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Zero-divisor graphs of idealizations

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Abstract

We consider zero-divisor graphs of idealizations of commutative rings. Specifically, we look at the preservation, or lack thereof, of the diameter and girth of the zero-divisor graph of a ring when extending to idealizations of the ring.

MSC: 13A99

1. Introduction

The concept of the graph of the zero-divisors of a ring was first introduced by Beck [5] when discussing the coloring of a commutative ring. In his work all elements of the ring were vertices of the graph. Anderson and Naseer used this same concept in [3]. We adopt the approach used by Anderson and Livingston [2] and consider only nonzero zero-divisors as vertices of the graph. Anderson and Livingston, Mulay [10], and DeMeyer and Schneider [6] examined, among other things, the diameter and girth of the zero-divisor graph of a commutative ring. For instance, Anderson and Livingston showed the zero-divisor graph of a commutative ring is connected with diameter less than or equal to three. In addition, they showed that the girth is either infinite or less than or equal to four when R is Artinian and conjectured that this would hold if R was not Artinian. DeMeyer and Schneider, and Mulay proved this conjecture independently, and a short proof of this can be found in [4].

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The authors (with Coykendall) examined the preservation, or lack thereof, of the diameter and girth of the graph of a commutative ring under extensions to polynomial and power series rings in [4]. In this paper we look at the preservation of the diameter and girth under idealizations of commutative rings. Specifically, in Section 2 we completely characterize the girth of the zero-divisor graph of an idealization. In Section 3 we completely characterize when the zero-divisor graph of an idealization will be complete and provide some conditions when the zero-divisor graph of the idealization will have diameter 2. We also present some questions that remain.

For the sake of completeness, we state some definitions and notations used throughout. We will use R to denote a commutative ring with identity, D to denote an integral domain, and M to denote a unitary R- or D-module. We use $Z(R)$ to denote the set of zero-divisors of R; we use $Z(R)^*$ to denote the set of nonzero zero-divisors of R. Additionally, we use $\mathrm{reg}(R)$ to denote the regular elements of R, i.e. $\mathrm{reg}(R) = R - Z(R)$. By the zero-divisor graph of R, denoted $\Gamma(R)$, we mean the graph whose vertices are the nonzero zero-divisors of R, and for distinct $r, s \in Z(R)^*$, there is an edge connecting r and s if and only if $rs = 0$. For two distinct vertices a and b in a graph Γ, the distance between a and b, denoted $d(a, b)$, is the length of a shortest path connecting a and b, if such a path exists; otherwise, $d(a, b) = \infty$. The *diameter* of a graph Γ is $\mathrm{diam}(\Gamma) = \sup\{d(a, b) \mid a \text{ and } b \text{ are distinct vertices of } \Gamma\}$. We will use the notation $\mathrm{diam}(\Gamma(R))$ to denote the diameter of the graph of $Z(R)^*$. The *girth* of a graph Γ, denoted $g(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise, $g(\Gamma) = \infty$. We will use the notation $g(\Gamma(R))$ to denote the girth of the graph of $Z(R)^*$. A graph is said to be *connected* if there exists a path between any two distinct vertices, and a graph is *complete* if it is connected with diameter less than or equal to one. A singleton graph is connected and of diameter zero. A graph is said to be a *star graph* if the graph is connected with all edges sharing a common vertex. We use the notation A^* to refer to the nonzero elements of A.

The idealization of M in R, denoted by $R(+)M$, is a ring with the following operations:

(i) $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$;
(ii) $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$.

We will assume that neither the ring nor the module is trivial. Observe that if $a \in Z(R)^*$, then $(a, m) \in Z(R(+)M)^*$ for all $m \in M$. To see this, consider $b \in Z(R)^*$ with $ab = 0$. If $bM = 0$, then $(a, m)(b, 0) = 0$. If $bM \neq 0$, then there exists some $n \in M$ such that $bn \neq 0$. Hence, $(a, m)(0, bn) = 0$. Thus we get the following result which is a special case of Theorem 25.3 of [8]:

Proposition 1.1. *Let R be a commutative ring and let M be an R-module. Then $Z(R(+)M)^* = \{(0, m) \mid m \in M^*\} \cup \{(a, n) \mid a \in R^*, n \in M$ and for some $m \in M^*$, $am = 0\} \cup \{(a, n) \mid a \in Z(R)^*, n \in M\}$.*

Proof. Clearly $(0, m) \in Z(R(+)M)^*$ for all $m \in M^*$, while from above we have $\{(a, n) \mid a \in Z(R)^*, n \in M\} \subset Z(R(+)M)^*$. If $(a, n) \in Z(R(+)M)^*$ and $a \in \mathrm{reg}(R)$, then we must have $(a, n)(0, m) = (0, 0)$. Hence $am = 0$, where $m \neq 0$. □

2. Girth of $\Gamma(R(+)M)$

When looking at the girth of $\Gamma(R(+)M)$, things are very simple if the module is large enough. For if $|M|\geqslant 4$, then $g(\Gamma(R(+)M)) = 3$, since $(0, m_1)$ — $(0, m_2)$ — $(0, m_3)$ – $(0, m_1)$ is a cycle of length 3 (where m_1, m_2, and m_3 are distinct nonzero elements of M). So, we only need to consider when the module has two or three elements. First we look at when $M \cong \mathbb{Z}_3$ and consider $R(+)\mathbb{Z}_3$. In most cases, the girth of $\Gamma(R(+)\mathbb{Z}_3)$ is three. One item worthy of note is that if R has more than three elements, there always exists a nonzero $r \in R$ such that $r \cdot \mathbb{Z}_3 = 0$. To see this, assume $r \cdot \mathbb{Z}_3 \neq 0$ for all $r \in R^*$. Then there exist distinct $r_1, r_2 \in R^*$ such that $r_1 \cdot 1 = r_2 \cdot 1$ and hence $(r_1 - r_2)1 = 0$ where $r_1 - r_2$ is nonzero, a contradiction. Also, since the module is unitary, the ring cannot have fewer than three elements. This is useful in our next result.

Theorem 2.1. *Let R a commutative ring with identity and* $M \cong \mathbb{Z}_3$ *an R-module. Then*

(i) $g(\Gamma(R(+)\mathbb{Z}_3)) = 3$ *if and only if* $\text{ann}(\mathbb{Z}_3) \neq \{0\}$.
(ii) $g(\Gamma(R(+)\mathbb{Z}_3)) = \infty$ *if and only if* $\text{ann}(\mathbb{Z}_3) = \{0\}$. *This occurs precisely when* $R \cong \mathbb{Z}_3$.

Proof. (i) Assume there exists a nonzero element $r \in R$ such that $r\mathbb{Z}_3 = 0$. Since $(r, 0) - (0, 1) - (0, 2) - (r, 0)$ is a cycle of length 3, the result is obvious. The other direction is proven by using the contrapositive of the implication proven below.

(ii) Assume that $r\mathbb{Z}_3 \neq 0$ for every nonzero element $r \in R$. Then $r \cdot 1 \neq 0$ for all $r \in R^*$. Thus $\text{ann}((0, 1)) = \text{ann}((0, 2)) = \{(0, 0), (0, 1), (0, 2)\}$. Since $\Gamma(R(+)\mathbb{Z}_3)$ is connected, we see that R has no nonzero zero divisors; hence, R is an integral domain. In light of the remark preceeding the theorem, $R \cong \mathbb{Z}_3$. Since $Z(R(+)\mathbb{Z}_3)^* = \{(0, 1), (0, 2)\}$, we have $g(\Gamma(R(+)\mathbb{Z}_3)) = \infty$. The other direction is proven by using the contrapositive of the implication proven in (i). Note that $\mathbb{Z}_3(+)\mathbb{Z}_3 \cong \mathbb{Z}_3[x]/(x^2)$. □

The above result classifies the girth of $R(+)\mathbb{Z}_3$, and it is somewhat surprising that the girth will never be 4. We now consider the situation when $M \cong \mathbb{Z}_2$. We will classify when the girth of $\Gamma(R(+)\mathbb{Z}_2)$ is 3 and when it is infinite. We begin with the girth 3 case.

Theorem 2.2. *The girth of* $\Gamma(R(+)\mathbb{Z}_2)$ *is three if and only if one of the following hold*:

(i) *The girth of* $\Gamma(R)$ *is three.*
(ii) *There exists an* $r \in R^*$ *such that* $r^2 = 0$.
(iii) *There exist distinct* $a, b \in Z(R)^*$ *such that* $ab = 0 = a\mathbb{Z}_2 = b\mathbb{Z}_2$.

Proof. ($\Leftarrow$) If (i) holds, the result is clear. If (ii) holds, note that $r \cdot 1 = 0$, lest $r \cdot (r \cdot 1) \neq r^2 \cdot 1$. Then, $(r, 0) - (r, 1) - (0, 1) - (r, 0)$ is a cycle of length three. If (iii) holds, then $(a, 0) - (b, 0) - (0, 1) - (a, 0)$ is a cycle of length 3.

($\Rightarrow$) *Case* 1: The element $(0, 1)$ is part of a minimal length cycle. Then the cycle has the form $(0, 1) - (a, i) - (b, j) - (0, 1)$. If $a \neq b$, we have distinct $a, b \in Z(R)^*$, $ab = 0$, and $a\mathbb{Z}_2 = b\mathbb{Z}_2 = 0$; if $a = b$, we have $a \in R^*$ such that $a^2 = 0$.

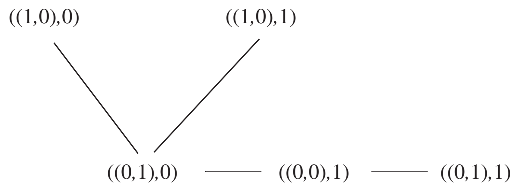
Case 2: The element (0, 1) is not part of a minimal length cycle. Then, the cycle has the form $(a, i) - (b, j) - (c, k) - (a, i)$. If a, b, and c are all distinct, then $a - b - c - a$ is a cycle in $\Gamma(R)$, and $g(\Gamma(R)) = 3$. If not, then either $a^2 = 0$ or $b^2 = 0$. □

We will now provide necessary and sufficient conditions for ensuring the girth of $R(+)\mathbb{Z}_2$ is infinite. We begin with some results to be used later.

Lemma 2.3. *Let $R \cong \mathbb{Z}_2 \times F$, where F is a field. Then, any module operation from R to $\mathbb{Z}_2$ is a canonical extension of a module operation either from $\mathbb{Z}_2$ to $\mathbb{Z}_2$, or from F to $\mathbb{Z}_2$ in the case where F is $\mathbb{Z}_2$.*

Proof. The annihilator of $\mathbb{Z}_2$ as an R-module is an ideal of R; thus $\text{ann}(\mathbb{Z}_2) = I_1 \times I_2 = I$, where I_1 is an ideal of $\mathbb{Z}_2$, and I_2 is an ideal of F. If $I_1 \times I_2 = \{0\}$ then $(1, 0) \cdot 1 = 1 = (0, 1) \cdot 1$, but this would then result in $(1, 1) \cdot 1 = ((1, 0) + (0, 1)) \cdot 1 = (1, 0) \cdot 1 + (0, 1) \cdot 1 = 0$, a contradiction. More easily, $I_1 \times I_2 \neq R$ since the module is unitary. Thus, $I = \{0\} \times F$ or $I = \mathbb{Z}_2 \times \{0\}$. If $I = \{0\} \times F$, then the operation is a canonical extension of the module operation from $\mathbb{Z}_2$ to $\mathbb{Z}_2$. Similarly, if $I = \mathbb{Z}_2 \times \{0\}$, then the operation is a canonical extension of the module operation from F to $\mathbb{Z}_2$. However, if $|F| \geqslant 3$, then there is no module operation from F to $\mathbb{Z}_2$since there are nonzero sums of units (which in turn are units), but in the module $u \cdot 1 = 1$. □

Example 2.4. Using Lemma 2.3, let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and consider $R(+)\mathbb{Z}_2$. Without loss of generality, the module operation is defined by $(0, 0) \cdot 1 = (0, 1) \cdot 1 = 0$ and $(1, 0) \cdot 1 = (1, 1) \cdot 1 = 1$. Note that $R(+)\mathbb{Z}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$. Then, $g(\Gamma(R(+)\mathbb{Z}_2)) = \infty$, as the zero-divisor graph below shows:



Proposition 2.5. *Let $R \cong \mathbb{Z}_2 \times F$, where F is a field and $|F| \geqslant 3$. Then, $g(\Gamma(R(+)\mathbb{Z}_2)) = 4$.*

Proof. Since F is a field and $|F| \geqslant 3$, by Lemma 2.3 the module operation from R to $\mathbb{Z}_2$ is an extension of the module operation from $\mathbb{Z}_2$ to $\mathbb{Z}_2$. We have $((0, 0), 1) - ((0, 1), 0) - ((1, 0), 1) - ((0, a), 0) - ((0, 0), 1)$ is a cycle of length 4 (where $a \in F$ is nonzero and not equal to 1). By Theorem 2.2, $\Gamma(R(+)\mathbb{Z}_2)$ cannot contain any cycles of length 3, since $\Gamma(R)$ is a star graph centered at $(1, 0)$. Hence $g(\Gamma(R(+)\mathbb{Z}_2)) = 4$. □

Lemma 2.6. *If* $\text{diam}(\Gamma(R)) = 3$, *then the girth of* $\Gamma(R(+)\mathbb{Z}_2)$ *is finite.*

Proof. Let $a - b - c - d$ be a path in $\Gamma(R)$ with a, b, c, d distinct. If $b\mathbb{Z}_2 \neq 0$ and $c\mathbb{Z}_2 \neq 0$, then $b \cdot 1 = 1$ and $c \cdot 1 = 1$, but $(bc) \cdot 1 = 0$, a contradiction. Thus, we must have either $b\mathbb{Z}_2 = 0$ or $c\mathbb{Z}_2 = 0$, or both. Assume $b\mathbb{Z}_2 = 0$. If $c\mathbb{Z}_2 = 0$, then we get the cycle $(b, 0) - (c, 0) - (b, 1) - (c, 1) - (b, 0)$. If $c\mathbb{Z}_2 \neq 0$, then $d\mathbb{Z}_2 = 0$; hence $(b, 0) - (c, 0) - (d, 0) - (c, 1) - (b, 0)$ is a cycle. □

Given the idealization $R(+)\mathbb{Z}_2$, it is easy to see that $|R/\text{ann}(\mathbb{Z}_2)| = 2$. Otherwise, let $r_1+\text{ann}(\mathbb{Z}_2)$ and $r_2+\text{ann}(\mathbb{Z}_2)$ be two cosets distinct from $0+\text{ann}(\mathbb{Z}_2)$. Thus $r_1, r_2 \notin \text{ann}(\mathbb{Z}_2)$ and so $r_1 \cdot 1 = r_2 \cdot 1 = 1$. Therefore $(r_1 - r_2) \in \text{ann}(\mathbb{Z}_2)$ and so $r_1 + \text{ann}(\mathbb{Z}_2) = r_2 + \text{ann}(\mathbb{Z}_2)$. This result will be useful in the proof of the following.

Theorem 2.7. *The girth of $\Gamma(R(+)\mathbb{Z}_2)$ is infinite if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or R is an integral domain.*

Proof. ($\Leftarrow$) If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, Example 2.4 shows $\Gamma(R(+)\mathbb{Z}_2)$ has no cycles. If R is an integral domain, then $\Gamma(R(+)\mathbb{Z}_2)$ is a star graph with center (0, 1).

($\Rightarrow$) Lemma 2.6 shows $\text{diam}(\Gamma(R)) \leqslant 2$ or $Z(R)^* = \emptyset$. If $Z(R)^* = \emptyset$, we are done. If $\text{diam}(\Gamma(R)) = 0$, then by Theorem 3.2 [1], we have $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[x]/(x^2)$. In either case, there exists a nonzero nilpotent element, and by Theorem 2.2, $g(\Gamma(R(+)\mathbb{Z}_2)) = 3$. If $\text{diam}(\Gamma(R)) = 1$, then $\Gamma(R)$ is complete. Thus, if $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then R contains a nilpotent element by Theorem 2.8 of [2], and by Theorem 2.2, $g(\Gamma(R(+)\mathbb{Z}_2)) = 3$, a contradiction.

If $\text{diam}(\Gamma(R)) = 2$ and $\Gamma(R)$ is not a star graph, then $g(\Gamma(R)) < \infty$, a contradiction. Thus, by Theorem 2.5 of [2], the only possibilities for R are $\mathbb{Z}_2 \times D$, where D is an integral domain, or $Z(R)$ is an annihilator ideal. If $Z(R)$ is an annihilator ideal, then R contains a nilpotent element, and we appeal to Theorem 2.2. Hence, $R \cong \mathbb{Z}_2 \times D$. If $|D| = 2$, we are done. If D is a finite integral domain, then D is a field and by Proposition 2.5, $g(\Gamma(R(+)\mathbb{Z}_2)) = 4$, a contradiction. The remaining case to investigate is when D is an infinite integral domain. In R, $\text{ann}(\mathbb{Z}_2)$ is an ideal and hence of one of the following three forms: $\mathbb{Z}_2 \times \{0\}$, $\{0\} \times I$, or $\mathbb{Z}_2 \times I$, where I is an nonzero ideal of D. If $\text{ann}(\mathbb{Z}_2) = \mathbb{Z}_2 \times \{0\}$, then $|R/\text{ann}(\mathbb{Z}_2)| > 2$ which contradicts the remarks preceeding this result. Again using the coset argument, if $\text{ann}(\mathbb{Z}_2) = \{0\} \times I$ or $\text{ann}(\mathbb{Z}_2) = \mathbb{Z}_2 \times I$, then there exist distinct, nonzero $a, b \in I$ such that $(0, a), (0, b) \in \text{ann}(\mathbb{Z}_2)$. Thus we form a cycle $((1, 0), 0) - ((0, a), 0) - ((1, 0), 1) - ((0, b), 0) - ((1, 0), 0)$. This contradicts $g(\Gamma(R(+)\mathbb{Z}_2)) = \infty$. □

The following theorem summarizes the results of this section.

Theorem 2.8. *Let R be a ring and M an R-module.*

(i) $g(\Gamma(R(+)M)) = 3$ *if and only if exactly one of the following hold*:

- (a) $|M| \geqslant 4$,
- (b) $M \cong \mathbb{Z}_3$ *and* $\text{ann}(M) \neq 0$, *or*
- (c) $M \cong \mathbb{Z}_2$ *and one of the following hold*:
  - (1) $g(\Gamma(R)) = 3$,
  - (2) *there exists a nonzero* $r \in R$ *such that* $r^2 = 0$, *or*
  - (3) *there exists distinct* $a, b \in Z(R)^*$ *such that* $ab = 0 = aM = bM$.

(ii) $g(\Gamma(R(+)M)) = \infty$ *if and only if exactly one of the following hold*:

- (a) $M \cong \mathbb{Z}_3$ *and* $\text{ann}(M) = 0$ (*and* $R \cong \mathbb{Z}_3$), *or*
- (b) $M \cong \mathbb{Z}_2$ *and either* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *or R is an integral domain.*

We observe that the only case in which $g(\Gamma(R(+)M))$ can be four is when $M \cong \mathbb{Z}_2$ and R does not meet any of the above conditions. For example, by Proposition 2.5 $g(\Gamma(R(+)\mathbb{Z}_2)) = 4$ when $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$.

3. Diameter of $\Gamma(R(+)M)$

In studying the zero-divisor graph of an idealization, it quickly becomes clear that the diameter of the ring's graph need not be preserved. It is also clear that if $\mathrm{diam}(\Gamma(R)) > 1$, then $\mathrm{diam}(\Gamma(R(+)M)) > 1$. However, it is possible to have a ring R such that $\Gamma(R)$ is complete and $\mathrm{diam}(\Gamma(R(+)M)) > 1$, as well as all other possible combinations between the diameter of R and that of $R(+)M$. In this section, we provide necessary and sufficient conditions to ensure $\Gamma(R(+)M)$ is complete and provide partial results concerning when $\mathrm{diam}(\Gamma(R(+)M)) = 2$.

We begin with some examples illustrating ways in which the diameter of $\Gamma(R)$ is not preserved upon moving to $R(+)M$.

Example 3.1. Consider $\mathbb{Z}_9(+)\mathbb{Z}_9$ under the usual module operation. Clearly $\mathrm{diam}(\Gamma(\mathbb{Z}_9)) = 1$. Observe that if $r \in \mathrm{reg}(\mathbb{Z}_9)$, then $rm = 0$ if and only if $m = 0$. Thus for $r \in \mathrm{reg}(\mathbb{Z}_9)$, we have $(r, m) \notin Z(\mathbb{Z}_9(+)\mathbb{Z}_9)^*$. Now, $(0, 3)(r, m) = 0$ for any $(r, m) \in Z(\mathbb{Z}_9(+)\mathbb{Z}_9)^*$ and $(3, 1)(3, 3) \neq 0$; hence, $\mathrm{diam}(\Gamma(\mathbb{Z}_9(+)\mathbb{Z}_9)) = 2$.

Example 3.2. Let $R = \mathbb{Z}[x]/(x^2)$. Then $\mathrm{diam}(\Gamma(R)) = 1$ since $Z(R)^* = \{ax \mid a \in \mathbb{Z}^*\}$. Let $M = \mathbb{Z}_6$, and define $(a + bx)m = am$. Consider $(2, 1), (3, 1) \in Z(R(+)M)^*$. Clearly, $\mathrm{ann}((2, 1)) = \{(0, 3), (0, 0)\}$, while $\mathrm{ann}((3, 1)) = \{(0, 2), (0, 0), (0, 4)\}$. Hence, $d((2, 1), (3, 1)) = 3$, so $\mathrm{diam}(\Gamma(R(+)M)) = 3$. We can also consider $\mathbb{Z}_4$ as an R-module under the same operation, in which case $\mathrm{diam}(\Gamma(R(+)\mathbb{Z}_4)) = 2$.

We now determine necessary and sufficient conditions on R and M to ensure $\Gamma(R(+)M)$ is complete. In [2] Anderson and Livingston show if $R \neq \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\Gamma(R)$ is complete if and only if $(Z(R))^2 = 0$. Notice in the previous two examples $(Z(R))^2 = 0$, yet $(Z(R(+)M))^2 \neq 0$. So, simply requiring $\Gamma(R)$ to be complete will not guarantee $\Gamma(R(+)M)$ will be complete.

We will state three properties that will be considered numerous times:

(a) $(Z(R))^2 = 0$.
(b) For every $r \in \mathrm{reg}(R)$, $rm \neq 0$ for all $m \in M^*$.
(c) If $r \in Z(R)^*$, then $rM = 0$.

We remark that in Example 3.1, the ring $\mathbb{Z}_9$ does not satisfy condition (c), and in Example 3.2, the ring $\mathbb{Z}[x]/(x^2)$ does not satisfy condition (b).

Theorem 3.3. *Let $\Gamma(R) \neq \emptyset$. Then, $\Gamma(R(+)M)$ is complete if and only if $R(+)M$ satisfies properties* (a), (b), *and* (c).

Proof. ($\Rightarrow$) Assume $\Gamma(R(+)M)$ is complete. Let $r, s \in Z(R)^*$ and $m \in M^*$. By Proposition 1.1, $(r, m), (s, 0) \in Z(R(+)M)^*$; hence, $(r, m)(s, 0) = (0, 0)$. So, $rs = 0$ for all $r, s \in Z(R)^*$, and $(Z(R))^2 = 0$. If for some $r \in \text{reg}(R)$ we had that $rm = 0$ for some $m \in M^*$, then $(r, 0) \in Z(R(+)M)^*$ since $(r, 0)(0, m) = (0, 0)$. So, let $a \in Z(R)^*$. Since $\Gamma(R(+)M)$ is complete, we have $(r, 0)(a, 0) = (0, 0)$, a contradiction. If for some $a \in Z(R)^*$ there exists an $m \in M^*$ so that $am \neq 0$, then $(a, 0), (0, m) \in Z(R(+)M)^*$, but $(a, 0)(0, m) = (0, am) \neq (0, 0)$, another contradiction.

($\Leftarrow$) Assume that properties (a), (b), and (c) hold. Therefore $Z(R(+)M)^* = \{(0, m) \mid m \in M^*\} \cup \{(a, n) \mid a \in Z(R)^*, n \in M\}$ by Proposition 1.1. Let $(r, m), (s, n) \in Z(R(+)M)^*$. If $r = s = 0$, then clearly $(r, m)(s, n) = 0$. If $s = 0$ and $r \in Z(R)^*$, then by (c) we have that $rM = 0$. So, $(r, m)(0, n) = (0, 0)$. If $r, s \in Z(R)^*$, then $rs = 0$ by (a) and $rM = sM = 0$ by (c), so $(r, m)(s, n) = (0, 0)$. Thus $\Gamma(R(+)M)$ is complete. □

Corollary 3.4. *$\Gamma(R(+)M)$ is complete if and only if $(Z(R(+)M))^2 = 0$.*

Proof. One direction is trivial while the other follows from the proof of Theorem 3.3. □

Theorem 2.8 of [2] along with this corollary shows $R(+)M \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ for any ring R and any M an R-module. (Though this result is easily proven without appealing to zero-divisor graphs, we believe this is an interesting sidenote.) We also remark that any domain satisfies condition (a) and vacuously satisfies condition (c). So, $\Gamma(D(+)M)$ is complete if and only if D satisfies condition (b).

We now focus on when $\Gamma(R)$ has diameter greater than one. In this case it is clear that $\text{diam}(\Gamma(R(+)M)) > 1$. However, it is possible to form idealizations with zero-divisor graphs having diameters of 2 and 3 from rings whose zero-divisor graphs have diameter 2 as well as rings whose zero-divisor graphs have diameter 3, as the following examples illustrate.

Example 3.5. Consider $\mathbb{Z}_8(+)\mathbb{Z}_2$. It is easy to verify that the diameter of $\Gamma(\mathbb{Z}_8)$ is 2, and the diameter of $\Gamma(\mathbb{Z}_8(+)\mathbb{Z}_2)$ is also 2.

Example 3.6. Let $R = \mathbb{Z}_6$ and $M = \mathbb{Z}_6$ and consider $R(+)M$ under the usual module operation. It is easy to see that $\text{diam}(\Gamma(R)) = 2$ and $Z(R(+)M)^* = \{(0, m) \mid m \in M^*\} \cup \{(a, n) \mid a \in Z(R)^*, n \in M\}$. However, $\text{diam}(\Gamma(R(+)M)) = 3$ since $\text{ann}((2, 1)) = \{(0, 3), (0, 0)\}$, but $(3, 1)(0, 3) = (0, 3) \neq (0, 0)$.

Example 3.7. Let K be a field, $S = K[Y, \{X_i\}_{i=0}^{\infty}]/(\{X_0Y\} \cup \{X_i - X_{i+1}Y\}_{i=0}^{\infty})$, and $R = S[[W]]$. Then, $\text{diam}(\Gamma(R)) = 3$ (see [7,4, Example 3.3].) Let $f(W) = f_0 + f_1W + f_2W^2 + \ldots$, where $f_i \in S$. Make K an R-module with the operation $f \cdot k = f_0(0) \cdot k$, where $f_0(0)$ is f_0 evaluated when all indeterminates appearing in f_0 are 0. Note $(f, k) \in Z(R(+)K)^*$ if and only if $f_0(0) = 0$. So, $(f, k)(0, l) = (0, 0)$ for all $(f, k) \in Z(R(+)K)^*$. Hence, $\text{diam}(\Gamma(R(+)K)) = 2$.

Example 3.8. Let $R = \mathbb{Z}_6 \times \mathbb{Z}_6$. Then, $\text{diam}(\Gamma(R)) = 3$. Make $\mathbb{Z}_6$ an R-module with the operation $(a, b) \cdot m = am$. Observe that $\text{ann}(((2, 3), 1)) = \{((0, a), b) \mid a = 0, 2, 4$ and $b = 0, 3\}$ and $((3, 1), 1)((0, a), b) \neq ((0, 0), 0)$ for $a = 0, 2, 4$ and $b = 0, 3$ unless $a = 0$ and $b = 0$. Hence, $\text{diam}(\Gamma(R(+)M)) = 3$.

If necessary and sufficient conditions for ensuring $\text{diam}(\Gamma(R(+)M))=2$ could be found, then the classification of the diameter of an idealization would be complete. This characterization is still however an open question. It may prove easier to classify the diameter 3 case instead. Below, we provide two results that ensure $\text{diam}(\Gamma(R(+)M)) = 2$ based on properties (a), (b), and (c) from earlier.

Theorem 3.9. *If $R(+)M$ satisfies properties* (b) *and* (c) *but not property* (a), *then* $\text{diam}(\Gamma(R(+)M)) = 2$.

Proof. First note that no $\mathbb{Z}_2 \times \mathbb{Z}_2$-module M can satisfy property (c), lest $((1, 0) + (0, 1)) \cdot m = (1, 1) \cdot m = m$ for some $m \in M^*$, but $(1, 0) \cdot m = 0_M = (0, 1) \cdot m$. Since $R \neq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $(Z(R))^2 \neq 0$, we must have $\text{diam}(R) > 1$, and hence $\text{diam}(\Gamma(R(+)M)) > 1$.

If $r \in Z(R)^*$, then $(r, m)(0, n) = 0$ for any $m, n \in M$. Thus, $d((r, m), (s, n)) \leqslant 2$ for all $(r, m), (s, n) \in Z(R(+)M)^*$. Thus, $\text{diam}(\Gamma(R(+)M)) = 2$. □

Theorem 3.10. *If $R(+)M$ satisfies properties* (a) *and* (b) *but not property* (c), *then* $\text{diam}(\Gamma(R(+)M)) = 2$.

Proof. Assume $R(+)M$ satisfies properties (a) and (b) and, for some $a \in Z(R)^*$ and $m \in M^*$, we have $am \neq 0$. For any $(r_1, n_1), (r_2, n_2) \in Z(R(+)M)^*$, we have $(r_1, n_1)(0, am) = (0, 0) = (r_2, n_2)(0, am)$. Hence, $\text{diam}(\Gamma(R(+)M)) \leqslant 2$. Since $(a, 0)(0, m) \neq (0, 0)$, we get that $\text{diam}(\Gamma(R(+)M)) = 2$. □

It is also worth briefly mentioning the diameter behavior of the idealization of non-domains R with $\text{diam}(\Gamma(R)) = 0$, namely $R \cong \mathbb{Z}_4$ and $R \cong \mathbb{Z}_2[x]/(x^2)$ [1]. If $R \cong \mathbb{Z}_4$, then any R-module M has the property that $2M = 0$ or $4M = 0$. It is easy to show using Theorem 3.3 that $\text{diam}(\Gamma(\mathbb{Z}_4(+)M)) = 1$ if and only if $2M = 0$. Thus, using Theorem 3.10, it is also clear that $\text{diam}(\Gamma(\mathbb{Z}_4(+)M)) = 2$ if and only if $2M \neq 0$.

Now, consider $R \cong \mathbb{Z}_2[x]/(x^2)$. The diameter of an idealization of R is less clear, though 1 and 2 are again the only possibilities. Clearly R satisfies property (a). Also note that any R-module M has the property that $2M = 0$, and so $m = -m$ for all $m \in M$. Observe that the only regular elements of R are 1 and $x + 1$. Clearly for all $m \in M^*$ we have $1m \neq 0$. If for some $m \in M^*$ it is the case that $(x + 1)m = 0$, then $xm = m$. Hence $x(xm) = xm$, and since $x^2m = 0m = 0$ we obtain $xm = m = 0$, a contradiction. Thus, $R(+)M$ satisfies property (b) also. Then by Theorems 3.3 and 3.10, we have that $\text{diam}(\Gamma(R(+)M)) = 1$ or 2 for $R \cong \mathbb{Z}_2[x]/(x^2)$. For example, if $M = \mathbb{Z}_2[x]/(x^2)$ using ring multiplication as the module operation, then $R(+)M$ does not satisfy property (c), and hence the diameter is 2. If $M = \mathbb{Z}_2$ with module operation defined by $(ax + b)m = bm$, then $R(+)M$ does satisfy property (c), and hence the diameter is 1.

The question of when an idealization can have diameter zero is also easily taken care of. By examining Proposition 1.1, it is clear that $\text{diam}(\Gamma(R(+)M)) = 0$ if and only if $R(+)M = \mathbb{Z}_2(+)\mathbb{Z}_2$. (Note that $\mathbb{Z}_2(+)\mathbb{Z}_2 \cong \mathbb{Z}_2[x]/(x^2)$.)

Example 3.1 serves as both an example of Theorem 3.10 and a counterexample to the converse of Theorem 3.9. Likewise, Example 3.5 serves as both an example of Theorem 3.9 and a counterexample to the converse of Theorem 3.10. In addition, Example 3.2 shows that if properties (a) and (c) hold, but not (b), then diameters of 2 and 3 are possible.

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