - \frac{1}{12} R^g \right) \text{dvol}.$$

Proof. By Lemma 4.7 we can assume without loss of generality that the Cartan-type component of the torsion vanishes. The orthogonal connection ∇ is given by $\nabla_X Y = \nabla_X^g Y + T(X, Y, \cdot)^\sharp$. Adapting [2, Theorem 6.2] into our notation we get the Bochner formula

$$D^2 = \Delta + \frac{3}{2} dT + \frac{1}{4} R^g - \frac{3}{4} \|T\|^2 \tag{41}$$

where Δ is the Laplacian associated to the spin connection

$$\tilde{\nabla}_X \psi = \nabla_X^g \psi + \frac{3}{2}(X \lrcorner T) \cdot \psi, \tag{42}$$

which is induced (as in Remark 4.2) by the orthogonal connection

$$\tilde{\nabla}_X Y = \nabla_X^g Y + 3T(X, Y, \cdot)^\sharp. \tag{43}$$

We notice that the trace of dT taken over the spinor space is zero due to Clifford relations. Inserting this into the general formulas for the Seeley–deWitt coefficients (see [18, Theorem 4.1.6]) the claim follows. \square

If we consider the spectral action given by the $a_2(D^2)$ and variations with respect to the torsion 3-form T we find that $T = 0$ is the only possibility for critical points. Therefore this spectral action detects the Dirac operator induced by the Levi-Civita connection within the class of Dirac operators induced by orthogonal connections without vectorial torsion. We note that this holds in any dimension. This is in complete accordance with [15, Section 18.2].

The computation of $a_4(D^2)$ is more involved, we will give it only for 4-dimensional manifolds. In [25] it has been noted that some terms given in [23] vanish. Similar results have been found before (compare [20,28,22]). The calculation given below is elementary, it takes place essentially in the tangent bundle and should therefore be easily accessible.

Proposition 5.5. *If M is 4-dimensional, the third Seeley–deWitt coefficient is*

$$a_4(D^2) = \frac{11}{720} \chi(M) - \frac{1}{320\pi^2} \int_M \|C\|^2 \text{dvol} - \frac{3}{32\pi^2} \int_M \|\delta T\|^2 \text{dvol}, \tag{44}$$

where C is the Weyl curvature of M (computed from the Levi-Civita connection).

Proof. We read (41) as $D^2 = \Delta - E$ with potential $E = -\frac{3}{2}dT - \frac{1}{4}R^g + \frac{3}{4}\|T\|^2$. From [18, Theorem 4.1.6, c)] we get

$$a_4(D^2) = \frac{1}{5760\pi^2} \int_M \left(\text{tr} \left(60R^g E + 180E^2 + 30 \sum_{i,j} \Omega_{ij} \Omega_{ij} \right) + 20(R^g)^2 - 8\|\text{ric}^g\|^2 + 8\|\text{Riem}^g\|^2 \right) \text{dvol},$$

where we have omitted the terms that integrate to zero over the closed manifold M (Laplacians of functions). The term Ω_{ij} is the curvature endomorphism for the spinor connection $\tilde{\nabla}$. The traces over the spinor space are

$$\begin{aligned} \text{tr}(E) &= -R^g + 3\|T\|^2, \\ \text{tr}(E^2) &= \frac{1}{4}(R^g)^2 - \frac{3}{2}R^g\|T\|^2 + \frac{9}{4}\|T\|^4 + \frac{9}{24}\|dT\|^2, \end{aligned}$$

where we note that $\text{tr}(\omega^2) = \frac{1}{6}\|\omega\|^2$ for any $\omega \in \wedge^4$. For the orthogonal connection $\tilde{\nabla}$ on the tangent bundle we denote the Riemannian curvature by Riem and apply Corollary 4.10. Then we get

$$\begin{aligned}
 a_4(D^2) &= \frac{1}{16\pi^2} \frac{1}{360} \int_M \left(5(R^g)^2 - 8\|\text{Ric}^g\|^2 + 8\|\text{Riem}^g\|^2 - 15\|\text{Riem}\|^2 \right. \\
 &\quad \left. - 90R^g\|T\|^2 + 405\|T\|^4 + \frac{135}{2}\|dT\|^2 \right) \text{dvol} \\
 &= \frac{1}{16\pi^2} \frac{1}{360} \int_M (5(R^g)^2 - 8\|\text{Ric}^g\|^2 - 7\|\text{Riem}^g\|^2) \text{dvol} \\
 &\quad - \frac{1}{16\pi^2} \frac{1}{8} \int_M \left(R^g\|T\|^2 - \frac{3}{4}\|dT\|^2 + 12B(T) + \frac{1}{3}\|\text{riem}^A\|^2 \right) \text{dvol}
 \end{aligned}$$

by means of Proposition 3.1. With Lemma 3.5 we identify the first integral as the Euler characteristic plus the square of the Weyl curvature. Lemma 3.6 shows that the second integral equals

$$\frac{1}{16\pi^2} \frac{1}{8} \int_M \left(R^g\|T\|^2 - \frac{3}{4}\|dT\|^2 + 12B(T) + \frac{1}{3}\|\text{riem}^A\|^2 \right) \text{dvol} = \frac{1}{16\pi^2} \frac{3}{2} \int_M \|\delta T\|^2 \text{dvol}.$$

This finishes the proof. \square

Next, we want to consider the Chamseddine–Connes spectral action (see [9]) for the Dirac operator D . For $\Lambda > 0$ it is defined as

$$I_{CC} = \text{Tr} F\left(\frac{D^2}{\Lambda^2}\right)$$

where Tr denotes the operator trace over \mathcal{H} as before, and $F : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a cut-off function with support in the interval $[0, +1]$ which is constant near the origin. Using the heat trace asymptotics one gets an asymptotic expression for I_{CC} as $\Lambda \rightarrow \infty$ (see [10] for details):

$$I_{CC} = \text{Tr} F\left(\frac{D^2}{\Lambda^2}\right) = \Lambda^4 F_4 a_0(D^2) + \Lambda^2 F_2 a_2(D^2) + \Lambda^0 F_0 a_4(D^2) + O(\Lambda^{-\infty}) \tag{45}$$

with the first three moments of the cut-off function which are given by $F_4 = \int_0^\infty s \cdot F(s) ds$, $F_2 = \int_0^\infty F(s) ds$ and $F_0 = F(0)$. Note that these moments are independent of the geometry of the manifold.

Now we want to deduce the equation of motion for I_{CC} . In analogy to the Riemannian Einstein–Hilbert case we consider variations with respect to the metric and the torsion 3-form while keeping the volume fixed. Then $a_0(D^2)$ and $\chi(M)$ are constant and their variation vanishes. In order to avoid further complications we neglect the contributions of the $O(\Lambda^{-\infty})$ -term. Therefore we will consider the following action functional

$$\tilde{I}_{CC} = -\alpha \int_M R^g \text{dvol} - \beta \int_M \|C\|^2 \text{dvol} + \gamma_1 \int_M \|T\|^2 \text{dvol} - \gamma_2 \int_M \|\delta T\|^2 \text{dvol}, \tag{46}$$

where $\alpha, \beta, \gamma_1, \gamma_2 > 0$.

Before we proceed let us briefly recall the standard scalar product on k -forms induced by a Riemannian metric g (compare e.g. with [5]). Let E_1, \dots, E_n be an orthonormal basis of some tangent space $T_p M$. Then the scalar product on $\bigwedge^k T_p^* M$ is uniquely determined by the requirement that $E_{i_1}^* \wedge \dots \wedge E_{i_k}^*$, $i_1 < \dots < i_k$, form an orthonormal basis. In local coordinates (x^1, \dots, x^n) it can be written as follows: for $\omega, \eta \in \bigwedge^k T_p^* M$ with

$$\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}, \quad \eta = \sum_{i_1 < \dots < i_k} \eta_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k},$$

where the coefficients $\omega_{i_1 \dots i_k}, \eta_{i_1 \dots i_k}$ are anti-symmetric in the indices $i_1 \dots i_k$, the scalar product is then given as

$$\langle \omega, \eta \rangle_g = \frac{1}{k!} \sum_{\substack{i_1, \dots, i_k \\ j_1, \dots, j_k}} g^{i_1 j_1} \dots g^{i_k j_k} \omega_{i_1 \dots i_k} \eta_{j_1 \dots j_k}. \tag{47}$$

Referring to the norm of 2-forms and 3-forms as used above we note that $\langle S, S \rangle_g = \frac{1}{2} \|S\|^2$, $\langle T, T \rangle_g = \frac{1}{6} \|T\|^2$ for $S \in \bigwedge^2 T_p^* M$ and $T \in \bigwedge^3 T_p^* M$, compare (58). With respect to this scalar product the Hodge $*$ -operator is an isometry, and on a 4-manifold the L^2 -adjoint of d is $\delta = - * d *$ independently of the degree k of the form. For 3-forms δ is given as in (27).

Let g be a Riemannian metric on M . For any k -form η on M we define a symmetric $(2, 0)$ -tensor g^η for $k \geq 2$ by

$$g^\eta(X, Y) = \langle X \lrcorner \eta, Y \lrcorner \eta \rangle_g$$

and for $k = 1$ by

$$g^\eta(X, Y) = X \lrcorner \eta \cdot Y \lrcorner \eta = \eta(X)\eta(Y) \quad \text{for any tangent vectors } X, Y.$$

For $(2, 0)$ -tensors a and h the natural scalar product defined in (57) reads as

$$\langle a, h \rangle = \sum_{\substack{r,s \\ i,j}} a_{ij} g^{ir} g^{js} h_{rs}, \tag{48}$$

in local coordinates, as above.

Lemma 5.6. *Let $(g(t))_t$ be a smooth family of Riemannian metrics on M with $g(0) = g$ and $\dot{g}(0) = h$, and let $k \geq 1$. Then for any k -form η on M we get*

$$\left. \frac{d}{dt} \right|_{t=0} \langle \eta, \eta \rangle_{g(t)} = -\langle g^\eta, h \rangle.$$

Proof. In coordinates we write $\eta = \sum_{i_1 < \dots < i_k} \eta_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$ with $\eta_{i_1 \dots i_k}$ anti-symmetric in the indices. We recall that $\left. \frac{d}{dt} \right|_{t=0} g^{ij}(t) = -\sum_{r,s} g^{ir} h_{rs} g^{js}$ and use (47) to obtain

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \langle \eta, \eta \rangle_{g(t)} &= -\frac{1}{k!} \sum_{r,s} \left(\sum_{\substack{i_1, \dots, i_k \\ j_1, \dots, j_k}} \eta_{i_1 \dots i_k} \eta_{j_1 \dots j_k} \sum_{m=1}^k g^{i_1 j_1} \dots \widehat{g^{i_m j_m}} \dots g^{i_k j_k} g^{i_m r} g^{j_m s} \right) h_{rs} \\ &= -\frac{1}{(k-1)!} \sum_{r,s} \left(\sum_{\substack{i_1, \dots, i_k \\ j_1, \dots, j_k}} \eta_{i_1 \dots i_k} \eta_{j_1 \dots j_k} g^{i_2 j_2} \dots g^{i_k j_k} \right) g^{i_1 r} g^{j_1 s} h_{rs} \\ &= -\sum_{\substack{r,s \\ i_1, j_1}} (g^\eta)_{i_1 j_1} g^{i_1 r} g^{j_1 s} h_{rs} \\ &= -\langle g^\eta, h \rangle, \end{aligned}$$

where we have used the total anti-symmetry of $\eta_{i_1 \dots i_k}$ in the indices for the second equality. \square

Theorem 5.7. Any critical point (M, g, T) of \tilde{I}_{CC} satisfies

$$0 = 3\gamma_1 T - \gamma_2 d\delta T, \tag{49}$$

$$0 = \alpha G^s - \beta B^s + \gamma_1 \left(-6g^{*T} + \frac{1}{2} \|T\|^2 g \right) - \gamma_2 \left(-2g^{d*T} + \frac{1}{2} \|\delta T\|^2 g \right) \tag{50}$$

where $G^s = \text{ric}^s - \frac{1}{2} R^s g$ is the Einstein tensor of the metric g and B^s denotes its Bach tensor (for a definition see [4, (4.77)]).

Proof. Let (M, g, T) be a critical point. We consider an arbitrary variation $T(t)$ of 3-forms with $T(0) = T$ and $\dot{T}(0) = \tau$. Then we have

$$\begin{aligned} 0 &= \frac{d}{dt} \Big|_{t=0} \int_M (\gamma_1 \|T(t)\|^2 - \gamma_2 \|\delta T(t)\|^2) \text{dvol} \\ &= \frac{d}{dt} \Big|_{t=0} \int_M (6\gamma_1 \langle T(t), T(t) \rangle_g - 2\gamma_2 \langle \delta T(t), \delta T(t) \rangle_g) \text{dvol} \\ &= \int_M \langle 3\gamma_1 T - \gamma_2 d\delta T, 4\tau \rangle_g \text{dvol} \end{aligned}$$

since d is the adjoint of δ . As τ can be chosen arbitrarily we have established (49).

Now we fix T and consider an arbitrary variation $g(t)$ of Riemannian metrics with $g(0) = g$ and $\dot{g}(0) = h$. In the following we label any object which depends on $g(t)$. First we note that $\frac{d}{dt} \Big|_{t=0} \text{dvol}_{g(t)} = \langle g, h \rangle \text{dvol}_g$ and by help of Lemma 5.6 we compute

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \int_M \|T\|_{g(t)}^2 \text{dvol}_{g(t)} \\ = \frac{d}{dt} \Big|_{t=0} \int_M 6 \langle T, T \rangle_{g(t)} \text{dvol}_{g(t)} \end{aligned}$$

$$\begin{aligned}
 &= \frac{d}{dt} \Big|_{t=0} \int_M 6 \langle *_{g(t)} T, *_{g(t)} T \rangle_{g(t)} \, \text{dvol}_{g(t)} \\
 &= 3 \int_M \left(\langle -2g^{*g} T + \langle *_{g(t)} T, *_{g(t)} T \rangle_{g(t)} g, h \rangle + 4 \left\langle *_{g(t)} T, \frac{d}{dt} \Big|_{t=0} (*_{g(t)} T) \right\rangle_g \right) \, \text{dvol}_g \\
 &= \int_M \left(\left\langle -6g^{*g} T + \frac{1}{2} \|T\|^2_{g(t)}, h \right\rangle + 12 \left\langle *_{g(t)} T, \frac{d}{dt} \Big|_{t=0} (*_{g(t)} T) \right\rangle_g \right) \, \text{dvol}_g. \tag{51}
 \end{aligned}$$

Now we calculate

$$\begin{aligned}
 \frac{d}{dt} \Big|_{t=0} \int_M \|\delta_{g(t)} T\|^2_{g(t)} \, \text{dvol}_{g(t)} &= \frac{d}{dt} \Big|_{t=0} \int_M 2 \langle \delta_{g(t)} T, \delta_{g(t)} T \rangle_{g(t)} \, \text{dvol}_{g(t)} \\
 &= \frac{d}{dt} \Big|_{t=0} \int_M 2 \langle d *_{g(t)} T, d *_{g(t)} T \rangle_{g(t)} \, \text{dvol}_{g(t)} \\
 &= \int_M \left(\langle -2g^{d*g} T + \langle d *_{g(t)} T, d *_{g(t)} T \rangle_{g(t)} g, h \rangle \right. \\
 &\quad \left. + 4 \left\langle \delta_g d *_{g(t)} T, \frac{d}{dt} \Big|_{t=0} (*_{g(t)} T) \right\rangle_g \right) \, \text{dvol}_g \\
 &= \int_M \left(\left\langle -2g^{d*g} T + \frac{1}{2} \|\delta_g T\|^2_{g(t)}, h \right\rangle \right. \\
 &\quad \left. + 12 \frac{\gamma_1}{\gamma_2} \left\langle *_{g(t)} T, \frac{d}{dt} \Big|_{t=0} (*_{g(t)} T) \right\rangle_g \right) \, \text{dvol}_g \tag{52}
 \end{aligned}$$

where we have inserted $3\gamma_1 *_{g(t)} T = \gamma_2 \delta_g d *_{g(t)} T$ which we obtained from (49).

Finally [4, Proposition 4.17] and [4, (4.77)] tell us that

$$\frac{d}{dt} \Big|_{t=0} \int_M (-\alpha R^{g(t)} - \beta \|C^{g(t)}\|^2_{g(t)}) \, \text{dvol}_{g(t)} = \int_M (\alpha G^g - \beta B^g, h) \, \text{dvol}_g. \tag{53}$$

Combining (51), (52) and (53) gives the assertion (50). \square

Remark 5.8. a) Eq. (49) is a Proca equation for a 3-form. This suggests a physical interpretation of torsion as a massive vector boson. This feature has been observed earlier in the Lorentzian context, see [29], and appears to be natural for dynamical Lagrangians of the torsion.

b) Equivalently, the Proca equation (49) can be expressed as $\Delta T = \frac{\gamma_2}{3\gamma_1} T$ under the condition that $dT = 0$.

Remark 5.9. Ricci flat manifolds (M, g) with $T = 0$ are critical points of \tilde{I}_{CC} . This follows from the fact that Ricci flat manifolds have vanishing Bach tensors (see [4, Proposition 4.78]).

Finding solutions for the equations of motions (49) and (50) with $T \neq 0$ is a challenge. The following lemmas show that classes of warped products with special choices of T can be excluded.

Lemma 5.10. *Let (N, h) be a compact oriented 3-dimensional Riemannian manifold with constant curvature, and let $f : S^1 \rightarrow (0, \infty)$ be some smooth function on the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Consider $M = S^1 \times N$ equipped with the warped product metric $g = dt^2 \oplus f(t)^2h$. Let $\tau : M \rightarrow \mathbb{R}$ be a smooth function and set the 3-form $T = \tau \cdot \pi^* \text{dvol}_{(N,h)}$ where $\text{dvol}_{(N,h)}$ is the Riemannian volume form of (N, h) and $\pi : S^1 \times N \rightarrow N$ is the canonical projection, and let this triple (M, g, T) solve the equations of motions (49) and (50). Then the torsion is zero: $T = 0$.*

Proof. As (N, h) is locally conformally flat, so is the warped product (M, g) . Therefore the Bach tensor of (M, g) vanishes, $B^g = 0$. In order to get the Einstein tensor G^g we need to calculate the curvatures of M . By X, Y, Z we denote vectors tangent to the leaves $\{t\} \times N$, $\frac{\partial}{\partial t}$ is the unit normal vector field. For the Levi-Civita connection we obtain

$$\nabla_X^g \frac{\partial}{\partial t} = \frac{\dot{f}(t)}{f(t)} X, \quad \nabla_{\frac{\partial}{\partial t}}^g \frac{\partial}{\partial t} = 0, \quad \nabla_X^g Y = \nabla_X^h Y - \dot{f}(t) f(t) h(X, Y) \frac{\partial}{\partial t},$$

where X, Y are also considered as tangent vectors fields of (N, h) and ∇^h is the corresponding Levi-Civita connection. The Riemann tensor of ∇^g is given by

$$\begin{aligned} \text{Riem}^g(X, Y)Z &= \text{Riem}^h(X, Y)Z + \left(\frac{\dot{f}(t)}{f(t)}\right)^2 \cdot \{g(X, Z)Y - g(Y, Z)X\}, \\ \text{Riem}^g\left(X, \frac{\partial}{\partial t}\right)Y &= \frac{\ddot{f}(t)}{f(t)} g(X, Y) \frac{\partial}{\partial t}, \\ \text{Riem}^g\left(X, \frac{\partial}{\partial t}\right) \frac{\partial}{\partial t} &= -\frac{\ddot{f}(t)}{f(t)} X. \end{aligned}$$

From that we get the Ricci curvature and the scalar curvature

$$\begin{aligned} \text{ric}^g &= \left(-3 \frac{\ddot{f}(t)}{f(t)}\right) dt^2 \oplus (\text{ric}^h - (\ddot{f}(t) f(t) + 2(\dot{f}(t))^2)h), \\ R^g &= \frac{1}{f(t)^2} (R^h - 6\ddot{f}(t) f(t) - 6(\dot{f}(t))^2). \end{aligned}$$

As (N, h) is assumed to have constant curvature there is a $\kappa \in \mathbb{R}$ such that $\text{ric}^h = 2\kappa h$ and $R^h = 6\kappa$. Hence the Einstein tensor of (M, g) is

$$G^g = \frac{3}{f(t)^2} ((\dot{f}(t))^2 - \kappa) dt^2 \oplus (2\ddot{f}(t) f(t) + (\dot{f}(t))^2 - \kappa) h. \tag{54}$$

Next, we consider normal coordinates (x^1, x^2, x^3) of N about p such that $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}$ form an orthonormal basis of $T_p N$ with respect to h . For the volume distortion \sqrt{h} , given by $\sqrt{h}(x) = \sqrt{\det(h_{ij}(x))}$, we get in p :

$$\sqrt{h} = 1, \quad \frac{\partial}{\partial x^i} \sqrt{h} = 0, \quad \sum_{i=1}^3 \frac{\partial^2}{(\partial x^i)^2} \sqrt{h} = -\frac{1}{3} R^h(p) = -\frac{1}{3} \kappa.$$

Now we take the product chart (t, x^1, x^2, x^3) of M . In these coordinates the torsion 3-form T reads as $\tau(t, x^1, x^2, x^3) \sqrt{h}(x^1, x^2, x^3) dx^1 \wedge dx^2 \wedge dx^3$. From (49) we get $dT = 0$ which implies that $\frac{\partial \tau}{\partial t} \equiv 0$.

As dx^1, dx^2, dx^3 is an orthonormal basis of T_p^*N w.r.t. h , we get that $dt, f dx^1, f dx^2, f dx^3$ form an orthonormal basis of $T_{(t,p)}^*M$ w.r.t. g . Hence, we get

$$\|T\|_g^2 = 6\langle T, T \rangle = \frac{6\tau^2}{f^6}. \tag{55}$$

In (t, p) we get $*T = \tau(x^1, x^2, x^3) \frac{1}{f(t)^3} dt$ and thus

$$g^{*T} = \frac{\tau^2}{f(t)^6} dt^2. \tag{56}$$

Furthermore we obtain $d * T = \frac{1}{f(t)^3} \sum_{i=1}^3 \frac{\partial \tau}{\partial x^i} dx^i \wedge dt$, and therefore

$$\begin{aligned} g^{d*T} \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) &= \frac{1}{f(t)^6} \|\text{grad}^g \tau\|_g^2, \\ g^{d*T} \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i} \right) &= 0, \\ g^{d*T} \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) &= \frac{1}{f(t)^6} d\tau \left(\frac{\partial}{\partial x^i} \right) \cdot d\tau \left(\frac{\partial}{\partial x^j} \right). \end{aligned}$$

If we now assume that Eq. (50) holds, we observe that after restricting the occurring $(2, 0)$ -tensors to TN every ingredient is a multiple of h except g^{d*T} . So (50) can only hold if $d\tau = 0$. Therefore the function τ is constant, and $d * T = 0$ and so $g^{d*T} = 0$. Then we decompose (50) into its dt^2 -component and its h -component:

$$\begin{aligned} \frac{3\alpha}{f(t)^2} ((\dot{f}(t))^2 - \kappa) &= \frac{6\gamma_1 \tau^2}{f(t)^6} - \frac{6\gamma_1 \tau^2}{2f(t)^6}, \\ \alpha(2\ddot{f}(t)f(t) + (\dot{f}(t))^2 - \kappa) &= -\frac{6\gamma_1 \tau^2}{2f(t)^4}. \end{aligned}$$

We divide the first equation by 3, the second one by $(f(t))^2$, and subtract the first from the second. We get:

$$\ddot{f}(t) = -\frac{\gamma_1 \tau^2}{\alpha} \cdot \frac{1}{f(t)^5}.$$

Since $f > 0$ and $\alpha > 0$ we conclude that $\ddot{f} \geq 0$ and therefore \dot{f} increases monotonically. On the other hand \dot{f} being a function on $S^1 = \mathbb{R}/\mathbb{Z}$ is periodic. Therefore \dot{f} is constant and hence τ is zero. \square

Remark 5.11. It should be noted that with the ansatz in Lemma 5.10 one can obtain restrictions on the geometry of (N, h) pointwise from Eq. (49). Namely, in the normal coordinates from above one finds

$$d\delta T = -\tau \sum_{i=1}^3 \frac{\partial^2}{(\partial x^i)^2} \sqrt{h} = \frac{1}{3} \kappa T.$$

By (49) we have $\frac{1}{3} \kappa = \frac{3\gamma_1}{\gamma_2} > 0$, therefore (N, h) is a spherical space form.

Remark 5.12. If we now consider formally the same equations of motions for Lorentzian manifolds, and if we admit also non-compact manifolds as solutions, the argument from the proof of Lemma 5.10 cannot discard the Robertson–Walker ansatz $M = \mathbb{R} \times N$ with $g = -dt^2 \oplus f(t)^2 h$ and $T = \tau \cdot \pi^* \text{dvol}_{(N,h)}$ because in the above proof the compactness of S^1 is essential.

One could argue that in the above examples the torsion T is not dynamical. In the last example we consider a situation where torsion may be dynamical, and we show that the torsion vanishes by other reasons.

Lemma 5.13. Let (N, h) be the flat torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$, let $f : S^1 \rightarrow (0, \infty)$ be some smooth function. Consider $M = S^1 \times N$ equipped with the warped product metric $g = dt^2 \oplus f(t)^2 h$. Let $\tau : S^1 \rightarrow (0, \infty)$ be a smooth function, and let

$$T = \tau(t)(dx^1 \wedge dx^2 + dx^2 \wedge dx^3 + dx^3 \wedge dx^1) \wedge dt.$$

Assume that the triple (M, g, T) solves the equations of motions (49) and (50). Then the torsion is zero: $T = 0$.

Proof. For the T as above we find

$$\begin{aligned} *T &= \frac{\tau(t)}{f(t)}(dx^3 + dx^1 + dx^2), \\ d*T &= \frac{f(t)\dot{\tau}(t) - \tau(t)\dot{f}(t)}{f(t)^2} dt \wedge (dx^3 + dx^1 + dx^2). \end{aligned}$$

Furthermore we get $\|T\|^2 = 18 \cdot \left(\frac{\tau(t)}{f(t)^2}\right)^2$ and $\|\delta T\|^2 = \frac{2}{f(t)^6} \cdot (f(t)\dot{\tau}(t) - \dot{f}(t)\tau(t))^2$ and

$$\begin{aligned} g^{*T} \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) &= 0, \\ g^{d*T} \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) &= 3 \cdot \frac{(f(t)\dot{\tau}(t) - \tau(t)\dot{f}(t))^2}{f(t)^6}. \end{aligned}$$

Now we insert twice $\frac{\partial}{\partial t}$ into the $(2, 0)$ -tensors of (50) and use the specific form of the Einstein tensor (54). This yields

$$0 = 3\alpha \cdot \frac{(\dot{f}(t))^2}{(f(t))^2} + 9\gamma_1 \cdot \frac{(\tau(t))^2}{(f(t))^4} + 5\gamma_2 \cdot \frac{(f(t)\dot{\tau}(t) - \tau(t)\dot{f}(t))^2}{(f(t))^6}.$$

Since $\alpha, \gamma_1, \gamma_2 > 0$ each summand is nonnegative, therefore each term vanishes individually, in particular the second one. From this we conclude $\tau = 0$. \square

Remark 5.14. In the Lorentzian setting actions similar to \tilde{I}_{CC} have already been considered and some cosmological consequences for possible critical points with non-vanishing torsion have been discussed (see e.g. [24] or [30] and the references therein).

Acknowledgments

The authors appreciate funding by the Deutsche Forschungsgemeinschaft, in particular by the SFB *Raum-Zeit-Materie*. We would like to thank Christian Bär and Thomas Schücker for their support and helpful discussions.

Appendix A. Proofs of Proposition 3.1, Theorem 3.4 and Lemma 3.6

We consider a manifold M equipped with a Riemannian metric g . For $(k, 0)$ -tensor fields S, T on M one has pointwise the natural scalar product

$$\langle S, T \rangle = \sum_{i_1, \dots, i_k} S(E_{i_1}, \dots, E_{i_k}) \cdot T(E_{i_1}, \dots, E_{i_k}), \tag{57}$$

where $(E_i)_i$ is an orthonormal basis of the tangent space. This coincides with the definition given in (8) for $(3, 0)$ -tensors. If S and T are k -forms there is another natural scalar product, as defined in (47), given by

$$\begin{aligned} \langle S, T \rangle_g &= \sum_{i_1 < \dots < i_k} S(E_{i_1}, \dots, E_{i_k}) \cdot T(E_{i_1}, \dots, E_{i_k}) \\ &= \frac{1}{k!} \sum_{i_1, \dots, i_k} S(E_{i_1}, \dots, E_{i_k}) \cdot T(E_{i_1}, \dots, E_{i_k}) = \frac{1}{k!} \langle S, T \rangle. \end{aligned} \tag{58}$$

Now let ∇^g denote the Levi-Civita connection on the tangent bundle. We fix some 3-form T on M and some $s \in \mathbb{R}$, and we are studying the connection ∇ with totally anti-symmetric torsion as in (22).

As in Section 2 we fix some point $p \in M$, and we extend any tangent vectors $X, Y, Z, W \in T_p M$ to vector fields again denoted by X, Y, Z, W being synchronous in p .

We recall the Ricci decomposition for Riem^g (compare [4, Chapter 1.G]): Let the trace-free part of the Ricci curvature be denoted by $\text{b}(X, Y) = \text{ric}^g(X, Y) - \frac{1}{n} R^g \langle X, Y \rangle$. Then, one has

$$\begin{aligned} \langle \text{Riem}^g(X, Y)Z, W \rangle &= \frac{1}{n(n-1)} R^g(\langle X, W \rangle \langle Y, Z \rangle - \langle Y, W \rangle \langle X, Z \rangle) \\ &\quad + \frac{1}{n-2} (\mathfrak{b}(X, W) \langle Y, Z \rangle - \mathfrak{b}(Y, W) \langle X, Z \rangle + \langle X, W \rangle \mathfrak{b}(Y, Z) \\ &\quad - \langle Y, W \rangle \mathfrak{b}(X, Z)) + \langle C(X, Y)Z, W \rangle, \end{aligned} \tag{59}$$

where the (3, 1)-tensor C is the Weyl tensor of the Riemannian metric g . The Weyl tensor possesses all the symmetries of the Riemann tensor Riem^g (e.g. the Bianchi identity holds), and in addition the contraction of $\langle C(X, Y)Z, W \rangle$ taken over any two slots is zero. Furthermore, it should be noted that this composition into scalar curvature, trace-free Ricci curvature and Weyl curvature is orthogonal with respect to the usual scalar product in the space of (4, 0)-tensors (see [4, Theorem 1.114]).

A.1. Proof of Proposition 3.1

Proof of Proposition 3.1. First, consider some Riemannian manifold (M, g) of arbitrary dimension n with Levi-Civita connection ∇^g . We fix some 3-form T on M , and we are studying the connection ∇ defined as in (22).

The symmetric part of riem is

$$\begin{aligned} \text{riem}^S(X, Y, Z, W) &= \frac{1}{2} (\text{riem}(X, Y, Z, W) + \text{riem}(Z, W, X, Y)) \\ &= \text{riem}^g(X, Y, Z, W) + \frac{S}{2} dT(X, Y, Z, W) \\ &\quad + s^2 (\langle T(X, Z, \cdot)^\sharp, T(Y, W, \cdot)^\sharp \rangle - \langle T(Y, Z, \cdot)^\sharp, T(X, W, \cdot)^\sharp \rangle), \end{aligned} \tag{60}$$

where we have used (23) and (26).

In the following we will abbreviate

$$\begin{aligned} C_{ijkl} &= \langle C(E_i, E_j)E_k, E_l \rangle, & \text{riem}^{(g/S/A)}_{ijkl} &= \text{riem}^{(g/S/A)}(E_i, E_j, E_k, E_l), \\ \text{ric}^{(g/S/A)}_{ij} &= \text{ric}^{(g/S/A)}(E_i, E_j), & \mathfrak{b}_{ij} &= \mathfrak{b}(E_i, E_j), \\ T_{ij} &= T(E_i, E_j, \cdot)^\sharp, & dT_{ijkl} &= dT(E_i, E_j, E_k, E_l) \end{aligned}$$

for all indices i, j, k, l .

We will calculate $\|\text{riem}^S\|^2 = \sum_{i,j,k,l=1}^n (\text{riem}^S_{ijkl})^2$ by means of (60). Three cross terms appear, one of them vanishes in the case of arbitrary dimension n : We choose some orthonormal frame E_1, \dots, E_n , which is defined on some neighbourhood of p and is synchronous in p . Using the Bianchi identity we compute

$$0 = \frac{1}{3} \sum_{i,j,k,l=1}^n (\text{riem}^g_{ijkl} + \text{riem}^g_{jkil} + \text{riem}^g_{kijl}) dT_{ijkl} = \sum_{i,j,k,l=1}^n (\text{riem}^g_{ijkl}) dT_{ijkl}. \tag{61}$$

From now on we restrict to the case of dimension $n = 4$. We note that $dT_{ijkl} \neq 0$ is only possible if $i, j, k, l \in \{1, \dots, 4\}$ are pairwise distinct. In that case one has $\langle T_{ij}, T_{kl} \rangle =$

$\sum_{m=1}^4 T(E_i, E_j, E_m)T(E_k, E_l, E_m) = 0$ as any $m \in \{1, \dots, 4\}$ is equal to one of i, j, k, l . Hence we get

$$0 = \sum_{i,j,k,l=1}^4 dT_{ijkl} \cdot (\langle T_{ik}, T_{jl} \rangle - \langle T_{jk}, T_{il} \rangle). \tag{62}$$

In order to identify the third cross term we impose the Ricci decomposition (59) which now reads as

$$\text{riem}_{ijkl}^g = \frac{1}{12}R^g(\delta_{ij}\delta_{jk} - \delta_{jl}\delta_{ik}) + \frac{1}{2}(\mathbf{b}_{il}\delta_{jk} - \mathbf{b}_{jl}\delta_{ik} + \delta_{il}\mathbf{b}_{jk} - \delta_{jl}\mathbf{b}_{ik}) + C_{ijkl}$$

and we get

$$\begin{aligned} \sum_{i,j,k,l} (\text{riem}_{ijkl}^g)(\langle T_{ik}, T_{jl} \rangle - \langle T_{jk}, T_{il} \rangle) &= \frac{1}{12}R^g \sum_{i,j} (\langle T_{ij}, T_{ji} \rangle + \langle T_{ji}, T_{ij} \rangle) \\ &\quad + \frac{1}{2} \left(\sum_{i,j,l} \mathbf{b}_{il} \langle T_{ij}, T_{jl} \rangle + \sum_{i,j,l} \mathbf{b}_{jl} \langle T_{ji}, T_{il} \rangle \right. \\ &\quad \left. + \sum_{i,j,k} \mathbf{b}_{jk} \langle T_{ik}, T_{ji} \rangle + \sum_{i,j,k} \mathbf{b}_{ik} \langle T_{jk}, T_{ij} \rangle \right) \\ &\quad + \sum_{i,j,k,l} C_{ijkl} (\langle T_{ik}, T_{jl} \rangle - \langle T_{jk}, T_{il} \rangle) \\ &= -\frac{1}{6}R^g \|T\|^2 + 2\mathbf{B}(T) + \mathbf{C}(T) \end{aligned}$$

where we have set

$$\begin{aligned} \mathbf{B}(T) &= \sum_{i,j,k} \mathbf{b}_{ik} \langle T_{ij}, T_{jk} \rangle = \sum_{i,j,k} \text{ric}_{ik}^g \langle T_{ij}, T_{jk} \rangle - \frac{1}{4}R^g \sum_{i,j} \langle T_{ij}, T_{ji} \rangle \\ &= \sum_{i,j,k} \text{ric}_{ik}^g \langle T_{ij}, T_{jk} \rangle + \frac{1}{4}R^g \|T\|^2, \end{aligned} \tag{63}$$

$$\mathbf{C}(T) = \sum_{i,j,k,l} C_{ijkl} (\langle T_{ik}, T_{jl} \rangle - \langle T_{jk}, T_{il} \rangle). \tag{64}$$

It follows that

$$\begin{aligned} \|\text{riem}^S\|^2 &= \|\text{riem}^g\|^2 + \frac{1}{4}s^2 \|dT\|^2 - \frac{1}{3}s^2 R^g \|T\|^2 + 4s^2 \mathbf{B}(T) + s^2 \mathbf{C}(T) \\ &\quad + s^4 \sum_{i,j,k,l=1}^4 (\langle T_{ik}, T_{jl} \rangle - \langle T_{jk}, T_{il} \rangle)^2 \\ &= \|\text{riem}^g\|^2 + \frac{1}{4}s^2 \|dT\|^2 - \frac{1}{3}s^2 R^g \|T\|^2 + 4s^2 \mathbf{B}(T) + \frac{1}{3}s^4 \|T\|^4 \end{aligned}$$

where for the last step we use the following technical Lemmas A.1 and A.2. \square

Table 1

$(T_a)_{ij} := T_a(E_i, E_j, \cdot)^{\sharp}$ for $a \in \{1, 2\}$.

	j	1	2	3	4
i					
$(T_1)_{ij}$:	1	0	$-t_1 E_3$	$t_1 E_2$	0
	2	$t_1 E_3$	0	$-t_1 E_1$	0
	3	$-t_1 E_2$	$t_1 E_1$	0	0
	4	0	0	0	0

	j	1	2	3	4
i					
$(T_2)_{ij}$:	1	0	0	0	0
	2	0	0	$-t_2 E_4$	$t_2 E_3$
	3	0	$t_2 E_4$	0	$-t_2 E_2$
	4	0	$-t_2 E_3$	$t_2 E_2$	0

Lemma A.1. Let M be a 4-dimensional Riemannian manifold and $p \in M$. For any $T \in \wedge^3 T_p^* M$ we consider $C(T)$ as defined in (64). Then one has $C(T) = 0$.

Proof. Let E_1, E_2, E_3, E_4 be an orthogonal basis of $T_p M$. For $T, S \in \wedge^3 T_p^* M$ we define

$$c(T, S) = 2 \cdot \sum_{i,j,k,l=1}^4 C_{ijkl} \langle T_{ik}, S_{jl} \rangle.$$

The Weyl tensor satisfies $C_{ijkl} = C_{jilk}$, and hence c is a symmetric bilinear form. For $T \in \wedge^3 T_p^* M$ we have

$$\begin{aligned} c(T, T) &= \sum_{i,j,k,l} C_{ijkl} \langle T_{ik}, T_{jl} \rangle - \sum_{i,j,k,l} C_{jikl} \langle T_{ik}, T_{jl} \rangle \\ &= \sum_{i,j,k,l} C_{jikl} (\langle T_{ik}, T_{jl} \rangle - \langle T_{jk}, T_{il} \rangle) = C(T). \end{aligned}$$

Hence, the lemma will be proved if we show that $c \equiv 0$. We will check this on the basis of $\wedge^3 T_p^* M$ consisting of $E_1^b \wedge E_2^b \wedge E_3^b, E_2^b \wedge E_3^b \wedge E_4^b, E_1^b \wedge E_3^b \wedge E_4^b, E_1^b \wedge E_2^b \wedge E_4^b$.

For $t_1, t_2 \in \mathbb{R}$ we consider $T_1, T_2 \in \wedge^3 T_p^* M$ defined by

$$T_1 := t_1 \cdot E_1^b \wedge E_2^b \wedge E_3^b, \quad T_2 := t_2 \cdot E_2^b \wedge E_3^b \wedge E_4^b.$$

We will verify that $c(T_1, T_1) = 0$ and $c(T_1, T_2) = 0$, this will suffice because the computations are the same if one inserts any elements of the above basis into c . For $a \in \{1, 2\}$ Table 1 gives $(T_a)_{ij} := T_a(E_i, E_j, \cdot)^{\sharp}$ explicitly.

Using Table 1 we compute

$$c(T_1, T_1) = \sum_{i,j,k,l=1}^4 C_{ijkl} \langle T_{ik}, T_{jl} \rangle = \sum_{i,j,k,l=1}^3 C_{ijkl} t_1^2 (\delta_{ij} \delta_{kl} - \delta_{il} \delta_{jk}) = -t_1^2 \sum_{i,j=1}^3 C_{ijji}. \tag{65}$$

The Weyl tensor is trace-free in any pair of indices. Hence for any $i \in \{1, 2, 3\}$ we get $-C_{i44i} = \sum_{j=1}^3 C_{ijji}$. Inserting this into (65) and using $C_{4444} = 0$ we obtain

$$c(T_1, T_1) = t_1^2 \sum_{i=1}^4 C_{i44i} = 0$$

as the Weyl tensor is trace-free in any pair of indices.

Table 2

$(T_a)_{ij} := T_a(E_i, E_j, \cdot)^\sharp$ for $a \in \{3, 4\}$.

	j	1	2	3	4
i					
$(T_3)_{ij}$:	1	0	0	$-t_3 E_4$	$t_3 E_3$
	2	0	0	0	0
	3	$t_3 E_4$	0	0	$-t_3 E_1$
	4	$-t_3 E_3$	0	$t_3 E_1$	0

	j	1	2	3	4
i					
$(T_4)_{ij}$:	1	0	$-t_4 E_4$	0	$t_4 E_2$
	2	$t_4 E_4$	0	0	$-t_4 E_1$
	3	0	0	0	0
	4	$-t_4 E_2$	$t_4 E_1$	0	0

With Table 1 we obtain

$$\begin{aligned}
 c(T_1, T_2) &= 2 \sum_{i,j,k,l=1}^4 C_{ijkl} \langle (T_1)_{ik}, (T_2)_{jl} \rangle \\
 &= -2t_1 t_2 (C_{1224} + C_{2412} + C_{1334} + C_{3413}) \\
 &= -4t_1 t_2 (C_{1224} + C_{1334}) \\
 &= -4t_1 t_2 \sum_{i=1}^4 C_{1i i 4} \\
 &= 0
 \end{aligned}$$

where we used the symmetry $C_{ijkl} = C_{klij}$ of the Weyl tensor in the second step and $C_{1114} = C_{1444} = 0$ in the third step, the last step is achieved as the Weyl tensor is trace-free. \square

Lemma A.2. *Let M be a Riemannian manifold of dimension 4, let $p \in M$ and let E_1, E_2, E_3, E_4 be an orthonormal basis of $T_p M$. Then for any $T \in \wedge^3 T_p^* M$ one has*

$$\sum_{i,j,k,l=1}^4 (\langle T_{ik}, T_{jl} \rangle - \langle T_{jk}, T_{il} \rangle)^2 = \frac{1}{3} \|T\|^4.$$

Proof. Choosing the parameters properly we have $T = T_1 + T_2 + T_3 + T_4$ with T_1 and T_2 as in Lemma A.1 and

$$T_3 := t_3 \cdot E_1^b \wedge E_3^b \wedge E_4^b, \quad T_4 := t_4 \cdot E_1^b \wedge E_2^b \wedge E_4^b.$$

Defining $(T_a)_{ij}$ as in Lemma A.1 we summarise $(T_3)_{ij}$ and $(T_4)_{ij}$ in Table 2.

In order to calculate $\|T\|^4 = (\|T\|^2)^2$ we note that

$$\sum_{i,j=1}^4 \langle (T_a)_{ij}, (T_b)_{ij} \rangle = \begin{cases} 0 & \text{if } a \neq b, \\ 6t_a^2 & \text{if } a = b \end{cases}$$

and therefore

$$\|T\|^2 = \sum_{i,j=1}^4 \langle T_{ij}, T_{ij} \rangle = \sum_{i,j,a=1}^4 \langle (T_a)_{ij}, (T_a)_{ij} \rangle = 6 \sum_{a=1}^4 t_a^2. \tag{66}$$

Next, we also want to express $\sum_{i,j,k,l=1}^4 (\langle T_{ik}, T_{jl} \rangle - \langle T_{jk}, T_{il} \rangle)^2$ in terms of t_1, \dots, t_4 .

Table 3
The summands $(\langle T_{ik}, T_{jl} \rangle - \langle T_{jk}, T_{il} \rangle)^2$ for $i < j, k < l$.

$(i, j) \backslash (k, l)$	(1, 2)	(1, 3)	(1, 4)	(2, 3)	(2, 4)	(3, 4)
(1, 2)	$(t_1^2 + t_4^2)^2$	$t_3^2 t_4^2$	$t_1^2 t_3^2$	$t_2^2 t_4^2$	$t_1^2 t_2^2$	0
(1, 3)	$t_3^2 t_4^2$	$(t_1^2 + t_3^2)^2$	$t_1^2 t_4^2$	$t_2^2 t_3^2$	0	$t_1^2 t_2^2$
(1, 4)	$t_1^2 t_3^2$	$t_1^2 t_4^2$	$(t_3^2 + t_4^2)^2$	0	$t_2^2 t_3^2$	$t_2^2 t_4^2$
(2, 3)	$t_2^2 t_4^2$	$t_2^2 t_3^2$	0	$(t_1^2 + t_2^2)^2$	$t_1^2 t_4^2$	$t_1^2 t_3^2$
(2, 4)	$t_1^2 t_2^2$	0	$t_2^2 t_3^2$	$t_1^2 t_4^2$	$(t_2^2 + t_4^2)^2$	$t_3^2 t_4^2$
(3, 4)	0	$t_1^2 t_2^2$	$t_2^2 t_4^2$	$t_1^2 t_3^2$	$t_3^2 t_4^2$	$(t_2^2 + t_3^2)^2$

We note that each summand $(\langle T_{ik}, T_{jl} \rangle - \langle T_{jk}, T_{il} \rangle)^2$ remains the same after interchanging i and j and is zero if $i = j$. Noticing the same for the indices k and l , we restrict ourselves to the case $i < j, k < l$.

For $(i, j) = (1, 2)$ and $(k, l) = (1, 2)$ we get

$$(\langle T_{11}, T_{22} \rangle - \langle T_{21}, T_{12} \rangle)^2 = (\langle T_{12}, T_{12} \rangle)^2 = ((\langle T_{11} \rangle_{12}, \langle T_{11} \rangle_{12}) + (\langle T_{44} \rangle_{12}, \langle T_{44} \rangle_{12}))^2 = (t_1^2 + t_4^2)^2,$$

where we have used that

$$\langle (T_a)_{ij}, (T_b)_{ij} \rangle = \begin{cases} 0 & \text{if } a \neq b, \\ t_a^2 & \text{if } a = b. \end{cases}$$

For $(i, j) = (1, 2)$ and $(k, l) = (1, 3)$ we find

$$(\langle T_{11}, T_{23} \rangle - \langle T_{21}, T_{13} \rangle)^2 = (\langle T_{12}, T_{13} \rangle)^2 = ((\langle T_{44} \rangle_{12}, \langle T_{33} \rangle_{13}))^2 = t_3^2 t_4^2,$$

and for $(i, j) = (1, 2)$ and $(k, l) = (3, 4)$ we have

$$(\langle T_{13}, T_{24} \rangle - \langle T_{23}, T_{14} \rangle)^2 = 0.$$

The same considerations apply to the other summands, and we give the results for the cases $i < j, k < l$ in Table 3.

Now we drop the condition $i < j, k < l$, and we get a factor 4 when we add up all summands:

$$\begin{aligned} \sum_{i,j,k,l=1}^4 (\langle T_{ik}, T_{jl} \rangle - \langle T_{jk}, T_{il} \rangle)^2 &= 4 \sum_{a \neq b} (t_a^2 + t_b^2)^2 + 16 \sum_{a \neq b} t_a^2 t_b^2 \\ &= 12 \sum_{a=1}^4 t_a^4 + 8 \sum_{a \neq b} t_a^2 t_b^2 + 16 \sum_{a \neq b} t_a^2 t_b^2 \\ &= 12 \left(\sum_{a=1}^4 t_a^4 + 2 \sum_{a \neq b} t_a^2 t_b^2 \right) \end{aligned}$$

$$= 12 \left(\sum_{a=1}^4 t_a^2 \right)^2,$$

from which in combination with (66) the claim follows. \square

Lemma A.3. *The square of the anti-symmetric part of the Riemann tensor of ∇ is pointwise given by*

$$\begin{aligned} \|\text{riem}^A\|^2 &= s^2 \sum_{i,j,k,l=1}^n ((\nabla_{E_i}^g T)(E_j, E_k, E_l))^2 \\ &\quad + s^2 \sum_{i,j,k,l=1}^n (\nabla_{E_i}^g T)(E_j, E_k, E_l)(\nabla_{E_j}^g T)(E_i, E_k, E_l). \end{aligned}$$

Proof. The anti-symmetric part of riem is then given by

$$\begin{aligned} \text{riem}^A(X, Y, Z, W) &= \frac{1}{2}(\text{riem}(X, Y, Z, W) - \text{riem}(Z, W, X, Y)) \\ &= \frac{s}{2}((\nabla_X^g T)(Y, Z, W) - (\nabla_Y^g T)(X, Z, W) \\ &\quad - (\nabla_Z^g T)(X, Y, W) + (\nabla_W^g T)(X, Y, Z)). \end{aligned} \tag{67}$$

The lemma follows by direct calculation. \square

This shows in particular that riem^A does only depend on the torsion T and not on the Riemannian curvature Riem^g .

A.2. *The square of the norm of the Ricci curvature*

Now we decompose the Ricci curvature into its symmetric and its anti-symmetric components

$$\text{ric}(X, Y) = \text{ric}^S(X, Y) + \text{ric}^A(X, Y).$$

With (24) we have

$$\text{ric}^S(X, Y) = \frac{1}{2}(\text{ric}(X, Y) + \text{ric}(Y, X)) = \text{ric}^g(X, Y) - s^2 \sum_{i=1}^n \langle T(X, E_i, \cdot)^\sharp, T(Y, E_i, \cdot)^\sharp \rangle \tag{68}$$

and

$$\text{ric}^A(X, Y) = \frac{1}{2}(\text{ric}(X, Y) - \text{ric}(Y, X)) = s \sum_{i=1}^n (\nabla_{E_i}^g T)(X, Y, E_i) = -s\delta T(X, Y). \tag{69}$$

To evaluate $\|\text{ric}\|^2$ of the Ricci curvature (24) we proceed as in the evaluation of $\|\text{Riem}\|^2$ and get

$$\begin{aligned} \|\text{ric}\|^2 &= \sum_{i,j=1}^n (\text{ric}(E_i, E_j))^2 = \sum_{i,j=1}^n \text{ric}_{ij}^2 = \sum_{i,j=1}^n (\text{ric}_{i,j}^S)^2 + \sum_{i,j=1}^n (\text{ric}_{i,j}^A)^2 \\ &= \|\text{ric}^S\|^2 + \|\text{ric}^A\|^2. \end{aligned} \tag{70}$$

We find

Theorem A.4. *Let M be a 4-dimensional manifold with Riemannian metric g and connection ∇ as given in (22). Then the norm of the Ricci tensor of ∇ is given by*

$$\|\text{ric}\|^2 = \|\text{ric}^g\|^2 + \frac{1}{3}s^4\|T\|^4 - \frac{1}{2}s^2R^g\|T\|^2 + 2s^2B(T) + s^2\|\delta T\|^2$$

with $B(T)$ as defined in (63).

Proof. In order to compute the squared norm $\|\text{ric}\|^2$ of the Ricci curvature defined in (24) we proceed as in the evaluation of $\|\text{Riem}\|^2$. With (68) and (69) we get

$$\|\text{ric}\|^2 = \sum_{i,j=1}^n (\text{ric}(E_i, E_j))^2 = \sum_{i,j=1}^n \text{ric}_{ij}^2 = \sum_{i,j=1}^n (\text{ric}_{ij}^S)^2 + \sum_{i,j=1}^n (\text{ric}_{ij}^A)^2.$$

We restrict ourselves again to dimension $n = 4$ and calculate $\|\text{ric}^S\|^2 = \sum_{i,j=1}^n (\text{ric}_{ij}^S)^2$ with (68). We find

$$\begin{aligned} \sum_{i,j=1}^4 (\text{ric}_{ij}^S)^2 &= \sum_{i,j=1}^4 (\text{ric}_{ij}^g)^2 - 2s^2 \sum_{i,j,k=1}^4 \text{ric}_{ij}^g \langle T_{ik}, T_{jk} \rangle + s^4 \sum_{i,j,k,l=1}^4 \langle T_{ik}, T_{jk} \rangle \langle T_{il}, T_{jl} \rangle \\ &= \|\text{ric}^g\|^2 - \frac{1}{2}s^2R^g\|T\|^2 + 2s^2B(T) + \frac{1}{3}s^4\|T\|^4 \end{aligned}$$

where we used Lemmas A.5 and A.6 in the last step. We conclude the proof noting that

$$\sum_{i,j=1}^n (\text{ric}_{ij}^A)^2 = s^2 \sum_{i,j=1}^n (\delta T(E_i, E_j))^2 = s^2\|\delta T\|^2. \quad \square$$

Lemma A.5. *Let M be an n -dimensional Riemannian manifold. For any 3-form T one has*

$$\sum_{i,j,k=1}^n \text{ric}_{ij}^g \langle T_{ik}, T_{jk} \rangle = -B(T) + \frac{1}{n}R^g\|T\|^2.$$

Proof. We use the decomposition of the Ricci curvature into its trace-free part and its trace (59) and find

$$\begin{aligned} \sum_{i,j,k=1}^n \text{ric}_{ij}^g \langle T_{ik}, T_{jk} \rangle &= \sum_{i,j,k=1}^n b(E_i, E_j) \langle T_{ik}, T_{jk} \rangle + \frac{1}{n} \sum_{i,j,k=1}^n R^g \delta_{ij} \langle T_{ik}, T_{jk} \rangle \\ &= - \sum_{i,j,k=1}^n b(E_i, E_j) \langle T_{ik}, T_{kj} \rangle + \frac{1}{n} R^g \sum_{i,k=1}^n \langle T_{ik}, T_{ik} \rangle \\ &= -B(T) + \frac{1}{n} R^g \|T\|^2 \end{aligned}$$

with $B(T)$ as defined in Theorem 3.1. \square

Lemma A.6. *Let M be a 4-dimensional Riemannian manifold. For any 3-form T one has*

$$\sum_{i,j,k,l=1}^4 \langle T_{ik}, T_{jk} \rangle \langle T_{il}, T_{jl} \rangle = \frac{1}{3} \|T\|^4.$$

Proof. We compute:

$$\begin{aligned} &\sum_{i,j,k,l=1}^4 \langle T_{ik}, T_{jk} \rangle \langle T_{il}, T_{jl} \rangle \\ &= \sum_{i,j,k,l,n,m=1}^4 T(E_i, E_k, E_n) T(E_j, E_k, E_n) T(E_i, E_l, E_m) T(E_j, E_l, E_m) \\ &= \sum_{i,j,k,l,n,m=1}^4 T(E_n, E_k, E_i) T(E_m, E_l, E_i) T(E_n, E_k, E_j) T(E_m, E_l, E_j) \\ &= \sum_{k,l,n,m=1}^4 \langle T_{kn}, T_{lm} \rangle^2. \tag{71} \end{aligned}$$

We decompose $T = T_1 + T_2 + T_3 + T_4$ according to the definitions of the T_a in Lemmas A.1 and A.2. Following precisely the arguments of Lemma A.2 we obtain for (71) from the Tables 1 and 2 the summands given in Table 3. It follows that

$$\sum_{k,l,n,m=1}^4 \langle T_{kn}, T_{lm} \rangle^2 = 12 \left(\sum_{a=1}^4 t_a^2 \right)^2.$$

Combining this with (66) and (71) finishes the proof. \square

A.3. Proofs of Theorem 3.4 and Lemma 3.6

Proof of Theorem 3.4. Choose an orthonormal basis E_1, \dots, E_4 of $T_p M$ at a point p and express the curvature 2-form in this basis

$$\begin{aligned} \text{Riem}_{ij} &= \sum_{n,m=1}^4 \text{Riem}_{ij}(E_n, E_m)(E_n)^b \wedge (E_m)^b \\ &= \sum_{n,m=1}^4 \langle \text{Riem}(E_i, E_j)E_n, E_m \rangle (E_n)^b \wedge (E_m)^b \\ &= \sum_{n,m=1}^4 \text{riem}_{ijnm}(E_n)^b \wedge (E_m)^b \end{aligned}$$

where *riem* is the (4, 0)-tensor defined in (29). Now the 4-form **K** reads

$$\begin{aligned} \mathbf{K} &= \frac{1}{32\pi^2} \sum_{i,\dots,p=1}^4 \epsilon_{ijkl} \text{riem}_{ijnm} \text{riem}_{klsp}(E_n)^b \wedge (E_m)^b \wedge (E_s)^b \wedge (E_p)^b \\ &= \frac{1}{32\pi^2} \sum_{i,\dots,p=1}^4 \epsilon_{ijkl} \epsilon_{nm sp} \text{riem}_{ijnm} \text{riem}_{klsp} \text{dvol}. \end{aligned}$$

With the standard equality

$$\epsilon_{ijkl} \epsilon_{nm sp} = \det \begin{pmatrix} \delta_{in} & \delta_{im} & \delta_{is} & \delta_{ip} \\ \delta_{jn} & \delta_{jm} & \delta_{js} & \delta_{jp} \\ \delta_{kn} & \delta_{km} & \delta_{ks} & \delta_{kp} \\ \delta_{ln} & \delta_{lm} & \delta_{ls} & \delta_{lp} \end{pmatrix}$$

and $\text{ric}_{ij} = \text{ric}(E_i, E_j)$ one finds

$$\mathbf{K} = \frac{1}{8\pi^2} \left(R^2 - 4 \sum_{j,k=1}^4 \text{ric}_{ij} \text{ric}_{ji} + \sum_{i,j,k,l=1}^4 \text{riem}_{ijkl} \text{riem}_{klij} \right) \text{dvol}.$$

Decomposing the Riemann and the Ricci curvature into their symmetric and anti-symmetric components we find

$$\begin{aligned} \sum_{j,k=1}^4 \text{ric}_{ij} \text{ric}_{ji} &= \sum_{j,k=1}^4 (\text{ric}_{ij}^S)^2 - \sum_{j,k=1}^4 (\text{ric}_{ij}^A)^2 - \sum_{j,k=1}^4 \text{ric}_{ij}^S \text{ric}_{ij}^A + \sum_{j,k=1}^4 \text{ric}_{ij}^A \text{ric}_{ij}^S \\ &= \|\text{ric}^S\|^2 - \|\text{ric}^A\|^2 \end{aligned}$$

and by the same argument

$$\sum_{i,j,k,l=1}^4 \text{riem}_{ijkl} \text{riem}_{klij} = \|\text{riem}^S\|^2 - \|\text{riem}^A\|^2.$$

Thus we have

$$\mathbf{K} = \frac{1}{8\pi^2} (R^2 - 4\|\text{ric}^S\|^2 + 4\|\text{ric}^A\|^2 + \|\text{riem}^S\|^2 - \|\text{riem}^A\|^2) \text{dvol}. \quad \square$$

Corollary A.7. *Let M be a closed 4-dimensional manifold with Riemannian metric g , ∇ an arbitrary orthogonal connection and ∇^g the Levi-Civita connection. Then*

$$\int_M ((R^g)^2 - 4\|\text{ric}^g\|^2 + \|\text{riem}^g\|^2) \text{dvol} = \int_M (R^2 - 4\|\text{ric}^S\|^2 + 4\|\text{ric}^A\|^2 + \|\text{riem}^S\|^2 - \|\text{riem}^A\|^2) \text{dvol}.$$

Proof. The Euler characteristic is a topological invariant and does not depend on the choice of the connection in the representation of the Euler class. \square

Proof of Lemma 3.6. Let riem , ric and R be the Riemann curvature, the Ricci curvature and the scalar curvature of ∇ . With the decomposition of the norm squared of the symmetric part of the Riemann curvature, Theorem 3.1, the Ricci curvature, Theorem A.4, and the scalar curvature, Eq. (25), we have from Corollary A.7

$$\begin{aligned} 0 &= \int_M (R^2 - (R^g)^2 - 4(\|\text{ric}^S\|^2 - \|\text{ric}^A\|^2 - \|\text{ric}^g\|^2) \\ &\quad + \|\text{riem}^S\|^2 - \|\text{riem}^A\|^2 - \|\text{riem}^g\|^2) \text{dvol} \\ &= \int_M \left(-2s^2 R^g \|T\|^2 + s^4 \|T\|^4 - \frac{4}{3}s^4 \|T\|^4 + 2s^2 R^g \|T\|^2 - 8s^2 \mathbf{B}(T) + 4s^2 \|\delta T\|^2 \right) \text{dvol} \\ &\quad + \int_M \left(\frac{1}{3}s^4 \|T\|^4 + \frac{1}{4}s^2 \|dT\|^2 - \frac{1}{3}s^2 R^g \|T\|^2 + 4s^2 \mathbf{B}(T) - \|\text{riem}^A\|^2 \right) \text{dvol} \\ &= s^2 \int_M \left(4\|\delta T\|^2 - \frac{1}{3}R^g \|T\|^2 + \frac{1}{4}\|dT\|^2 - 4\mathbf{B}(T) - \frac{1}{s^2}\|\text{riem}^A\|^2 \right) \text{dvol}. \end{aligned}$$

For any $s \neq 0$ the assertion of the lemma follows. \square

Remark A.8. At first glance the cancelation of the term $\|T\|^4$ in the proof of Lemma 3.6 might surprise. But it has to vanish because it is the only term scaling in the 4th power in this pre-factor s while all other terms scale quadratically in s .

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