An International Journal
computers & mathematics with applications
PERGAMON Computers and Mathematics with Applications 43 (2002) 1027–1033
www.elsevier.com/locate/camwa

Conjugacy in the Discretized Fold Bifurcation

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(Received October 2000; revised and accepted May 2001)

Abstract—In this paper, we construct a conjugacy between the time-1-map of the solution flow generated by an ordinary differential equation and its numerical approximation in a neighborhood of a fold bifurcation point. Our main result is that the conjugacy is $O(h^p)$-close to the identity on the center manifold where h is the step size and p is the order of the numerical method. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Generalized Hartman-Grobman theorem, Fold bifurcation, Discretization.

1. INTRODUCTION

It is well known that conjugacies play a fundamental role in the qualitative theory of ordinary differential equations. Indeed, when a conjugacy exists between two dynamical systems, then the dynamical systems have the same orbit structure; they are qualitatively the same.

The discretization of a dynamical system is a family of maps (depending on the step size h) which is close to the time-h-map of the dynamical system. We want to claim that under certain conditions, the dynamics of the discretization considered as a discrete dynamical system and of the original system are the same. Thus, it is natural to seek for conjugacies between a dynamical system and its numerical approximation.

In the vicinity of a hyperbolic equilibrium point, this was done in [1] by putting the problem in the general framework of the Hartman-Grobman theorem. A similar approach was carried out in [2] in the case of delay differential equations. Structural stability results were obtained in [3] (for Morse-Smale systems without periodic orbits) and in [4] (for systems satisfying Axiom A and the strong transversality condition). The construction of the conjugacies uses the various type of hyperbolicity conditions of the dynamical system.

However, hyperbolicity is usually lost in a bifurcation point. So these results cannot be applied to a bifurcation problem. We note that, in general, we cannot expect that a conjugacy exists in

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Supported by DAAD Project 323-PPP, Qualitative Theory of Numerical Methods for Evolution Equations in Infinite Dimensions and by the Hungarian National Research Fund under number T 031807.
This work was done while the author was a visitor at the University of Bielefeld. The author would like to thank SFB 343 for the hospitality and Prof. W.-J. Beyn for the stimulating discussions.

PII: S0898-1221(01)00343-1

a neighborhood of a nonhyperbolic equilibrium point, as the simple example of the planar linear center and the Euler method shows. (Indeed, consider the planar linear center $\dot{x} = y$, $\dot{y} = -x$, and its Euler discretization $X = x + hy$, $Y = y - hx$. A simple calculation shows that the origin becomes unstable under Euler discretization for all step size h.) On the other hand, under certain conditions, the existence of a conjugacy can be saved. Namely, we show in this paper that in the neighborhood of a fold bifurcation point, the desired conjugacy exists. Moreover, the conjugacy is $O(h^p)$-close to the identity on the center manifold where p is the order of the method.

The proof of our main result works via the generalized Hartman-Grobman theorem (see [5,6]), the center manifold reduction (see [7,8]), and the method of fundamental domains. The use of fundamental domains was inspired by a lecture by Y.A. Kuznetsov, where the topological normal form of the fold bifurcation was constructed in a similar way. The center manifold reduction played a fundamental role in [9] where a numerical Hopf bifurcation theorem was proved for partial differential equations.

The paper is organized as follows. Preliminaries are placed into Section 2. Section 3 contains our main result. We end this note with some final remarks.

2. PRELIMINARIES

Let $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ be a globally Lipschitzian C^j function with $j \geq 4$. Consider the following ordinary differential equation depending on a single parameter α:

$$\dot{z} = f(z; \alpha). \tag{1}$$

Denote the solution flow of (1) with parameter value α by $\Phi(\cdot, \cdot; \alpha) : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$.

By the h-discretized equation of (1), we mean equation

$$Z = \phi(h, z; \alpha), \qquad z, Z \in \mathbf{R}^n, \quad h > 0, \tag{2}$$

where ϕ is a fixed one-step method with step size h. Assume that ϕ is smooth and is of order $p \geq 1$; i.e., there exist a constant h_0 and a constant K_1 (depending only on f) such that

$$|\Phi(h, z; \alpha) - \phi(h, z; \alpha)|_j \leq K_1 h^{p+1}, \qquad \text{for all } h \in (0, h_0], \quad z \in \mathbf{R}^n, \tag{3}$$

where $\Phi(h, \cdot; \alpha) : \mathbf{R}^n \to \mathbf{R}^n$ is the time-h-map of the induced solution flow of (1) with parameter value α and $|\cdot|_j$ denotes the usual C^j-norm of the space $C^j(\mathbf{R}^n \times \mathbf{R}, \mathbf{R}^n)$.

In the usual definition of the order of the method, the $|\cdot|_0$ norm is used instead of the $|\cdot|_j$ norm. Since property (3) is a consequence of the C^0-closeness, for sufficiently smooth systems we use (3) as a definition of the order of the method. A more detailed treatment of this property can be found in [1].

With $[\cdot]$ denoting the integer part, for fixed $t > 0$ the approximation of the time-t-map of the induced solution flow, i.e., $\Phi(t)$, is

$$\phi^{[t/h]}(h, \cdot; \alpha),$$

and if $t/h \in \mathbf{N}$, then

$$\left|\Phi(t, z; \alpha) - \phi^{[t/h]}(h, z; \alpha)\right|_j \leq K_2 h^p \tag{4}$$

holds with some constant $K_2 > 0$ (depending only on f and t). For a detailed treatment of inequality (4), we refer to [10].

Assume that $\Phi(t, 0; 0) = 0$ and $\phi(h, 0; 0) = 0$ for all $t \in \mathbf{R}$ and all $h \in (0, h_0]$, respectively. Assume further that $\alpha = 0$ is a fold bifurcation point for both (1) and (2). To be concrete, assume that there are no equilibria for $\alpha > 0$ and there are two equilibria for $\alpha < 0$. We note that a simple analysis of (4) shows that ϕ must have a nearby fold bifurcation point whenever

$\alpha = 0$ is a fold bifurcation point for Φ. We only assume for simplicity that this point is shifted into 0.

By enlarging the dimension by 1, i.e., by adding $\dot{\alpha} = 0$ and $A = \alpha$ to (1) and to (2), respectively, we have local center manifolds around 0 in the enlarged phase space denoted by

$$W^C_{\text{loc}}(0) = \{(x, \xi(x,\alpha), \alpha) : x \in \mathbf{R},\ |x|, |\alpha| \text{ are sufficiently small}\}$$

and

$$W^{C_h}_{\text{loc}}(0) = \{(x, \xi_h(x,\alpha), \alpha) : x \in \mathbf{R},\ |x|, |\alpha| \text{ are sufficiently small}\},$$

where $\xi, \xi_h : \mathbf{R} \times \mathbf{R} \to \mathbf{R}^{n-1}$ are C^j functions.

Applying the result of [11] (or of [1]), we have that these manifolds are C^j-close; i.e., the functions ξ and ξ_h are C^j-close, and moreover, their C^j-distance is bounded by $O(h^p)$. For the sake of simplicity, we denote the solution flow of the enlarged system and its discretization simply by Φ and ϕ, respectively. Finally, denote the reduced maps on their center manifolds by Φ_C and ϕ_{C_h}, respectively; i.e.,

$$\Phi_C(t, x; \alpha) = y, \qquad \text{where } (y, \xi(y,\alpha), \alpha) = \Phi(t, (x, \xi(x,\alpha)); \alpha)$$

and

$$\phi_{C_h}(h, x; \alpha) = y, \qquad \text{where } (y, \xi_h(y,\alpha), \alpha) = \phi(h, (x, \xi_h(x,\alpha)); \alpha).$$

From the C^j-closeness of the center manifolds and from (4), it follows that

$$\left| \Phi_C(t, x; \alpha) - \phi^{[t/h]}_{C_h}(h, x; \alpha) \right|_j = O\left(h^p\right), \tag{5}$$

where $t/h \in \mathbf{N}$. From now on, we restrict ourselves to the case $1/h \in \mathbf{N}$.

Following [7], we see that the construction of the normal form of the fold bifurcation works via Taylor expansion, implicit function theorem (to eliminate the parameter dependent first-order term), and inverse function theorem (to introduce a new parameter). Thus, our closeness property (5) yields the following lemma.

LEMMA 1. *There are positive numbers ε, α_0, and smooth invertible coordinate transforms τ and τ_h, such that τ transforms $\Phi_C(1)$ into*

$$X = x + \alpha + ax^2 + x^3\psi(x,\alpha) =: f^1(x;\alpha), \tag{6}$$

while τ_h transforms $\phi^{[1/h]}_{C_h}(h)$ into

$$X = x + \alpha + a_h x^2 + x^3\psi_h(x,\alpha) =: f^2_h(x;\alpha), \tag{7}$$

where $a > 0$, ψ, and ψ_h are smooth functions of x and α provided $|x| < \varepsilon$ and $|\alpha| < \alpha_0$ holds. Moreover, we have that

$$|a - a_h| \le K_3 h^p, \qquad |\psi(x,\alpha) - \psi_h(x,\alpha)| \le K_3 h^p, \qquad |\tau(x,\alpha) - \tau_h(x,\alpha)| \le K_3 h^p,$$

for all $|x| < \varepsilon$, $|\alpha| < \alpha_0$.

3. MAIN RESULT

Assume all the conditions listed in Section 2 hold true. We prove the following theorem.

THEOREM 1. *There are positive numbers h_1, ε_1, α_1, and a real function J defined on $(0, h_1] \times (-\varepsilon_1, \varepsilon_1) \times (-\alpha_1, \alpha_1)$ such that $J(h, \cdot, \alpha)$ is a homeomorphism,*

$$f^1(J(h,x,\alpha);\alpha) = J\left(h, f_h^2(x;\alpha), \alpha\right), \tag{8}$$

and

$$|J(h,\cdot,\alpha) - \mathrm{id}|_0 \le Kh^p \tag{9}$$

holds with some constant $K > 0$ independent of h and α.

PROOF. Set

$$X = x + \alpha + ax^2 =: g(x;\alpha).$$

Our method is to construct homeomorphisms $H(\cdot,\alpha)$ and $G(h,\cdot,\alpha)$ such that

$$f^1(H(x,\alpha);\alpha) = H(g(x;\alpha),\alpha), \tag{10}$$

$$f_h^2(G(h,x,\alpha);\alpha) = G(h, g(x;\alpha),\alpha), \tag{11}$$

and

$$|H(\cdot,\alpha) - G(h,\cdot,\alpha)|_0 \le Kh^p \tag{12}$$

hold. Then it remains to set $J = H \circ G^{-1}$.

Let N be a neighborhood of $x = 0$ and $0 < h \le h_2$ such that f^1, f_h^2, and g have the same number of fixed points with the same stability, provided $|\alpha|$ is sufficiently small. Fix $0 > y_0 \in N$ such that $g(y_0;\alpha) < f^1(g(y_0;\alpha);\alpha)$, $g(y_0;\alpha) < f_h^2(g(y_0;\alpha);\alpha)$, and if $\alpha \le 0$, then $g(-y_0;\alpha) \in N$, $g(-y_0;\alpha) > (f^1)^{-1}(g(-y_0;\alpha);\alpha)$, $g(-y_0;\alpha) > (f_h^2)^{-1}(g(-y_0;\alpha);\alpha)$. We divide the construction of H and G into three parts according to $\alpha < 0$, $\alpha = 0$, or $\alpha > 0$.

CASE $\alpha < 0$. Fix $x_0 = 0$ and set $x_k = g^k(x_0;\alpha)$, $k \in \mathbf{Z}$. Note that $x_1 = \alpha$. Set $H(x_0,\alpha) = G(h,x_0,\alpha) = g(x_0;\alpha)$ and $H(x_k,\alpha) = (f^1)^k(x_1;\alpha)$, $G(h,x_k,\alpha) = (f_h^2)^k(x_1;\alpha)$, $k \in \mathbf{Z}$. On $[x_1,x_0]$, extend both H and G linearly. For $y \in [x_2,x_1]$, set $H(y,\alpha) = f^1(H(g^{-1}(y;\alpha),\alpha);\alpha)$ and $G(h,y,\alpha) = f_h^2(G(h,g^{-1}(y;\alpha),\alpha);\alpha)$. Recursively, in both directions, we see that H and G extend continuously to the interval (x^-,x^+), where x^-, x^+ are the negative and positive fixed points of g, respectively. Finally, set $H(x^-,\alpha) = x_1^-$, $G(h,x^-,\alpha) = x_2^-$, $H(x^+,\alpha) = x_1^+$, and $G(h,x^+,\alpha) = x_2^+$, where x_1^-, x_1^+ are the negative and positive fixed points of f^1; x_2^-, x_2^+ are the negative and positive fixed points of f_h^2, respectively.

From initial points y_0 and $-y_0$, the same construction can be carried out (by taking the inverse when necessary). Note that here the assumptions on y_0 enter. As a result, we obtain functions H and G defined on some neighborhood of $x = 0$ for all $\alpha < 0$, $|\alpha|$ sufficiently small, and all $0 < h \le h_2$.

From the construction, it is easy to see that H and G are homeomorphisms (since they are continuous, strictly monotone functions) and are indeed the desired conjugacies; i.e., equations (10) and (11) hold.

It remains to prove the closeness of H and G, i.e., inequality (9). We restrict ourselves to estimate the distance between H and G on $[y_0, 0]$; the complementary part can be treated similarly.

First, we estimate $|H - G|$ on $[x^-, 0]$. It is clear that $|H(x,\alpha) - G(h,x,\alpha)| \le K_4 h^p$ holds for $x \in [x_1, x_0]$. Note that

$$\left|f^1(x;\alpha) - f_h^2(x;\alpha)\right| \le |a - a_h| \cdot |x|^2 + |\psi(x,\alpha) - \psi_h(x,\alpha)| \cdot |x|^3 \le K_5 h^p |x|^2, \tag{13}$$

provided N and α_1 are sufficiently small. Consequently,

$$\left|f^1(x;\alpha) - f_h^2(x;\alpha)\right| \le K_5 h^p \left|x^-\right|^2 = K_5 h^p \left(-\frac{\alpha}{a}\right), \tag{14}$$

for all $x \in [x^-, x_0]$. On the other hand, the derivative of f^1 (and f_h^2) is strictly monotone increasing, and thus,

$$\left|\left(f^1\right)_x'(y;\alpha)\right| \le \left|\left(f^1\right)_x'(x_1;\alpha)\right| \le (1+2\tilde{a}\alpha) < 1, \tag{15}$$

with some nonzero constant $\tilde{a}$, for all $y \le x_1$ (provided $|\alpha|$ small enough).

Now estimate $|H-G|$ on $[x_2, x_1]$ as

$$\begin{aligned}\sup_{y\in[x_2,x_1]} |H(y,\alpha) - G(h,y,\alpha)| &\le \sup_{y\in[x_2,x_1]} \left|f^1\left(H\left(g^{-1}(y;\alpha),\alpha\right);\alpha\right) - f^1\left(G\left(h, g^{-1}(y;\alpha),\alpha\right);\alpha\right)\right| \\ &\quad + \sup_{y\in[x_1,x_0]} \left|f^1(y;\alpha) - f_h^2(y;\alpha)\right| \\ &\le (1+2\tilde{a}\alpha) \sup_{y\in[x_1,x_0]} |H(y,\alpha) - G(h,y,\alpha)| + K_5 h^p \left(-\frac{\alpha}{a}\right).\end{aligned}$$

Repeating inductively, we see that

$$|H(y,\alpha) - G(h,y,\alpha)| \le K_4 h^p + \frac{K_5(-\alpha/a)}{-2\tilde{a}\alpha} h^p = K_6 h^p,$$

for all $y \in (x^-, x_0]$. Finally, at x^- this inequality holds as well.

Finally, we estimate $|H-G|$ on $[y_0, x^-]$. By setting $y_k = g^k(y_0;\alpha)$, $k \in \mathbf{N}$, we have that $\sup_{y\in[y_0,y_1]} |H(y,\alpha) - G(h,y,\alpha)| \le K_6 h^p$; on the other hand,

$$\begin{aligned}\sup_{y\in[y_1,y_2]} |H(y,\alpha) - G(h,y,\alpha)| \le \sup_{y\in[y_1,y_2]} &\left|f^1\left(H\left(g^{-1}(y;\alpha),\alpha\right);\alpha\right) - f^1\left(G\left(h, g^{-1}(y;\alpha),\alpha\right);\alpha\right)\right| \\ &+ \sup_{y\in[y_0,y_1]} \left|f^1(y;\alpha) - f_h^2(y;\alpha)\right|.\end{aligned}$$

Define $a_k = |y_k|^2$. Since

$$\left|\left(f^1\right)_x'(y;\alpha)\right| \le \left|\left(f^1\right)_x'(x^-;\alpha)\right| \le q < 1 \qquad \text{(by (15))}$$

and

$$\sup_{y\in[y_k,y_{k+1}]} \left|f^1(y;\alpha) - f_h^2(y;\alpha)\right| \le K_5 h^p |y_k|^2 = a_k K_5 h^p \qquad \text{(by (13))},$$

inductive application of the above estimate yields

$$\sup_{y\in[y_0,y_{k+1}]} |H(y,\alpha) - G(h,y,\alpha)| \le q^k K_6 h^p + \left(q^{k-1}a_0 + q^{k-2}a_1 + \cdots + a_{k-1}\right) K_5 h^p.$$

Set $c_k = q^k a_0 + \cdots + a_k$ and $b_k = a_k - |x^-|^2$. Then $b_k \to 0$ as $k \to \infty$ and $c_k \le |x^-|^2/(1-q) + \sum_{i=0}^{\infty} b_i$ (for all k). We show that $\sum_{k=0}^{\infty} b_k \le K_7$ with some constant $K_7 > 0$ independent of α and h. This will finish the proof of case $\alpha < 0$ since $|x^-|^2/(1-q) \le K_8$ with some constant K_8 independent of α (and h). We note here that the trivial estimate $c_k \le a_0/(1-q)$ does not work since $1/(1-q) \to \infty$ as $\alpha \to 0$.

We construct a sequence z_k of negative numbers such that $z_0 = y_0$, $z_k > -1/(2a)$ for all $k \in \mathbf{N}$ and

$$z_{k+1} \le z_k + \alpha + a z_k^2, \qquad k = 0, 1, \ldots, \tag{16}$$

hold. With such a sequence in hand (by using that $g(x;\alpha)$ is strictly monotone increasing for $x > -1/(2a)$) we get that $y_k \ge z_k$, and thus $a_k = |y_k|^2 \le |z_k|^2$. To this end, let $z_0 = y_0$, $z_k = -\sqrt{-\alpha/a} + \delta y_0/k^{1-\gamma}$, where $0 < \gamma < 1/2$ and $\delta \ge 1$ will be chosen later. It is easy to see (note that $\delta \geq 1$) that the desired inequality (16) holds for $k = 0$ provided $|\alpha|$ is sufficiently small. It remains to check that

$$-\sqrt{-\frac{\alpha}{a}} + \frac{\delta y_0}{(k+1)^{1-\gamma}} \leq -\sqrt{-\frac{\alpha}{a}} + \frac{\delta y_0}{k^{1-\gamma}} + \alpha + a\left(-\sqrt{-\frac{\alpha}{a}} + \frac{\delta y_0}{k^{1-\gamma}}\right)^2,$$

or equivalently,

$$\frac{k^{2(1-\gamma)}}{(k+1)^{1-\gamma}} \geq \left(1 - 2a\sqrt{-\frac{\alpha}{a}}\right) k^{1-\gamma} + a\delta y_0 \tag{17}$$

holds. We show a slightly stronger inequality, namely,

$$\frac{k^{2(1-\gamma)}}{(k+1)^{1-\gamma}} \geq k^{1-\gamma} + a\delta y_0.$$

It is easy to see that (since $a > 0$)

$$d_k(\gamma) := \frac{k^{1-\gamma}\left(k^{1-\gamma} - (k+1)^{1-\gamma}\right)}{a(k+1)^{1-\gamma}} \to 0, \qquad \text{as } k \to \infty.$$

On the other hand, if γ is sufficiently close to $1/2$, then $d_k(\gamma)$ is strictly monotone increasing with respect to k ($\gamma = 0.4$ works). Note that $d_1(\gamma) > -1/(2a)$. With such a fixed γ, now choose $\delta \geq 1$ such that $d_1(\gamma) \geq \delta y_0$ holds. Note that δ is independent of α. Thus,

$$d_k(\gamma) \geq d_1(\gamma) \geq \delta y_0,$$

and the desired inequality (16) follows. We remark that similarly, the exact asymptotic behavior can be studied about nonhyperbolic equilibria; see [12].

Now we are in a position to prove the convergence of $\sum b_k$. Since $a_k \leq |z_k|^2 \leq |x^-|^2 + |\delta y_0|^2/k^{2(1-\gamma)}$, we have that $b_k \leq |\delta y_0|^2/k^{2(1-\gamma)}$, and the convergence of $\sum b_k$ follows from $2(1-\gamma) > 1$. Finally, note that δ and γ were chosen independently of α, which completes the proof of case $\alpha < 0$.

CASE $\alpha = 0$. The construction of H and G is the same as in the case $\alpha < 0$. The only difference is that $[x^-, x^+] = \{0\}$. Since $|(f^1)'_x(y;0)| \leq 1$ for all $y \in [y_0, 0]$, we arrive at the following estimate:

$$\sup_{y \in [y_0, y_{k+1}]} |H(y,0) - G(h,y,0)| \leq K_8 h^p + (a_{k-1} + \cdots + a_0) K_5 h^p,$$

where as before $a_k = |y_k|^2 = |g^k(y_0;0)|^2$. By using the $\alpha = 0$ variant of the estimate of b_k from Case $\alpha < 0$, we obtain $a_k \leq |\delta y_0|^2/k^{2(1-\gamma)}$ with suitably chosen $0 < \gamma < 1/2$ and $\delta \geq 1$, and thus

$$\sup_{y \in [y_0, 0]} |H(y,0) - G(h,y,0)| \leq K_9 h^p.$$

CASE $\alpha > 0$. The construction of H and G is the same as in the case $\alpha < 0$. The only difference is that we do not make use of $-y_0$; i.e., only one initial point is necessary. Although we reach $x = 0$ in a finite number of steps for all $\alpha > 0$, the number of these steps tends to infinity as $\alpha > 0$ tends to zero.

Since $|(f^1)'_x(y;\alpha)| \leq 1$ for all $y \in [y_0, 0]$, we arrive at the following estimate:

$$\sup_{y \in [y_0, y_{k+1}]} |H(y,\alpha) - G(h,y,\alpha)| \leq K_8 h^p + (a_{k-1} + \cdots + a_0) K_5 h^p,$$

where $a_k = |y_k|^2 = |g^k(y_0;\alpha)|^2$. For $y_k \leq 0$, we show that $a_k \leq |g^k(y_0;0)|^2$. But this holds because $g^k(y_0;\alpha) > g^k(y_0;0)$. (Case $k = 0$ is clear ($\alpha > 0$). By induction, using that $g(x;\alpha)$ is monotone increasing, we have that $g^{k+1}(y_0;\alpha) = g^k(y_0;\alpha) + \alpha + a(g^k(y_0,\alpha))^2 > g^k(y_0;0) + \alpha + a(g^k(y_0,0))^2 > g^k(y_0;0) + a(g^k(y_0,0))^2 = g^{k+1}(y_0,0)$.) Thus, $a_{k-1} + \cdots + a_0 \leq \sum_{k=0}^{\infty} |g^k(y_0;0)|^2$, and as a result

$$\sup_{y \in [y_0, 0]} |H(y,\alpha) - G(h,y,\alpha)| \leq K_{10} h^p,$$

which completes the proof of the theorem. ∎

We end this section with a consequence of Theorem 1 claiming that $\Phi(1)$ and $\phi^{[1/h]}$ conjugate.

COROLLARY 1. *$\Phi(1)$ and $\phi^{[1/h]}$ conjugate in a neighborhood of the 0 equilibrium in $\mathbf{R}^n \times \mathbf{R}$.*

PROOF. By using the generalized Hartman-Grobman theorem for maps, see, e.g., [5,6], we get that $\Phi(1)$ conjugates with $\Phi_C(1)$ times a standard linear saddle and $\phi^{[1/h]}$ conjugates with $\phi_{C_h}^{[1/h]}$ times a standard linear saddle. Moreover, using the C^j-closeness the linear saddles are the same. From Theorem 1, it follows that $\Phi_C(1)$ and $\phi_{C_h}^{[1/h]}$ conjugate since their normal forms conjugate. Thus, we obtain the desired result. ∎

4. FINAL REMARKS

We conjecture that the conjugacy appearing in Corollary 1 is $O(h^p)$-close to the identity. However, we admit that we cannot prove this closeness result by using the techniques of [6] or [5]. On the other hand, it is proved that partial linearization, see [13], can be carried out within the order of $O(h^p)$. Moreover, certain invariant foliations (which are the main tool in proving the generalized Hartman-Grobman theorem) are preserved by the numerical method in the C^j-norm to the order of $O(h^p)$; see [14].

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