Global Attractivity for a Nonlinear Difference Equation with Variable Delay

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Abstract—In this paper, we give sufficient conditions under which every solution of the nonlinear difference equation with variable delay
\[ x(n+1) - x(n) + p_n f(x(g(n))) = 0, \quad n = 0, 1, 2, \ldots \]
tends to zero as \( n \to \infty \). Here, \( \{p_n\} \) is a nonnegative sequence, \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function with \( xf(x) > 0 \) if \( x \neq 0 \), and \( g : \mathbb{N} \to \mathbb{Z} \) is nondecreasing and satisfies \( g(n) \leq n \) for \( n \geq 0 \) and \( \lim_{n \to \infty} g(n) = \infty \). © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In the last decade, there has been much literature on qualitative properties of delay difference equations, whose study has been deeply inspired by the development of the study of delay differential equations. For the general background of delay difference equations, one can refer to recent books [1-3] and [4, Chapter 7].

In this paper, we consider a nonlinear difference equation with variable delay
\[ x(n+1) - x(n) + p_n f(x(g(n))) = 0, \quad n = 0, 1, 2, \ldots \] (1.1)

Here \( \{p_n\} \) is a nonnegative sequence, \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function with \( xf(x) > 0 \) if \( x \neq 0 \), and \( g : \mathbb{N} \to \mathbb{Z} \) is nondecreasing and satisfies \( g(n) \leq n \) for \( n \geq 0 \) and \( \lim_{n \to \infty} g(n) = \infty \).\( \mathbb{N} = \{0, 1, 2, \ldots \} \) and \( \mathbb{Z} = \{ \ldots, -1, 0, 1, \ldots \} \).

By a solution of (1.1), we mean a sequence \( \{x(n)\} \) which is defined for \( n \geq g(0) \) and satisfies (1.1) for \( n \geq 0 \). Concerning the variable delay in (1.1), we consider two cases.

(i) \( n - g(n) \) is bounded, that is, there exists a positive integer \( k \) such that
\[ n - g(n) \leq k, \quad \text{for all large } n. \] (1.2)
(ii) $n - g(n)$ is unbounded, that is,

$$\lim_{n \to \infty} (n - g(n)) = \infty. \quad (1.3)$$

We see that $g(n) = n - k$ and $g(n) = \lceil n/2 \rceil$, where $\lceil \cdot \rceil$ denotes the greatest integer function, are typical examples of (1.2) and (1.3), respectively.

For the linear difference equation

$$x(n + 1) - x(n) + p_n x(g(n)) = 0, \quad n = 0, 1, 2, \ldots, \quad (1.4)$$

many authors [5-9] have studied the asymptotic behavior of solutions. Recently, Zhang et al. [9] gave the following result.

**Theorem A.** Assume that

$$\limsup_{n \to \infty} \sum_{i=g(n)}^{n} p_i < \begin{cases} \frac{3}{2} + \frac{1}{2(k+1)}, & \text{if (1.2) holds,} \\ \frac{3}{2}, & \text{if (1.3) holds} \end{cases}$$

and

$$\sum_{i=0}^{\infty} p_i = \infty.$$

Then every solution of (1.4) tends to zero as $n \to \infty$.

For the nonlinear equation (1.1), however, a few results on the asymptotic behavior of solutions are obtained [10,11]. The purpose of this paper is to give sufficient conditions for global attractivity of (1.1) under the following strict nonlinearity on $f(x)$:

$$|f(x)| < |x|, \quad \text{if } x \neq 0. \quad (1.5)$$

The main result is stated as follows.

**Theorem 1.1.** Assume that (1.5) holds. Suppose

$$\sum_{i=g(n)}^{n} p_i \leq \begin{cases} \frac{3}{2} + \frac{1}{2(k+1)}, & \text{if (1.2) holds,} \\ \frac{3}{2}, & \text{if (1.3) holds} \end{cases} \quad (1.6)$$

for all large $n$, and

$$\sum_{i=0}^{\infty} p_i = \infty. \quad (1.7)$$

Then every solution of (1.1) tends to zero as $n \to \infty$.

Our research is motivated by a corresponding result on delay differential equations in [12].

**2. PROOF OF THEOREM 1.1**

Throughout this paper, we will use the convention

$$\sum_{i=n}^{m} p_i \equiv 0, \quad \text{whenever } m \leq n - 1.$$

Before stating the proof of Theorem 1.1, we give some remarks. First, in view of the property of $g$, we see that there exists a sufficiently large integer $n_0$ such that $g(n) \geq 0$ for $n \geq n_0$. Note that

$$g(g(n)) \leq g(n) \leq n, \quad \text{for } n \geq n_0.$$
Next, for \( n \geq 0 \), let \( g^{-1}(n) = \sup\{m: g(m) \leq n\} \). Then
\[
n \leq g^{-1}(n), \quad \text{for } n \geq n_0.
\]

As a beginning, we notice that if the solution of (1.1) is nonoscillatory, then the following result holds.

**Lemma 2.1.** If (1.7) holds, then every nonoscillatory solution of (1.1) tends to zero as \( n \to \infty \).

**Proof.** Let \( x(n) \) be a nonoscillatory solution of (1.1). Then there exists a sufficiently large integer \( l_1 \geq g^{-1}(n_0) \) such that
\[
x(n) > 0 \quad \text{or} \quad x(n) < 0, \quad \text{for } n \geq g(l_1).
\]
Assume first that \( x(n) > 0 \) for \( n \geq g(l_1) \). (In case \( x(n) < 0 \), the proof is similar.) Then we have
\[
x(n + 1) - x(n) = -p_n f(x(g(n))) \leq 0, \quad \text{for } n \geq l_1.
\]
Hence, \( x(n) \) is nonincreasing for \( n \geq l_1 \) and there exists a nonnegative real number \( \alpha \) such that
\[
\lim_{n \to \infty} x(n) = \alpha.
\]
If \( \alpha > 0 \), there exists a positive integer \( l_2 = l_2(\alpha) \geq g^{-1}(l_1) \) such that
\[
\frac{\alpha}{2} \leq x(n) \leq \frac{3}{2} \alpha, \quad \text{for } n \geq g(l_2),
\]
which implies
\[
x(n + 1) - x(n) \leq -p_n v, \quad \text{for } n \geq l_2,
\]
where \( v = \min_{\alpha/2 \leq u \leq 3/2} f(u) > 0 \). Thus, summing up (2.1) from \( l_2 \) to \( n \) and using (1.7), we obtain
\[
x(n + 1) - x(l_2) \leq -v \sum_{i=l_2}^{n} p_i \to -\infty, \quad \text{as } n \to \infty.
\]
This contradicts the fact that the left-hand side tends to a finite limit as \( n \to \infty \), and so \( \alpha = 0 \). The proof is complete.

**Proof of Theorem 1.1.** By virtue of Lemma 2.1, we have only to consider the case where solutions of (1.1) are oscillatory.

Let \( x(n) \) be an oscillatory solution of (1.1). First, we consider (1.1) with bounded delay. By (1.2), there exists a sufficiently large integer \( n_1 \geq n_0 \) such that
\[
n - g(n) \leq k, \quad \text{for } n \geq n_1.
\]
We remark that the definition of \( g^{-1} \) yields
\[
g^{-1}(n) - n \leq k, \quad \text{for } n \geq n_1.
\]
From (1.6) and the oscillatory property of \( x(n) \), there exists a sufficiently large integer \( n_2 \geq g^{-1}(n_0) \) with \( g(g(n_2 - 1)) \geq 0 \) such that
\[
x(n_2 - 1) x(n_2) \leq 0 \quad \text{and} \quad |x(n_2)| > 0,
\]
and
\[
\sum_{i=g(n)}^{n} p_i \leq \frac{3}{2} + \frac{1}{2(k + 1)}, \quad \text{for } n \geq g(g(n_2 - 1)).
\]
Then, by (2.3), there exists a real number \( \xi \in [n_2 - 1, n_2) \) such that
\[
x(n_2 - 1) + \{x(n_2) - x(n_2 - 1)\} (\xi - n_2 + 1) = 0.
\]
Let $M$ be a positive constant satisfying
\[
\max_{g(n^2-1) \leq n \leq n^2-1} |x(n)| \leq M.
\]
We here define
\[
\tilde{f}(x) = \max \left\{ \sup_{0 \leq u \leq x} f(u), \sup_{0 \leq u \leq x} (-f(-u)) \right\}, \quad \text{for } x \geq 0,
\]
then, from the nondecreasing property of $\tilde{f}$, it turns out that
\[
|f(x(n))| \leq \tilde{f}(|x(n)|) \leq \tilde{f}(M), \quad \text{for } g(g(n^2-1)) \leq n \leq n^2-1.
\]
Hence, we have
\[
|x(n+1) - x(n)| = p_n |f(x(g(n)))| \leq p_n \tilde{f}(M), \quad \text{for } n^2-1 \leq n \leq g^{-1}(n^2-1).
\]
Also since
\[
|x(n^2-1) - x(g(n))| \leq \sum_{j=g(n)}^{n^2-2} |x(j+1) - x(j)|
= \sum_{j=g(n)}^{n^2-2} p_j |f(x(g(j)))| \leq \tilde{f}(M) \sum_{j=g(n)}^{n^2-2} p_j, \quad \text{for } n^2-1 \leq n \leq g^{-1}(n^2-1),
\]
we get, together with (2.5) and (2.6),
\[
|x(g(n))| \leq |x(n^2-1)| + \tilde{f}(M) \sum_{j=g(n)}^{n^2-2} p_j
= |x(n^2) - x(n^2-1)| (\xi - n^2 + 1) + \tilde{f}(M) \sum_{j=g(n)}^{n^2-2} p_j
\leq p_{n^2-1} \tilde{f}(M) (\xi - n^2 + 1) + \tilde{f}(M) \sum_{j=g(n)}^{n^2-2} p_j \quad \text{for } n^2-1 \leq n \leq g^{-1}(n^2-1).
\]
Thus, it follows from (1.5) and (2.7) that
\[
|x(n+1) - x(n)| \leq p_n |x(g(n))| \leq \tilde{f}(M) p_n \sum_{j=g(n)}^{n^2-1} p_j (\xi - n^2 + 1),
\quad \text{for } n^2-1 \leq n \leq g^{-1}(n^2-1).
\]
Now we will show that
\[
|x(n)| \leq \tilde{f}(M), \quad \text{for } n^2 \leq n \leq g^{-1}(n^2-1) + 1.
\]
There are two possible cases to consider. Hereafter, for simplicity, we put $\beta = 3/2 + 1/2(k+1)$. 

CASE (I). $d = \sum_{i=n_2}^{g^{-1}(n_2 - 1)} p_i + (n_2 - \xi)p_{n_2 - 1} \leq 1$. By using (2.4), (2.5), and (2.8), we see that for $n_2 \leq n \leq g^{-1}(n_2 - 1) + 1,$

$$|x(n)| \leq |x(n_2)| + \sum_{i=n_2}^{n-1} |x(i + 1) - x(i)|$$

$$\leq |x(n_2) - x(n_2 - 1)|(n_2 - \xi) + \sum_{i=n_2}^{g^{-1}(n_2 - 1)} |x(i + 1) - x(i)|$$

$$\leq (n_2 - \xi)\tilde{f}(M)p_{n_2 - 1} \left\{ \sum_{j=g(n_2 - 1)}^{n_2 - 1} p_j - (n_2 - \xi)p_{n_2 - 1} \right\}$$

$$+ \tilde{f}(M) \sum_{i=n_2}^{g^{-1}(n_2 - 1)} p_i \left\{ \sum_{j=g(i)}^{n_2 - 1} p_j - (n_2 - \xi)p_{n_2 - 1} \right\}$$

$$\leq \tilde{f}(M) \left[ (n_2 - \xi)p_{n_2 - 1} \left\{ \beta - (n_2 - \xi)p_{n_2 - 1} \right\} \right.$$  

$$+ \sum_{i=n_2}^{g^{-1}(n_2 - 1)} p_i \left\{ \beta - \sum_{j=n_2}^{i} p_j - (n_2 - \xi)p_{n_2 - 1} \right\}$$

$$= \tilde{f}(M) \left[ \beta \left\{ \sum_{i=n_2}^{g^{-1}(n_2 - 1)} p_i + (n_2 - \xi)p_{n_2 - 1} \right\} - \sum_{i=n_2}^{g^{-1}(n_2 - 1)} p_i \sum_{j=n_2}^{i} p_j \right.$$  

$$- (n_2 - \xi)p_{n_2 - 1} \sum_{i=n_2}^{g^{-1}(n_2 - 1)} p_i - (n_2 - \xi)^2 p_{n_2 - 1}^2$$

$$= \tilde{f}(M) \left[ \beta d - \frac{1}{2} \left( \sum_{i=n_2}^{g^{-1}(n_2 - 1)} p_i \right)^2 - \frac{1}{2} \sum_{i=n_2}^{g^{-1}(n_2 - 1)} p_i^2 \right.$$  

$$- (n_2 - \xi)p_{n_2 - 1} \sum_{i=n_2}^{g^{-1}(n_2 - 1)} p_i - (n_2 - \xi)^2 p_{n_2 - 1}^2$$

$$= \tilde{f}(M) \left[ \beta d - \frac{1}{2} d^2 - \frac{1}{2} \left( \sum_{i=n_2}^{g^{-1}(n_2 - 1)} p_i^2 + (n_2 - \xi)^2 p_{n_2 - 1}^2 \right) \right].$$

Taking account of (2.2) and the inequality

$$\left( \sum_{i=1}^{m} y_i \right)^2 \leq m \sum_{i=1}^{m} y_i^2, \quad \text{for } y_i \in \mathbb{R}, \quad 1 \leq i \leq m,$$

we obtain

$$|x(n)| \leq \tilde{f}(M) \left\{ \beta d - \frac{1}{2} d^2 - \frac{1}{2} \cdot \frac{1}{g^{-1}(n_2 - 1) - (n_2 - 1) + 1} \left( \sum_{i=n_2}^{g^{-1}(n_2 - 1)} p_i + (n_2 - \xi)p_{n_2 - 1} \right)^2 \right\}$$

$$\leq \tilde{f}(M) \left\{ \beta d - \frac{1}{2} d^2 - \frac{1}{2(k + 1)} \right\}.$$
\[= \tilde{f}(M) \left\{ \beta d - \frac{k + 2}{2(k + 1)} d^2 \right\} \]
\[\leq \tilde{f}(M) \left\{ \beta - \frac{k + 2}{2(k + 1)} \right\} = \tilde{f}(M), \quad \text{for } n_2 \leq n \leq g^{-1}(n_2 - 1) + 1.\]

**CASE (II).** \(d = \sum_{i=n_2}^{g^{-1}(n_2-1)} p_i + (n_2 - \xi) p_{n_2-1} > 1.\) In this case, there exists a positive integer \(n_3 \in [n_2, g^{-1}(n_2 - 1) + 1]\) such that
\[g^{-1}(n_2-1) \sum_{i=n_3} p_i \leq 1 \quad \text{and} \quad g^{-1}(n_2-1) \sum_{i=n_3} p_i > 1.\]

Therefore, there exists a real number \(\eta \in (n_3 - 1, n_3]\) such that
\[g^{-1}(n_3-1) \sum_{i=n_3} p_i + (n_3 - \eta) p_{n_3-1} = 1. \quad (2.11)\]

It follows from (2.5) that for \(n_2 \leq n \leq g^{-1}(n_2 - 1) + 1,\)
\[x(n) = x(n_2) + \sum_{i=n_2}^{n_3-2} \{x(i + 1) - x(i)\} + x(n_3) - x(n_3 - 1) + \sum_{i=n_3}^{n-1} \{x(i + 1) - x(i)\}\]
\[= \{x(n_2) - x(n_2 - 1)\} (n_2 - \xi) + \sum_{i=n_2}^{n_3-2} \{x(i + 1) - x(i)\} + (1 - n_3 + \eta) \{x(n_3) - x(n_3 - 1)\}\]
\[+ (n_3 - \eta) \{x(n_3) - x(n_3 - 1)\} + \sum_{i=n_3}^{n-1} \{x(i + 1) - x(i)\},\]

and hence, we get
\[|x(n)| \leq S_1 + S_2, \quad \text{for } n_2 \leq n \leq g^{-1}(n_2 - 1) + 1, \quad (2.12)\]

where
\[S_1 = |x(n_2) - x(n_2 - 1)| (n_2 - \xi) + \sum_{i=n_2}^{n_3-2} |x(i + 1) - x(i)| + (1 - n_3 + \eta) |x(n_3) - x(n_3 - 1)|,\]
\[S_2 = (n_3 - \eta) |x(n_3) - x(n_3 - 1)| + g^{-1}(n_2-1) \sum_{i=n_3}^{n-1} |x(i + 1) - x(i)|.\]

In view of (2.6), we have
\[S_1 \leq \tilde{f}(M)p_{n_2-1} (n_2 - \xi) + \sum_{j=n_2}^{n_3-2} \tilde{f}(M)p_j + (\eta - n_3 + 1) \tilde{f}(M)p_{n_3-1}.\]

By (2.8),
\[S_2 \leq (n_3 - \eta) \tilde{f}(M)p_{n_3-1} \left\{ \sum_{i=g(n_3-1)}^{n_3-1} p_i - (n_2 - \xi) p_{n_2-1} \right\} \]
\[+ \sum_{i=n_3}^{g^{-1}(n_2-1)} \tilde{f}(M)p_i \left\{ \sum_{j=g(i)}^{n_3-1} p_j - (n_2 - \xi) p_{n_2-1} \right\}.\]
From (2.4), (2.11), and the above inequalities, thus, (2.12) implies that for \( n_2 \leq n \leq g^{-1}(n_2 - 1) + 1 \),

\[
|x(n)| \leq \tilde{f}(M) \left[ \frac{\sum_{i=n_3}^{g^{-1}(n_2-1)} p_i + (n_3 - \eta) p_{n_3-1}}{\sum_{i=n_3}^{g^{-1}(n_2-1)} p_i + (n_3 - \eta) p_{n_3-1}} \right]
\]

Using (2.2) and (2.10) again, we arrive at

\[
|x(n)| \leq \tilde{f}(M) \left[ \frac{\sum_{i=n_3}^{g^{-1}(n_2-1)} p_i + (n_3 - \eta) p_{n_3-1}}{\sum_{i=n_3}^{g^{-1}(n_2-1)} p_i + (n_3 - \eta) p_{n_3-1}} \right] = \tilde{f}(M),
\]
Furthermore, we claim that

$$|x(n)| \leq \hat{f}(M), \quad \text{for } n \geq n_2. \quad (2.13)$$

Suppose, for the sake of contradiction, that

$$|x(N)| > \hat{f}(M), \quad \text{for some } N > g^{-1}(n_2 - 1) + 1.$$

Then there exist positive integers $m_1$ and $m_2$ such that

$$m_1 = \sup \left\{ n > g^{-1}(n_2 - 1) + 1 : |x(j - 1)| \leq \hat{f}(M) \text{ for } g^{-1}(n_2 - 1) + 1 \leq j \leq n \text{ and } \right\},$$

$$m_2 = \sup \left\{ n_2 \leq n \leq m_1 : \frac{x(n - 1)x(n)}{|x(n)|} \leq 0 \text{ and } |x(n)| > 0 \right\}.$$

In case $m_1 - 1 \leq g^{-1}(m_2 - 1)$, noticing that

$$\max_{g(g(n; - 1)) \leq n \leq m_2 - 1} |x(n)| \leq M,$$

we get

$$|x(n)| \leq \hat{f}(M), \quad \text{for } m_2 \leq n \leq g^{-1}(m_2 - 1) + 1,$$

which contradicts the definition of $m_1$. In case $m_1 - 1 > g^{-1}(m_2 - 1)$, it follows from the choice of $m_1$ and $m_2$ that:

$$x(n) > 0 \quad \text{or} \quad x(n) < 0, \quad \text{for } m_2 \leq n \leq m_1 - 1.$$

Assume that $x(n) > 0$ for $m_2 \leq n \leq m_1 - 1$. (In case $x(n) < 0$, the proof is similar.) If $x(m_1) > 0$, then we have

$$0 < x(m_1) = x(m_1 - 1) - p_{m_1 - 1}f(x(g(m_1 - 1))) \leq x(m_1 - 1) \leq \hat{f}(M)$$

because of $m_2 \leq g(m_1 - 1) \leq m_1 - 1$. This contradicts the definition of $m_1$. If $x(m_1) < 0$, then we must have $m_1 = m_2$, which contradicts $m_1 - 1 > g^{-1}(m_2 - 1)$. Consequently, (2.13) is valid.

From the argument above, we can establish an increasing sequence $\{ n_j^* \}$ with $n_j^* = n_2$ such that $\lim_{j \to \infty} n_j^* = \infty$,

$$x(n_j^* - 1)x(n_j^*) \leq 0 \quad \text{and} \quad |x(n_j^*)| > 0,$$

and a sequence $\{ X_j \}$ with $X_1 = M$, $X_{j+1} = \hat{f}(X_j)$ such that

$$\max_{g(g(n_j^* - 1)) \leq n \leq n_j^* - 1} |x(n)| \leq X_j \quad \text{and} \quad \sup_{n \geq n_j^*} |x(n)| \leq X_{j+1}. \quad (2.14)$$

Here, in view of (1.5) and the definition of $\hat{f}$, it is easily seen that

$$\hat{f}(x) < x, \quad \text{for } x > 0,$$

which implies $X_j$ tends to zero as $j \to \infty$. Therefore, by (2.14), we conclude that $x(n)$ tends to zero as $n \to \infty$.

For equation (1.1) with unbounded delay, replacing $k$ with $\infty$, we can easily carry out the proof in a similar way to the case with bounded delay. This completes the proof of Theorem 1.1.
Chen and Liu [10] gave sufficient conditions for global attractivity of (1.1) with constant delay, that is,

\[
x(n + 1) - x(n) + p_n f(x(n - k)) = 0, \quad n = 0, 1, 2, \ldots,
\]

(2.15)

where \( k \) is a positive integer. Their result can be stated as follows.

**THEOREM B.** Assume that

\[
|f(x)| \leq |x|, \quad \text{for } x \in \mathbb{R}.
\]

Suppose that (1.7) holds and

\[
\limsup_{n \to \infty} \sum_{i=n-2k}^{n} p_i < 2. \quad (2.16)
\]

Then every solution of (2.15) tends to zero as \( n \to \infty \).

In case \( p_n \equiv p > 0 \), we notice that condition (2.16) implies

\[
p < \frac{2}{2k+1}.
\]

On the other hand, condition (1.6) is reduced to

\[
p \leq \frac{1}{k+1} \left\{ \frac{3}{2} + \frac{1}{2(k+1)} \right\},
\]

which means that (1.6) is sharper than (2.16) in some sense.

**REFERENCES**